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FRACTIONAL SPACE-TIME VARIATIONAL FORMULATIONS OF (NAVIER–) STOKES EQUATIONS*

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Abstract. Well-posed space-time variational formulations in fractional order Bochner–Sobolev spaces are proposed for parabolic partial differential equations, and in particular for the instationary Stokes and Navier–Stokes equations on bounded Lipschitz domains. The latter formulations include the pressure variable as a primal unknown and so account for the incompressibility constraint via a Lagrange multiplier. The proposed new variational formulations can be the basis of adaptive numerical solution methods that converge with the best possible rate, which, by exploiting the tensor product structure of a Bochner space, equals the rate of best approximation for the corresponding stationary problem. Unbounded time intervals are admissible in many cases, permitting an optimal adaptive solution of long-term evolution problems.

Key words. Navier–Stokes equations, parabolic PDEs, space-time variational formulations, interpolation spaces, wavelets

AMS subject classifications. 35A15, 35Q30, 65M12, 65T60, 76D05

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1. Introduction.

The topic of this paper is the development of *well-posed* space-time variational formulations of parabolic partial differential equations (PDEs) and instationary Stokes and Navier–Stokes equations. Here and below, well-posed means that the corresponding operator is *boundedly invertible*, or in the case of a nonlinear equation, that its Fréchet derivative at the solution has this property.

We emphasize that the question about well-posedness is different from the (intensively studied) questions about existence, uniqueness, and regularity of solutions for right-hand sides in appropriate spaces. Indeed, for the latter questions it is not an issue whether the corresponding parabolic operator is onto, i.e., whether its range is equal to appropriate spaces of right-hand sides.

The present investigation of well-posedness is motivated by the development of numerical solution methods. First, a numerical discretization can only lead to a matrix-vector equation that is well-conditioned, uniformly in its size, when a continuous, infinite-dimensional operator equation of the evolution is well-posed. In that case, it is known by now how numerical solution algorithms of optimal asymptotic computational complexity can be developed.

Second, for a well-posed problem, the norm of the residual of an approximate solution is proportional to the norm of its error. Such an equivalence is paramount for the development of adaptive solution methods that converge with the best possible rates, in linear computational complexity. Although our interest mainly lies in the construction of adaptive wavelet schemes, these observations about the necessity of well-posedness apply equally well to other numerical solution methods, like finite element methods.

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For evolution problems, as parabolic problems and instationary (Navier–) Stokes equations, traditionally time marching schemes are applied. By applying an (implicit) time semidiscretization, a sequence of stationary, elliptic PDEs in the spatial domain is obtained that have to be solved sequentially. One drawback of this approach is that it is inherently serial and not well suited for a parallel implementation. Recent years have seen the emergence of methods that aim at overcoming or at least reducing this disadvantage, e.g., the parareal method (see [LMT01]). Successive time stepping entails, moreover, that the time increment Δt at time t is (essentially) independent of the spatial location. In particular, successive time stepping does not generally allow an efficient approximation of singularities that are localized in space *and* in time. Finally, in applications where an approximation of the whole time evolution is needed, as with problems of optimal control or in visualizations, successive time stepping requires a huge amount of storage.

Having a well-posed space-time variational formulation of an evolutionary PDE at hand, we advocate to solve the evolution problem numerically *as one operator equation on the space-time cylinder* with an adaptive wavelet scheme [CDD01, SS09, CS11, Ste14]. Such methods are "embarrassingly parallel" and converge with the best possible rate from the basis. Moreover, since these bases are constructed as tensor products of bases in space and time, under mild (Besov) smoothness conditions on the solution this best possible rate is equal to that when solving one instance of the corresponding stationary problem. The latter property induces the reduction in computational cost and storage not afforded by sequential time stepping.

Although we have wavelet schemes in mind, we emphasize that the advantages of starting from a well-posed space-time variational formulation apply equally well to other space-time solution schemes; see, e.g., [BJ89, BJ90, Tan13, UP14, And14, Mol14, AT15, LMN15, Ste15].

The interest in simultaneous space-time solution methods mainly arose in recent years. Therefore, it still has to be seen to which extent the (mathematically provable) asymptotic superiority of these discretization methods materalizes in ranges of accuracy which are relevant in practical applications. Another issue is that the present results for the instationary (Navier–) Stokes equations are not shown to hold uniformly in the viscosity parameter.

This paper is organized as follows. In sections 2–4, we consider parabolic PDEs. By application of the method of real interpolation, we derive well-posed space-time variational formulations w.r.t. scales of spaces, being intersections of Bochner spaces, with which we generalize results known from the literature.

In section 5, the core of this paper, we construct well-posed space-time variational formulations for the instationary Stokes problem. Although we build on results obtained for the parabolic problem, we are not content with a formulation of the flow problem as a parabolic problem for the divergence-free velocities. Indeed, only in special cases can (Sobolev) spaces of divergence-free functions be equipped with wavelet Riesz bases or, for other solution methods, with a dense nested sequence of trial spaces. Therefore, well-posed variational formulations are constructed for the saddle-point problem involving the pair of velocities and pressure.

In our previous work [GSS14] we arrived at a formulation that contains Sobolev spaces of smoothness index 2. The same holds true for the formulations derived in [Köh13] that allow more general boundary conditions and that extend to Banach spaces. Such Sobolev spaces require trial spaces of globally C^1-functions whose construction is cumbersome on nonproduct domains. In the current work, such spaces are avoided, and the arising spaces can be conveniently equipped with continuous piecewise polynomial wavelet Riesz bases, for general polytopal spatial domains (see, e.g., [DS99]). Related to this is that, unlike [GSS14, Köh13], we avoid making a "full-regularity" assumption on the stationary Stokes operator, so that the current space-time variational formulations are well-posed on general bounded spatial Lipschitz domains. To establish the necessary inf-sup conditions, a key role is played by the right-inverse of the divergence operator that was introduced in [Bog79].

Finally, in section 6 the results are extended to the instationary Navier–Stokes equations. The results concerning the bounded invertibility of the instationary Stokes operator are extended to the Oseen operator, being the Fréchet derivative of the instationary Navier–Stokes operator. The spaces with respect to which we show well-posedness satisfy all requirements to lead to easily implementable discretizations in $n = 2$ space dimensions, but not in $n = 3$. In the latter case, some function spaces still mandate trial functions that are continuously differentiable as a function of the spatial variable. Furthermore, our formulations for the instationary Navier–Stokes operator do not allow for a convenient incorporation of nonhomogeneous initial conditions.

In this work, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For normed linear spaces E and F, by $\mathcal{L}(E, F)$ we will denote the normed linear space of bounded linear mappings $E \to F$ and by $\mathcal{L}\mathrm{is}(E, F)$ its subset of boundedly invertible linear mappings $E \to F$. We write $E \hookrightarrow F$ to denote that E is continuously embedded into F. For simplicity only, we exclusively consider linear spaces over the scalar field $\mathbb{R}$.

2. "Classical" variational formulations of linear parabolic problems.

We recall known results on well-posedness of space-time variational formulations of parabolic PDEs and extend them to unbounded time intervals.

Let V, H be separable Hilbert spaces of functions on some "spatial domain" such that $V \hookrightarrow H$ with dense and compact embedding. Identifying H with its dual, we obtain the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$.

We use the notation $\langle \cdot, \cdot \rangle$ to denote both the scalar product on $H \times H$ and its unique extension by continuity to the duality pairing on $W' \times W$ for any densely embedded $W \hookrightarrow H$.

Let $-\infty \leq \alpha < \beta \leq \infty$ and denote, for a.e.

$$t \in I := (\alpha, \beta),$$

by $a(t; \cdot, \cdot)$ a bilinear form on $V \times V$ such that for any $\eta, \zeta \in V$, $t \mapsto a(t; \eta, \zeta)$ is measurable on I and such that, for some constants $M, \gamma > 0$ and $\lambda \geq 0$, for a.e. $t \in I$,

$$|a(t; \eta, \zeta)| \leq M \|\eta\|_V \|\zeta\|_V \quad (\eta, \zeta \in V) \quad (\textit{boundedness}), \tag{2.1}$$

$$a(t; \eta, \eta) + \lambda \|\eta\|_H^2 \geq \gamma \|\eta\|_V^2 \quad (\eta \in V) \quad (\textit{Gårding inequality}). \tag{2.2}$$

For $|I| = \infty$, we will need (2.2) for $\lambda = 0$, i.e.,

$$a(t; \eta, \eta) \geq \gamma \|\eta\|_V^2 \quad (\eta \in V) \quad (\textit{coercivity}). \tag{2.3}$$

With $A(t) \in \mathcal{L}(V, V')$ being defined by $(A(t)\eta)(\zeta) = a(t;\eta,\zeta)$, we are interested in solving the *parabolic initial value problem* to find u such that

$$\left\{\begin{array}{c} \frac{du}{dt}(t) + A(t)u(t) = g(t) \quad (t \in I), \\ u(\alpha) = u_\alpha, \end{array}\right. \tag{2.4}$$

where for $\alpha = -\infty$, the initial condition should be omitted.

In a simultaneous space-time variational formulation, the parabolic PDE reads as finding u from a suitable space of functions of space and time such that

$$(Bw)(v) := \int_I \langle \tfrac{dw}{dt}(t), v(t)\rangle + a(t; w(t), v(t))dt = \int_I \langle g(t), v(t)\rangle =: g(v) \tag{2.5}$$

for all v from another suitable space of functions of space and time.

In [SS09], the initial condition was appended by testing it against additional test functions. There, the following result was proved (see also [DL92, Chap. XVIII, sect. 3] and [Wlo82, Chap. IV, sect. 26] for slightly different statements).

THEOREM 2.1. *For $-\infty < \alpha < \beta < \infty$, and under conditions* (2.1)–(2.2)*, with*

$$(B_e w)(v_1, v_2) := (Bw)(v_1) + \langle w(\alpha), v_2\rangle,$$

it holds that

$$B_e \in \mathcal{L}\mathrm{is}(L_2(I;V) \cap H^1(I;V'), (L_2(I;V) \times H)'),$$

with the norm of B_e^{-1} being bounded by an increasing function of γ^{-1}, M, $|I|^{-1}$, and $\max(0, \lambda|I|)$.

Using this theorem, for given $g \in L_2(I;V')$ and $u_\alpha \in H$, a valid, well-posed variational formulation of (2.4) reads as finding $u \in L_2(I;V) \cap H^1(I;V')$ such that

$$(B_e u)(v_1, v_2) = g(v_1) + \langle u_\alpha, v_2\rangle \quad ((v_1, v_2) \in L_2(I;V) \times H), \tag{2.6}$$

or, in operator form, as $B_e u = [g \ \ u_\alpha]^\top$. For completeness, with a well-posed weak formulation, we mean one that corresponds to a boundedly invertible mapping.

A necessary ingredient for Theorem 2.1 is that

$$L_2(I;V) \cap H^1(I;V') \hookrightarrow C(\bar{I}, H); \tag{2.7}$$

see, e.g., [DL92, Chap. XVIII, sect. 1, Thm. 1] for a proof of this continuous embedding result. By definition of the norms involved, the norm of the embedding depends only on $\beta - \alpha$ when it tends to zero.

From the norm of B_e^{-1} being uniformly bounded for $|I| \to \infty$ when $\lambda = 0$ one infers the following.

COROLLARY 2.2. *For $-\infty < \alpha < \beta \le \infty$, and under conditions* (2.1) *and* (2.2)*, or* (2.3) *when $\beta = \infty$,*

$$B_e \in \mathcal{L}\mathrm{is}(L_2(I;V) \cap H^1(I;V'), (L_2(I;V) \times H)').$$

As a preparation for handling the case that $\alpha = -\infty$, next we focus on the case of having a *homogeneous* initial condition. For $s \ge 0$, and $\delta \in \{\alpha, \beta\}$, let

$$H^s_{0,\{\delta\}}(I) := \mathrm{clos}_{H^s(I)}\{v \in C^\infty(\bar{I})\colon \operatorname{supp} v \cap \{\delta\} = \emptyset\}.$$

Note that $H^s_{0,\{\delta\}}(I) = H^s(I)$ when $\delta = \pm\infty$.

Noting that for $w \in H$, $w = 0$ in H is equivalent to $w = 0$ in V', from (2.7) we infer that for $-\infty < \alpha$,

$$L_2(I;V) \cap H^1_{0,\{\alpha\}}(I;V') = \{w \in L_2(I;V) \cap H^1(I;V') \colon w(\alpha) = 0 \text{ in } H\}.$$

From Corollary 2.2, we conclude the following.

THEOREM 2.3. *For $-\infty < \alpha < \beta \leq \infty$, and under conditions* (2.1) *and* (2.2)*, or* (2.3) *when $\beta = \infty$, for the operator B defined in* (2.5)*, it holds that*

$$B \in \mathcal{L}\mathrm{is}(L_2(I;V) \cap H^1_{0,\{\alpha\}}(I;V'), L_2(I;V)').$$

The norms of B and B^{-1} are bounded by those of B_e and B_e^{-1}, respectively.

With this result, a valid, well-posed weak formulation of the parabolic initial value problem (2.4) with *homogeneous* initial condition $u(\alpha) = 0$ reads as finding $u \in L_2(I;V) \cap H^1_{0,\{\alpha\}}(I;V')$ such that

$$(Bu)(v) = g(v) \quad (v \in L_2(I;V)$$

or, in operator form, as $Bu = g$.

Finally, the same argument that led to Corollary 2.2 yields the following result.

COROLLARY 2.4. *For $-\infty \leq \alpha < \beta \leq \infty$, and under conditions* (2.1) *and* (2.2)*, or* (2.3) *when $|I| = \infty$,*

$$B \in \mathcal{L}\mathrm{is}(L_2(I;V) \cap H^1_{0,\{\alpha\}}(I;V'), L_2(I;V)').$$

3. Well-posed variational formulations w.r.t. scales of spaces.

By using well-posedness of variational formulations of parabolic problems with a reversed time direction, duality, and the Riesz–Thorin interpolation theorem, we derive well-posed variational formulations with respect to scales of spaces. The exposition will mainly be used for our subsequent treatment of the instationary Navier–Stokes, but the results are also relevant for their own sake. In this section we consider homogeneous initial conditions. We defer the discussion of inhomogeneous initial data to the next section.

Let $S(t) := -t$ for $t \in \mathbb{R}$, and let $\bar{B}$ denote the parabolic operator B with $I = (\alpha, \beta)$ reading as $S(I) = (-\beta, -\alpha)$ and $a(t;\eta,\zeta)$ reading as $a(S(t);\zeta,\eta)$.

Remark 3.1. For finite I, or $I = \mathbb{R}$, it would be more convenient to replace $S(t)$ by $t \mapsto \beta + \alpha - t$, in which case $S(I) = I$. The current setting, however, allows us to include the case of I being a half-line.

Throughout this and the next section, let $\varrho \in [0,1]$ and W be a separable Hilbert space with $W \hookrightarrow V$ and dense embedding, such that

$$V \simeq [H, W]_{\frac{1}{1+\varrho}}, \tag{3.1}$$

where the right-hand side denotes the (real) interpolation space of "exponent" $\frac{1}{1+\varrho}$, i.e., $V = W$ when $\rho = 0$, and V is halfway between H and W when $\rho = 1$. We define

$$V^{s(1+\varrho)} := [H, W]_s, \quad V^{-s(1+\varrho)} = (V^{s(1+\varrho)})' \quad (s \in [0,1]), \tag{3.2}$$

which generally involves a harmless redefinition of V. We assume that H, W, I, ϱ, and B are such that

$$B \in \mathcal{L}\mathrm{is}(L_2(I;V^{1+\varrho}) \cap H^1_{0,\{\alpha\}}(I;V^{\varrho-1}), L_2(I;V^{1-\varrho})'), \tag{3.3}$$

$$\bar{B} \in \mathcal{L}\mathrm{is}(L_2(S(I);V^{1+\varrho}) \cap H^1_{0,\{-\beta\}}(S(I);V^{\varrho-1}), L_2(S(I);V^{1-\varrho})'). \tag{3.4}$$

Note that for $\boxed{\varrho = 0}$, and thus for

$$W = V,$$

(3.3) and so equivalently (3.4) follow from (2.1) and (2.2), or (2.3) when $|I| = \infty$, as shown in Corollary 2.4. Validity of (3.3) and (3.4) for $\varrho > 0$ will be discussed after the next lemma.

Lemma 3.2. *The statement* (3.4) *is equivalent to*

$$B \in \mathcal{L}\mathrm{is}(L_2(I; V^{1-\varrho}), (L_2(I; V^{1+\varrho}) \cap H^1_{0,\{\beta\}}(I; V^{\varrho-1}))'), \tag{3.5}$$

and the norm of $\bar{B}$ *or* $\bar{B}^{-1}$ *implied in* (3.4) *is equal to that of* B *or* B^{-1} *in* (3.5)*.*

Proof. For $w : I \to V^{\varrho-1}$ being smooth and compactly supported in I, and $v \in L_2(I; V^{1+\varrho}) \cap H^1_{0,\{\beta\}}(I; V^{1-\varrho})$, integration-by-parts followed by a change of variables that reverses the time direction shows that

$$\begin{aligned}(Bw)(v) &= \int_I -\left\langle w(t), \frac{dv}{dt}(t)\right\rangle + a(t; w(t), v(t))dt \\ &= \int_{S(I)} \left\langle (S^*w)(t), \frac{d(S^*v)}{dt}(t)\right\rangle + a(S(t), (S^*w)(t), (S^*v)(t))dt \\ &= (\bar{B}S^*v)(S^*w) = (\bar{B}'S^*w)(S^*v).\end{aligned}$$

Here, S^* is defined by $(S^*w)(t) = w(S(t))$. The operator S^* is an isomorphism between $L_2(S(I); V^{1-\varrho})$ and $L_2(I; V^{1-\varrho})$ and between $L_2(S(I); V^{1+\varrho}) \cap H^1_{0,\{\beta\}}(S(I); V^{\varrho-1})$ and $L_2(I; V^{1+\varrho}) \cap H^1_{0,\{-\beta\}}(I; V^{\varrho-1})$. By the density of the smooth, compactly supported functions $w : I \to V^{\varrho-1}$ in $L_2(I; V^{\varrho-1})$, the proof is completed. ∎

In the forthcoming Theorem 3.5, from (3.3) and (3.5) we will derive bounded invertibility of B w.r.t. "intermediate" spaces using the Riesz–Thorin theorem. Until this theorem, we will discuss the validity of conditions (3.3)–(3.4) for $\varrho > 0$.

Under mild additional conditions, for a suitable W, (3.3), and equivalently, (3.4), can be expected to hold for $\boxed{\varrho = 1}$, which will be particularly relevant for our treatment of the instationary Stokes problem. Indeed, considering the case that $A(t) = A(t)'$ for a.e. $t \in I$, let

$$D(A(t)) := \{u \in H \colon A(t)u \in H\},$$

equipped with $\sqrt{\|A(t)u\|_H^2 + \|u\|_H^2}$. Under the condition that $D(A(t))$ is independent of $t \in I$, picking some $t_0 \in I$, let

$$W := D(A(t_0)). \tag{3.6}$$

Then (2.1)–(2.2) imply that $V \simeq D((A(t_0) + \lambda I)^{\frac{1}{2}}) = [H, W]_{\frac{1}{2}}$, i.e., (3.1) is valid for $\varrho = 1$. Moreover, the property (3.3), known under the name *maximal regularity*, and equivalently (3.4), holds true when $|I| < \infty$ and $t \mapsto A(t) \in C(I, \mathcal{L}(H, W))$ (see [PS01], also for the addition of possible nonsymmetric lower order terms to the operator $A(t)$). Being defined in terms of the domain of $A(t_0)$, and not in terms of higher order Sobolev spaces, we note that maximal regularity does not require smoothness of the underlying spatial domain or that of the coefficients of the operator $A(t)$ as a function of the spatial variables.

Assuming (2.3), conditions (3.3)–(3.4) for $\varrho = 1$ can also hold for $|I| = \infty$. In particular, in the autonomous case $A(t) \equiv A = A' > 0$, (3.3)–(3.4) with $W = D(A)$ can be verified by direct calculations by expanding functions w on the space-time cylinder as $w = \sum_\phi w_\phi(t) \otimes \varphi$ with $\{\varphi\}$ being an orthonormal basis for H of eigenfunctions of A (cf. [CS11, Thms. 7.1, 7.3]).

Remark 3.3. With the choice (3.6) for W, the spaces V^s for $|s| \in (1,2]$ depend on $A(t_0)$, and it is therefore a priori not clear how to equip them with a (wavelet) Riesz basis. Therefore, let $\breve{W}$ be another separable Hilbert space, with $\breve{W} \hookrightarrow W$ and dense embedding. In applications, for V being a Sobolev space of order m on a domain $\Omega \subset \mathbb{R}^n$, e.g., $V = H^m(\Omega)$ or $V = H_0^m(\Omega)$, typically $\breve{W}$ will be given by $H^{2m}(\Omega) \cap V$. Corresponding to $\breve{W}$, let

$$\breve{V}^{2s} := [H, \breve{W}]_s, \quad \breve{V}^{-2s} := (\breve{V}^{2s})' \quad (s \in [0,1]) .$$

Then $\breve{V}^s \hookrightarrow V^s$ for $s \in [0,2]$. Moreover, it holds that $A(t_0) + \lambda I \in \mathcal{L}(W, H)$. Since from (2.1), $V \hookrightarrow H$ and $A(t_0)+\lambda I \in \mathcal{L}(V, V')$, we infer that for $s \in [1,2]$, $A(t_0)+\lambda I \in \mathcal{L}(V^s, V^{s-2})$. Now *assume* that for some $\theta \in (0,1]$,

$$\|u\|_{\breve{V}^{1+\theta}} \lesssim \|(A(t_0) + \lambda I)u\|_{\breve{V}^{-1+\theta}} \quad (u \in \breve{V}^{1+\theta}),$$

known as an *elliptic regularity condition.* In the example of $(A(t)\eta)(\zeta) = \int_\Omega \nabla\eta \cdot \nabla\zeta \, dx$, $V = H_0^1(\Omega)$, and $\breve{W} = H^2(\Omega) \cap H_0^1(\Omega)$, it is known to be satisfied for $\theta \in (0, \frac{1}{2})$ ("moderate" elliptic regularity) for Ω being a bounded Lipschitz domain, and for $\theta \in (0,1]$ ("full" elliptic regularity) for Ω being bounded and convex. Then for any $u \in \breve{V}^{1+\theta}$, using $V^{-1+\theta} \hookrightarrow \breve{V}^{-1+\theta}$ we have

$$\|u\|_{\breve{V}^{1+\theta}} \lesssim \|(A(t_0) + \lambda I)u\|_{\breve{V}^{-1+\theta}} \lesssim \|(A(t_0) + \lambda I)u\|_{V^{-1+\theta}} \lesssim \|u\|_{V^{1+\theta}},$$

or $V^{1+\theta} \hookrightarrow \breve{V}^{1+\theta}$, and so

$$V^s \simeq \breve{V}^s \quad (|s| \leq 1 + \theta). \tag{3.7}$$

As stated before, from (3.3) and (3.4) we are going to derive, in Theorem 3.5, boundedly invertibility of B w.r.t. a whole range of "intermediate" spaces. A subrange of these results will only involve spaces V^s for $|s| \leq 1 + \theta$, which in applications therefore can be equipped with (wavelet) Riesz bases.

Our discussion about (3.3)–(3.4) is finished by the following remark.

Remark 3.4. For $(J, \delta) \in \{(I, \alpha), (S(I), -\beta)\}$, it holds that $[L_2(J; V^1)', L_2(J; V^0)']_\varrho = L_2(J; V^{1-\varrho})'$, and

$$\Big[L_2(J; V^1) \cap H^1_{0,\{\delta\}}(J; V^{-1}), L_2(J; V^2) \cap H^1_{0,\{\delta\}}(J; V^0)\Big]_\varrho \simeq L_2(J; V^{1+\varrho}) \cap H^1_{0,\{\delta\}}(J; V^{\varrho-1}).$$

Consequently, if (3.3)–(3.4) are valid for $\varrho = 1$, then from the fact that (3.3)–(3.4) are always valid for $\varrho = 0$, one infers that (3.3)–(3.4) hold for any intermediate $\varrho \in (0,1)$.

We are ready to present the result about bounded invertibility of B w.r.t. a scale of spaces.

THEOREM 3.5. *Let* (3.3) *and* (3.4) *be valid. For* $s \in [0,1]$ *and* $\delta \in \{\alpha, \beta\}$*, let*

$$\breve{H}^s_{0,\{\delta\}}(I) := [L_2(I), H^1_{0,\{\delta\}}(I)]_s$$

and

$$\mathscr{H}^s_{\varrho,\delta} := L_2(I;V^{1-\varrho+2s\varrho}) \cap \breve{H}^s_{0,\{\delta\}}(I;V^{(1-2s)(1-\varrho)}).$$

Then for $s \in [0,1]$ *it holds that*

$$B \in \mathcal{L}\mathrm{is}(\mathscr{H}^s_{\varrho,\alpha}, (\mathscr{H}^{1-s}_{\varrho,\beta})').$$

Remark 3.6. We recall that $\breve{H}^s_{0,\{\delta\}}(I) = H^s_{0,\{\delta\}}(I) = H^s(I)$ for $s \in [0,\frac{1}{2})$ or $\delta = \pm\infty$. For $\delta \neq \pm\infty$, $\breve{H}^s_{0,\{\delta\}}(I) = H^s_{0,\{\delta\}}(I) \subsetneq H^s(I)$ for $s \in (\frac{1}{2},1]$, and $\breve{H}^{\frac{1}{2}}_{0,\{\delta\}}(I) \hookrightarrow H^{\frac{1}{2}}_{0,\{\delta\}}(I)$ with the norm on $\breve{H}^{\frac{1}{2}}_{0,\{\delta\}}(I)$ being strictly stronger than that on $H^{\frac{1}{2}}_{0,\{\delta\}}(I)$ (and so $\breve{H}^{\frac{1}{2}}_{0,\{\delta\}}(I) \subsetneq H^{\frac{1}{2}}_{0,\{\delta\}}(I)$); see, e.g., [LM68, Thm. 11.7]). In the literature, sometimes the space $\breve{H}^{\frac{1}{2}}_{0,\{\delta\}}(I)$ is also denoted as $H^{\frac{1}{2}}_{00,\{\delta\}}(I)$.

Proof. Note that $\mathscr{H}^1_{\varrho,\delta} = L_2(I;V^{1+\varrho}) \cap H^1_{0,\{\delta\}}(I;V^{\varrho-1}) \hookrightarrow L_2(I;V^{1-\varrho}) = \mathscr{H}^0_{\varrho,\delta}$, and so $(\mathscr{H}^0_{\varrho,\delta})' \hookrightarrow (\mathscr{H}^1_{\varrho,\delta})'$, with dense embeddings. By the application of interpolation, from (3.3) and Lemma 3.2 we infer that

$$B \in \mathcal{L}\mathrm{is}([\mathscr{H}^0_{\varrho,\alpha}, \mathscr{H}^1_{\varrho,\alpha}]_s, [(\mathscr{H}^1_{\varrho,\beta})', (\mathscr{H}^0_{\varrho,\beta})']_s).$$

Since

$$[(\mathscr{H}^1_{\varrho,\beta})', (\mathscr{H}^0_{\varrho,\beta})']_s = [\mathscr{H}^0_{\varrho,\beta}, \mathscr{H}^1_{\varrho,\beta}]'_{1-s}, \tag{3.8}$$

the proof follows from the characterization of these interpolation spaces given in the following lemma. ∎

In view of the definition of $\mathscr{H}^s_{\varrho,\delta}$, note that for $\varrho < 1$, $L_2(I;V^{1-\varrho}) \not\hookrightarrow H^1_{0,\{\delta\}}(I;V^{\varrho-1})$ and $H^1_{0,\{\delta\}}(I;V^{\varrho-1}) \not\hookrightarrow L_2(I;V^{1-\varrho})$. Nevertheless, we have the following result.

LEMMA 3.7. *For* $s \in [0,1]$ *and* $\delta \in \{\alpha,\beta\}$*, it holds that*

$$\mathscr{H}^s_{\varrho,\delta} \simeq [\mathscr{H}^0_{\varrho,\delta}, \mathscr{H}^1_{\varrho,\delta}]_s\ .$$

Proof. In the last part of this proof, we will demonstrate the claim for the case $I = \mathbb{R}$. We start with showing that this result implies the result for $I \subsetneq \mathbb{R}$. In the following, let $\bar{\mathscr{H}}^s$ denote the space $\mathscr{H}^s_{\varrho,\delta}$ with I reading as $\mathbb{R}$.

There exists an extension E of functions on I to $\mathbb{R}$ with $E \in \mathcal{L}(L_2(I), L_2(\mathbb{R}))$ and $E \in \mathcal{L}(H^1_{0,\{\delta\}}(I), H^1(\mathbb{R}))$. Furthermore, there exists a mapping R of functions on $\mathbb{R}$ to functions on I with $R \in \mathcal{L}(L_2(\mathbb{R}), L_2(I))$, $R \in \mathcal{L}(H^1(\mathbb{R}), H^1_{0,\{\delta\}}(I))$, and $RE = \mathrm{Id}$. To show the latter, it is sufficient to discuss the construction of R for $\delta = \alpha$ and $I = (\alpha,\infty)$. Let $\bar{E}$ be an extension of functions on $(-\infty,\alpha)$ to functions on $\mathbb{R}$ such that $\bar{E} \in \mathcal{L}(L_2(-\infty,\alpha), L_2(\mathbb{R}))$ and $\bar{E} \in \mathcal{L}(H^1(-\infty,\alpha), H^1(\mathbb{R}))$. Then R defined by $Ru = u - \bar{E}(u|_{(-\infty,\alpha)})$ satisfies the assumptions.

By interpolation and a tensor product argument, we have for $s \in [0,1]$,

$$R \otimes \mathrm{Id} \in \mathcal{L}(H^s(\mathbb{R};V^{(1-2s)(1-\varrho)}), \breve{H}^s_{0,\{\delta\}}(I;V^{(1-2s)(1-\varrho)})),$$
$$E \otimes \mathrm{Id} \in \mathcal{L}(\breve{H}^s_{0,\{\delta\}}(I;V^{(1-2s)(1-\varrho)}), H^s(\mathbb{R};V^{(1-2s)(1-\varrho)})),$$

from which we infer that

$$R \otimes \mathrm{Id} \in \mathcal{L}(\bar{\mathscr{H}}^s, \mathscr{H}^s_{\varrho,\delta}), \quad E \otimes \mathrm{Id} \in \mathcal{L}(\mathscr{H}^s_{\varrho,\delta}, \bar{\mathscr{H}}^s).$$

Writing for $u \in \mathscr{H}^s_{\varrho,\delta}$, $u = (R \otimes \mathrm{Id}) \circ (E \otimes \mathrm{Id})u$, the last result together with the claim for $I = \mathbb{R}$ shows that

$$\|u\|_{\mathscr{H}^s_{\varrho,\delta}} \lesssim \|E \otimes \mathrm{Id}\|_{\bar{\mathscr{H}}^s} \eqsim \|E \otimes \mathrm{Id}\|_{[\bar{\mathscr{H}}^0,\bar{\mathscr{H}}^1]_s} \lesssim \|u\|_{[\mathscr{H}^0_{\varrho,\delta},\mathscr{H}^1_{\varrho,\delta}]_s},$$
$$\|u\|_{[\mathscr{H}^0_{\varrho,\delta},\mathscr{H}^1_{\varrho,\delta}]_s} \lesssim \|E \otimes \mathrm{Id}\|_{[\bar{\mathscr{H}}^0,\bar{\mathscr{H}}^1]_s} \eqsim \|E \otimes \mathrm{Id}\|_{\bar{\mathscr{H}}^s} \lesssim \|u\|_{\mathscr{H}^s_{\varrho,\delta}},$$

which proves the claim for a general interval $I \subseteq \mathbb{R}$.

There remains to prove the claim for $I = \mathbb{R}$. With $(\Lambda w)(v) := \langle w, v\rangle_{V^{1+\varrho}}$ and $D(\Lambda) := \{w \in V^{1+\varrho} : \Lambda w \in H\}$, let $\{\varphi\}$ be an orthonormal basis for H consisting of eigenfunctions, with eigenvalues λ_φ, of the unbounded symmetric operator $\Lambda : H \subset D(\Lambda) \to H$, which basis exists in virtue of the compactness of the embedding $V^{1+\varrho} \hookrightarrow H$ (see, e.g., [DL90, Chap. VIII, sect. 2.6, Thm. 7]). With $\hat{u}(\xi,\cdot) := \int_{\mathbb{R}} u(t,\cdot)e^{-i2\pi t\xi}dt$, writing $\hat{u}(\xi,\cdot) = \sum_\varphi \hat{u}_\varphi(\xi)\varphi$, it holds that

$$\|u\|^2_{\bar{\mathscr{H}}^0} = \int_{\mathbb{R}} \sum_\varphi |\hat{u}_\varphi(\xi)|^2 \lambda_\varphi^{\frac{1-\varrho}{1+\varrho}} d\xi,$$
$$\|u\|^2_{\bar{\mathscr{H}}^1} = \int_{\mathbb{R}} \sum_\varphi |\hat{u}_\varphi(\xi)|^2 \left(\lambda_\varphi + \frac{(1+|\xi|^2)^{\frac{1}{2}}}{\lambda_\varphi^{\frac{1-\varrho}{1+\varrho}}}\right) d\xi,$$

and so

$$\begin{aligned}\|u\|^2_{[\bar{\mathscr{H}}^0,\bar{\mathscr{H}}^1]_s} &= \int_{\mathbb{R}} \sum_\varphi |\hat{u}_\varphi(\xi)|^2 \lambda_\varphi^{\frac{1-\varrho}{1+\varrho}} \left(\frac{\lambda_\varphi + \frac{(1+|\xi|^2)^{\frac{1}{2}}}{\lambda_\varphi^{\frac{1-\varrho}{1+\varrho}}}}{\lambda_\varphi^{\frac{1-\varrho}{1+\varrho}}}\right)^s d\xi \\ &\eqsim \int_{\mathbb{R}} \sum_\varphi |\hat{u}_\varphi(\xi)|^2 \left(\lambda_\varphi^{\frac{1-\varrho+2s\varrho}{1+\varrho}} + (1+|\xi|^2)^{\frac{s}{2}} \lambda_\varphi^{\frac{(1-2s)(1-\varrho)}{1+\varrho}}\right) \\ &= \|u\|^2_{\bar{\mathscr{H}}^s},\end{aligned}$$

where we have used that for $s \in [0,1]$ and $\eta, \zeta \geq 0$, $\frac{1}{2}(\eta^s + \zeta^s) \leq (\eta+\zeta)^s \leq \eta^s + \zeta^s$. ∎

Remark 3.8. In view of the application of Theorem 3.5 to construct an adaptive wavelet scheme, let us briefly comment on the construction of tensor product wavelet Riesz bases for the spaces $\mathscr{H}^s_{\varrho,\alpha}$ and $\mathscr{H}^{1-s}_{\varrho,\beta}$. For more information, we refer to [SS09] and, for the case $s = \frac{1}{2}$, to [LS15]. If Θ (Σ) is a collection of temporal (spatial) wavelets that, when normalized in the corresponding norm, is a Riesz basis for $L_2(I)$ ($V^{1-\varrho+2\varsigma\varrho}$) and $\breve{H}^\varsigma_{0,\{\delta\}}(I)$ ($V^{(1-2\varsigma)(1-\varrho)}$), then, normalized, the collection $\Theta \otimes \Sigma$ is a Riesz basis for $\mathscr{H}^\varsigma_{\varrho,\delta}$.

Suitable collections Θ are amply available. When V is a Sobolev space of order $m = 1$ on a general polytopal domain $\Omega \subset \mathbb{R}^n$, then the same holds true for Σ when the smoothness indices $1-\varrho+2\varsigma\varrho, (1-2\varsigma)(1-\varrho) \in (-\frac{3}{2}, \frac{3}{2})$. Indeed, for those indices, Σ can be a collection of *continuous* piecewise polynomial wavelets, whereas smoothness indices outside $(-\frac{3}{2}, \frac{3}{2})$ require smoother (primal or dual) wavelets, whose construction is troublesome on nonrectangular domains. For $\varsigma \in \{s, 1-s\}$, these conditions are fulfilled when $\varrho = 0$, whereas for $\varrho = 1$ they read as $|s| < \frac{3}{4}$ (where $|2s| < 1+\theta$ might already be needed to guarantee that $V^{2\varsigma}$ is (isomorphic to) a Sobolev space of order 2ς; cf. (3.7)).

When the bases Θ and Σ have polynomial reproduction orders d_t and d_x, respectively, then functions in $\mathscr{H}^s_{\varrho,\alpha}$ that satisfy a mild (Besov) smoothness condition can be (nonlinearly) approximated from the tensor product basis at an algebraic rate $\min(d_t - s, \frac{d_x-1}{n})$ when $\varrho = 0$ (up to a logarithmic factor when $s = 0$ and $d_t = \frac{d_x-1}{n}$), or, when $\varrho = 1$, at rate $\min(d_t - s, \frac{d_x-2s}{n})$ (up to a logarithmic factor when $s = 0$ and $d_t = \frac{d_x}{n}$). Note that for $d_t - s \geq \frac{d_x-1}{n}$ or $d_t - s \geq \frac{d_x-2s}{n}$, these rates are *equal* to the generally best possible rates of best approximation in the spaces V or V^{2s} for the corresponding stationary elliptic problem. Consequently, using the adaptive wavelet scheme the time evolution problem can be solved at an asymptotic error versus work rate which is equal to solving the stationary problem.

Noting that for ϱ, $s \in [0,1]$, $1-\varrho+2s\varrho \geq 0$, and for $\varrho \in [0,1]$, $(1-2s)(1-\varrho) \geq 0$ if and only if $s \leq \frac{1}{2}$, we infer that for all $s \in [0,1]\backslash\{\frac{1}{2}\}$, either $\mathscr{H}^s_{\varrho,\alpha}$ or $\mathscr{H}^{1-s}_{\varrho,\beta}$, which spaces appear in the statement $B \in \mathcal{L}\mathrm{is}(\mathscr{H}^s_{\varrho,\alpha}, (\mathscr{H}^{1-s}_{\varrho,\beta})')$ given in Theorem 3.5, involve a "spatial" Sobolev space of negative order. Interestingly, for the special case $s = \frac{1}{2}$, no Sobolev spaces of negative order enter the formulation. This is in particular convenient for numerical schemes that are not based on wavelets. Below we repeat the variational formulation for this case. Note that $\mathscr{H}^{\frac{1}{2}}_{\varrho,\delta}$ is independent of ϱ, which fact we use to formulate the next corollary under the mildest conditions which correspond to the case $\varrho = 0$.

COROLLARY 3.9. *Under conditions* (2.1) *and* (2.2)*, or* (2.3) *when* $|I| = \infty$*, it holds that*

$$B \in \mathcal{L}\mathrm{is}\big(L_2(I;V) \cap \breve{H}^{\frac{1}{2}}_{0,\{\alpha\}}(I;H), (L_2(I;V) \cap \breve{H}^{\frac{1}{2}}_{0,\{\beta\}}(I;H))'\big) .$$

This latter result generalizes corresponding known results for the half-line $I = (\alpha,\infty)$ to $I = (\alpha,\beta)$ for general $-\infty \leq \alpha < \beta \leq \infty$, so in particular to finite I. (See, e.g., [BB83, Thm. 2.2], [Fon09, Thm. 4.3] (also for nonlinear spatial operators), and [LS15]. The proofs of well-posedness in these references are based on the application of a Hilbert transform in the temporal direction.

Remark 3.10. For $|I| < \infty$ or $I = \mathbb{R}$, the problem of finding $w \in L_2(I;V) \cap \breve{H}^{\frac{1}{2}}_{0,\{\alpha\}}(I;H)$ such that $(Bw)(v) = g(v)$ $(v \in L_2(I;V) \cap \breve{H}^{\frac{1}{2}}_{0,\{\beta\}}(I;H))$ is equivalent to

$$\text{(3.9)} \quad \int_I \left\langle \frac{dw}{dt}(t), v(\beta+\alpha-t) \right\rangle + a(t; w(t), v(\beta+\alpha-t))dt = \int_I \langle g(t), v(\beta+\alpha-t)\rangle$$

$(v \in L_2(I;V) \cap \breve{H}^{\frac{1}{2}}_{0,\{\alpha\}}(I;H))$. Now if, for a.e. $t \in I$, $a(\beta+\alpha-t,\cdot,\cdot) = a(t,\cdot,\cdot)$, and $a(t,\cdot,\cdot)$ is symmetric, then the bilinear form at the left-hand side of (3.9) is *symmetric*. It can, however, not be expected to be coercive.

4. Inhomogeneous initial condition. A valid, well-posed weak formulation of the parabolic initial value problem (2.4) with a possibly *inhomogeneous* initial condition at $\alpha > -\infty$ was already given in (2.6). In this section, we investigate whether for $(s,\varrho) \neq (1,0)$ such a problem can also be solved in $L_2(I;V^{1-\varrho+2s\varrho}) \cap H^s(I;V^{(1-2s)(1-\varrho)})$, that for $s \in [0,\frac{1}{2})$ equals the space $\mathscr{H}^s_{\varrho,\delta}$. In order to do so, we need a substitute for the embedding from (2.7).

PROPOSITION 4.1. *For any* $s \in (\frac{1}{2},1]$ *and* $\varepsilon > 0$*,*

$$L_2(I;V^{1-\varrho+2s\varrho}) \cap H^s(I;V^{(1-2s)(1-\varrho)}) \hookrightarrow C(\bar{I}, V^{(2s-1)\varrho-\varepsilon}),$$

with a compact embedding when $|I| < \infty$.

Proof. We apply [Ama00, Thm. 5.2] (see also [Sim87, Cor. 9]) with $n = 1$, $X = I$, $E_0 = V^{(1-2s)(1-\varrho)}$, $E_1 = V^{1-\varrho+2s\varrho}$, $s_0 = s$, $s_1 = 0$, and $p_0 = p_1 = 2$ (so that $p_\theta = 2$). Thanks to $s > \frac{1}{2}$, there exists a $\theta \in [0, 1]$ with $s_\theta := s(1-\theta) = \frac{1+\varepsilon}{2} > \frac{1}{2}$, where we use that it is sufficient to consider $-\varepsilon \geq 1 - 2s$. Taking $E = [V^{(1-2s)(1-\varrho)}, V^{1-\varrho+2s\varrho}]_\theta = V^{(2s-1)\varrho-\varepsilon}$, all conditions are satisfied, and our statement follows. ∎

Unfortunately, given an $s \in (\frac{1}{2}, 1]$, for any $\varepsilon > 0$, the mapping $L_2(I; V^{1-\varrho+2s\varrho}) \cap H^s(I; V^{(1-2s)(1-\varrho)}) \to V^{(2s-1)\varrho-\varepsilon} : u \mapsto u(\alpha)$ cannot be surjective, since $u(\alpha)$ is also in $V^{(2s-1)\varrho-\varepsilon/2}$. Consequently, on the basis of Proposition 4.1, for $s < 1$ it is not possible to solve the parabolic problem with inhomogeneous initial condition in $L_2(I; V^{1-\varrho+2s\varrho}) \cap H^s(I; V^{(1-2s)(1-\varrho)})$ from a well-posed weak formulation of type (2.6), so where the inhomogeneous initial condition is appended as an equation in $V^{(2s-1)\varrho-\varepsilon}$. Note that the well-posedness of (2.6) implies that $L_2(I; V) \cap H^1(I; V') \to H : u \mapsto u(\alpha)$ *is* surjective.

With a formulation of type (2.6), the initial condition plays the role of an *essential* condition. As proposed in [CS11] (see also the earlier works [BJ89, BJ90] for $A(t) \equiv A = A' > 0$) it is, however, possible, and computationally more convenient, to impose it as a *natural* condition by applying integration-by-parts over time.

Let u be a "classical" solution of problem (2.4). Multiplying the PDE with smooth test functions $t \mapsto v(t) \in V^{1+\varrho-2s\varrho}$ that vanish at β, integrating over space and time, applying integration-by-parts over time, and substituting the initial condition yields

$$\int_I -\left\langle u(t), \frac{dv}{dt}(t) \right\rangle_H + a(t; u(t); v(t))dt = \int_I \langle g(t), v(t) \rangle_H + \langle u_\alpha, v(\alpha) \rangle_H.$$

For v as above, and $t \mapsto u(t) \in V^{1-\varrho+2s\varrho}$ being smooth on I and vanishing at $\{\alpha\}$, the bilinear form at the left-hand side reads as $(Bu)(v)$ as defined in (2.5). For $s \in [0, 1]$, such functions u and v form dense subsets in $\mathscr{H}^s_{\varrho,\alpha}$ and $\mathscr{H}^{1-s}_{\varrho,\beta}$, respectively. We conclude that the unique extension of the left-hand side to a bilinear form on $\mathscr{H}^s_{\varrho,\alpha} \times \mathscr{H}^{1-s}_{\varrho,\beta}$ is $(Bu)(v)$. For $s \in [0, \frac{1}{2})$, $g \in (\mathscr{H}^{1-s}_{\varrho,\beta})'$, and $u_\alpha \in V^{(2s-1)\varrho+\varepsilon}$, Proposition 4.1 shows that the right-hand side is an element of $(\mathscr{H}^{1-s}_{\varrho,\beta})'$. For $s = 0$, this holds true even when $u_\alpha \in V^{-\varrho}$ by (a generalization of) (2.7). In line with these facts, we recall that for $s \in [0, \frac{1}{2})$, $\mathscr{H}^s_{\varrho,\alpha} = L_2(I; V^{1-\varrho+2s\varrho}) \cap H^s(I; V^{(1-2s)(1-\varrho)})$, so without the incorporation of a homogeneous boundary condition at α. From Theorem 3.5 we conclude the following result.

THEOREM 4.2. *Assume conditions* (3.3)–(3.4)*. Then for any $s \in [0, \frac{1}{2})$, assuming that $g \in (\mathscr{H}^{1-s}_{\varrho,\beta})'$, and, for some $\varepsilon > 0$, $u_\alpha \in V^{(2s-1)\varrho+\varepsilon}$, or even $u_\alpha \in V^{-\varrho}$ when $s = 0$, a valid, well-posed weak formulation of the parabolic initial value problem* (2.4) *with possibly* inhomogeneous *initial condition $u(\alpha) = u_\alpha$ reads as finding $u \in \mathscr{H}^s_{\varrho,\alpha}$ such that*

$$(Bu)(v) = g(v) + \langle u_\alpha, v(\alpha) \rangle_H \quad (v \in \mathscr{H}^{1-s}_{\varrho,\beta}).$$

For alternative well-posed variational formulations of the inhomogeneous parabolic initial value problem (on the half-line) we refer to [Tom83, eqs. (0.15), (0.16) and Thm. 2] and [Fon09, Thm. 4.10].

5. Instationary Stokes. For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a *bounded Lipschitz domain*, and $-\infty < \alpha < \beta \leq \infty$. Given a constant $\nu > 0$, a vector field $\mathbf{f}$ on $I \times \Omega$, and a function g on $I \times \Omega$, we consider the instationary inhomogeneous Stokes problem with *no-slip* boundary conditions and, for the moment, *homogeneous initial condition* to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$(5.1)\qquad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \mathbf{\Delta_x} \mathbf{u} + \nabla_{\mathbf{x}}\, p = \mathbf{f} & \text{on } I \times \Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } I \times \Omega, \\ \mathbf{u} = 0 & \text{on } I \times \partial\Omega, \\ \mathbf{u}(\alpha, \cdot) = 0 & \text{on } \Omega, \\ \int_\Omega p \, d\mathbf{x} = 0. \end{cases}$$

Remark 5.1. By introducing the new unknowns $\sqrt{\nu}\,\mathbf{u}$ and $\frac{p}{\sqrt{\nu}}$, and $\tilde{t} = \nu t$, one arrives at a Stokes problem with $\nu = 1$, but time interval $(\nu\alpha, \nu\beta)$. So in view of Theorem 2.1, unless $\beta = \infty$ one cannot expect to arrive at results that hold uniformly for $\nu \downarrow 0$.

With

$$(5.2)\qquad \begin{cases} c(\mathbf{u}, \mathbf{v}) &:= \int_I \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} + \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} \, d\mathbf{x} dt, \\ d(p, \mathbf{v}) &:= -\int_I \int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x} dt, \\ \mathbf{f}(\mathbf{v}) &:= \int_I \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} dt, \\ g(q) &:= \int_I \int_\Omega g \, q \, d\mathbf{x} dt, \end{cases}$$

in variational form it reads as finding $(\mathbf{u}, p)$ in some suitable space, that "incorporates" the homogeneous initial/boundary conditions for $\mathbf{u}$ and $\int_\Omega p \, d\mathbf{x} = 0$ such that

$$(\mathrm{S}(\mathbf{u}, p))(\mathbf{v}, q) := c(\mathbf{u}, \mathbf{v}) + d(p, \mathbf{v}) + d(q, \mathbf{u}) = \mathbf{f}(\mathbf{v}) - g(q)$$

for all $(\mathbf{v}, q)$ from another suitable space. As always, the bilinear forms should be interpreted as the unique extensions to the arising spaces of the bilinear forms on dense subsets of sufficiently smooth functions in these spaces. Consequently, in particular, sometimes it will be more natural to read $-\int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x}$ as $\int_\Omega \mathbf{v} \cdot \nabla p \, d\mathbf{x}$ and $\int_I \frac{\partial \mathbf{u}}{\partial t} \mathbf{v} \, dt$ as $-\int_I \frac{\partial \mathbf{v}}{\partial t} \mathbf{u} \, dt$.

We start with collecting results about the *stationary* Stokes problem. Let, for $s \in [0, 2]$,

$$\hat{H}^s(\Omega) := [L_2(\Omega), H^2(\Omega) \cap H_0^1(\Omega)]_{\frac{s}{2}},$$
$$\bar{H}^{s-1}(\Omega) := [(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s}{2}},$$

where the second definition should be interpreted w.r.t. the embedding of $H^1(\Omega)/\mathbb{R}$ into $(H^1(\Omega)/\mathbb{R})'$ by means of $H^1(\Omega)/\mathbb{R} \hookrightarrow L_2(\Omega)/\mathbb{R} \simeq (L_2(\Omega)/\mathbb{R})' \hookrightarrow (H^1(\Omega)/\mathbb{R})'$. Let

$$a(\mathbf{u}, \mathbf{v}) := \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x}, \; (\widetilde{\operatorname{div}}\, \mathbf{v})(p) := -b(p, \mathbf{v}) := -\int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x}.$$

For $s \in [0, 2]$, a is bounded on $\hat{H}^s(\Omega)^n \times \hat{H}^{2-s}(\Omega)^n$, and b is bounded on $\bar{H}^{1-s}(\Omega)^n \times \hat{H}^s(\Omega)$, i.e., $\widetilde{\operatorname{div}} \in \mathcal{L}(\hat{H}^s(\Omega)^n, \bar{H}^{s-1}(\Omega))$. We set

$$\hat{H}^0(\operatorname{div} 0; \Omega) : = \left\{ \mathbf{u} \in L_2(\Omega)^n : \; \widetilde{\operatorname{div}}\, \mathbf{u} = 0 \right\},$$
$$\hat{H}^2(\operatorname{div} 0; \Omega) : = \left\{ \mathbf{u} \in \hat{H}^2(\Omega)^n : \; \widetilde{\operatorname{div}}\, \mathbf{u} = 0 \right\}.$$

For smooth fields $\mathbf{u}$, it holds that $\widetilde{\operatorname{div}}\,\mathbf{u} = \operatorname{div}\mathbf{u}$. On the other hand, it is known that $\hat{H}^0(\operatorname{div} 0;\Omega) = \{\mathbf{u} \in L_2(\Omega)^n\colon\ \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}|_{\partial\Omega}\cdot\mathbf{n} = 0\}$.

We will be interested in "inf-sup conditions" satisfied by b, i.e., in surjectivity of $\widetilde{\operatorname{div}}$, for which operator we will construct a right-inverse. It will be relevant that the same right-inverse is bounded *simultaneously* w.r.t. different norms.

LEMMA 5.2. *There exists a mapping F with $F \in \mathcal{L}(H_0^k(\Omega), H_0^{k+1}(\Omega)^n)(k \in \mathbb{N}_0)$, $(\operatorname{div}\circ F)v = v$ when $\int_\Omega v(\mathbf{x})\,d\mathbf{x} = 0$, and $F' \in \mathcal{L}(L_2(\Omega)^n, H^1(\Omega))$.*

For $\Omega \subset \mathbb{R}^n$ being bounded and star-shaped with respect to a ball K, F was constructed in [Bog79] as $Fv(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y})v(\mathbf{y})d\mathbf{y}$ with $G = (G_1,\dots,G_n)$ being defined as $G(\mathbf{x},\mathbf{y}) = \int_0^1 \frac{1}{s^{n+1}}(\mathbf{x}-\mathbf{y})\omega(\mathbf{y}+\frac{\mathbf{x}-\mathbf{y}}{s})ds$ for arbitrary $\omega \in C_0^\infty(K)$ with $\int_K \omega\,d\mathbf{x} = 1$. Membership of $F \in \mathcal{L}(H_0^k(\Omega), H_0^{k+1}(\Omega))$ $(k \in \mathbb{N}_0)$ was shown in [Bog79], [Gal94, Lem. III.3.1], and $F' \in \mathcal{L}(L_2(\Omega)^n, H^1(\Omega))$ in [GHH06]. In [Gal94, GHH06], the construction was generalized to any bounded Lipschitz domain Ω using the property that such an Ω can be written as a finite union of star-shaped domains.

COROLLARY 5.3. *There exists a mapping* $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^s(\Omega), \hat{H}^{1+s}(\Omega)^n)(s \in [-1,\frac{1}{2}))$ *with* $\widetilde{\operatorname{div}}\circ\operatorname{div}^+ = I$.

Proof. For $s \in [0,1]$, we have $F \in \mathcal{L}([L_2(\Omega), H_0^1(\Omega)]_s, [H_0^1(\Omega)^n, H_0^2(\Omega)^n]_s)$ by Lemma 5.2. For $s \in [0,\frac{1}{2})$, it is known that $[L_2(\Omega), H_0^1(\Omega)]_s \simeq H^s(\Omega)$ as well as $[H_0^1(\Omega)^n, H_0^2(\Omega)^n]_s \simeq \hat{H}^{1+s}(\Omega)^n$. Defining div^+ as the restriction of F to functions with vanishing mean, in particular we have $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^s(\Omega), \hat{H}^{1+s}(\Omega)^n)$.

With $\mathbb{1} := x \mapsto \operatorname{vol}(\Omega)^{\frac{-1}{2}}$, for $\mathbf{u} \in C_0^\infty(\Omega)^n$ and $v \in C^\infty(\bar{\Omega})/\mathbb{R}$, we have

$$\langle \mathbf{u}, \operatorname{div}^+ v\rangle_{L_2(\Omega)^n} = \langle F'\vec{u} - \langle F'\vec{u}, \mathbb{1}\rangle_{L_2(\Omega)}\mathbb{1}, v\rangle_{L_2(\Omega)},$$

i.e., $(\operatorname{div}^+)' = \vec{u} \mapsto F'\vec{u} - \langle F'\vec{u}, \mathbb{1}\rangle_{L_2(\Omega)}\mathbb{1}$. From Lemma 5.2, we infer that $(\operatorname{div}^+)' \in \mathcal{L}(L_2(\Omega)^n, \bar{H}^1(\Omega))$, i.e., $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{-1}(\Omega), L_2(\Omega)^n)$.

By an application of interpolation, we conclude that $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^s(\Omega), \hat{H}^{1+s}(\Omega)^n)$ for $s \in [-1,\frac{1}{2})$. Since for smooth v with zero mean, $(\widetilde{\operatorname{div}}\circ\operatorname{div}^+)v = v$, the proof is completed. ∎

The next corollary of Lemma 5.2 says that, under conditions, first interpolating between $L_2(\Omega)^n$ and $(H^2(\Omega)\cap H_0^1(\Omega))^n$, and subsequently taking the subspace of divergence-free functions yields the same space as when both operations are applied in reversed order. This result will be essential for the well-posedness proof.

COROLLARY 5.4. *With*

$$\hat{H}^s(\operatorname{div} 0;\Omega) := [\hat{H}^0(\operatorname{div} 0;\Omega), \hat{H}^2(\operatorname{div} 0;\Omega)]_{\frac{s}{2}} \quad (s \in [0,2]),$$

it holds that

$$\hat{H}^s(\operatorname{div} 0;\Omega) \simeq \{\mathbf{u} \in \hat{H}^s(\Omega)^n\colon\ \widetilde{\operatorname{div}}\,\mathbf{u} = 0\} \quad (s \in [0,\tfrac{3}{2})).$$

Proof. By the definition of a real interpolation space using the K-functional, it holds that

$$\hat{H}^s(\operatorname{div} 0;\Omega) \hookrightarrow \{\mathbf{u} \in \hat{H}^s(\Omega)^n\colon\ \widetilde{\operatorname{div}}\,\mathbf{u} = 0\} \quad \text{for } s \in [0,2]. \tag{5.3}$$

We have $\widetilde{\operatorname{div}} \in \mathcal{L}(H_0^2(\Omega)^n, \bar{H}^1(\Omega)\cap H_0^1(\Omega))$ and $\widetilde{\operatorname{div}} \in \mathcal{L}(L_2(\Omega)^n, \bar{H}^{-1}(\Omega))$. As we have shown in Lemma 5.2 and Corollary 5.3, $\widetilde{\operatorname{div}}$ admits a right-inverse div^+ with

$\operatorname{div}^+ \in \mathcal{L}(\bar{H}^1(\Omega) \cap H_0^1(\Omega), H_0^2(\Omega)^n)$ and $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{-1}(\Omega), L_2(\Omega)^n)$. By an abstract interpolation result from [LM68] (cf. [MM08, Lemma 2.13]), the existence of such a right-inverse guarantees that

$$\begin{aligned}&\Big\{\mathbf{u} \in \big[L_2(\Omega)^n, H_0^2(\Omega)^n\big]_{\frac{s}{2}} : \widetilde{\operatorname{div}}\,\mathbf{u} = 0\Big\}\\ &\quad\simeq \Big[\Big\{\mathbf{u} \in L_2(\Omega)^n \colon \widetilde{\operatorname{div}}\,u = 0\Big\}, \Big\{\mathbf{u} \in H_0^2(\Omega)^n \colon \widetilde{\operatorname{div}}\,\mathbf{u} = 0\Big\}\Big]_{\frac{s}{2}} \quad (s \in [0,2]).\end{aligned} \tag{5.4}$$

For $s \in [0, \frac{3}{2})$ the space on the left in (5.4) is isomorphic to $\{\mathbf{u} \in \hat{H}^s(\Omega)^n \colon \widetilde{\operatorname{div}}\,\mathbf{u} = 0\}$. By $H_0^2(\Omega) \hookrightarrow \hat{H}^2(\Omega)$, the space on the right in (5.4) is continuously embedded in $\hat{H}^s(\operatorname{div} 0; \Omega)$, which completes the proof. ∎

The next result is the analogue for the Stokes operator on divergence-free functions of Remark 3.3 dealing with standard elliptic operators.

Proposition 5.5 (“moderate” elliptic regularity). *With $A \in \mathcal{L}(\hat{H}^1(\operatorname{div} 0; \Omega), \hat{H}^1(\operatorname{div} 0; \Omega)')$ defined by*

$$(A\mathbf{u})(\mathbf{v}) = a(\mathbf{u}, \mathbf{v}) \quad (\mathbf{u},\, \mathbf{v} \in \hat{H}^1(\operatorname{div} 0; \Omega)),$$

let $D(A) := \{\mathbf{u} \in \hat{H}^0(\operatorname{div} 0; \Omega) \colon A\mathbf{u} \in \hat{H}^0(\operatorname{div} 0; \Omega)\}$, equipped with the graph norm. Then

$$[\hat{H}^0(\operatorname{div} 0; \Omega), D(A)]_{\frac{s}{2}} \simeq \hat{H}^s(\operatorname{div} 0; \Omega) \quad (s \in [0, \tfrac{3}{2})).$$

Proof. As shown in [MM08, Thm. 5.1], for Ω being a bounded Lipschitz domain, and $s \in (\frac{1}{2}, \frac{3}{2})$,

$$D(A^{\frac{s}{2}}) \eqsim \{\mathbf{u} \in \hat{H}^s(\Omega)^n \colon \widetilde{\operatorname{div}}\,\mathbf{u} = 0\}.$$

Since the space on the left is isomorphic to $[\hat{H}^0(\operatorname{div} 0; \Omega), D(A)]_{\frac{s}{2}}$, the proof is completed by Corollary 5.4 and the application of interpolation. ∎

For the analysis of the *instationary* Stokes operator, for $s \in [0,1]$ and $\delta \in \{\alpha, \beta\}$ we set

$$\begin{aligned}\mathscr{U}_\delta^s &:= L_2(I; \hat{H}^{2s}(\Omega)^n) \cap \breve{H}^s_{0,\{\delta\}}(I; L_2(\Omega)^n),\\ \mathscr{P}_\delta^s &:= (L_2(I; \bar{H}^{2s-1}(\Omega)') \cap \breve{H}^{1-s}_{0,\{\delta\}}(I; \bar{H}^1(\Omega)'))'.\end{aligned}$$

Applications of Lemma 3.7 for $\varrho = 1$, show that for $\delta \in \{\alpha, \beta\}$ and $s \in [0,1]$,

$$\begin{aligned}[\mathscr{U}_\delta^0, \mathscr{U}_\delta^1]_s &= [L_2(I; L_2(\Omega)^n), L_2(I; \hat{H}^2(\Omega)^n) \cap \breve{H}^1_{0,\{\delta\}}(I; L_2(\Omega)^n)]_s\\ &\simeq L_2(I; \hat{H}^{2s}(\Omega)^n) \cap \breve{H}^s_{0,\{\delta\}}(I; L_2(\Omega)^n) = \mathscr{U}_\delta^s,\end{aligned}$$

and

$$\begin{aligned}[\mathscr{P}_\delta^0, \mathscr{P}_\delta^1]_s &= \big[(L_2(I; \bar{H}^1(\Omega)) \cap \breve{H}^1_{0,\{\delta\}}(I; \bar{H}^{-1}(\Omega)))', L_2(I; \bar{H}^1(\Omega))\big]_s\\ &= \big[L_2(I; \bar{H}^{-1}(\Omega)), L_2(I; \bar{H}^1(\Omega)) \cap \breve{H}^1_{0,\{\delta\}}(I; \bar{H}^{-1}(\Omega))\big]'_{1-s}\\ &\simeq (L_2(I; \bar{H}^{1-2s}(\Omega)) \cap \breve{H}^{1-s}_{0,\{\delta\}}(I; \bar{H}^{-1}(\Omega)))' = \mathscr{P}_\delta^s.\end{aligned}$$

Well-posedness of the instationary Stokes operator is established next.

Theorem 5.6. *Recalling that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, for $s \in (\frac{1}{4}, \frac{3}{4})$ it holds that*

$$\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})').$$

Proof. One easily verifies that $\mathrm{S} \in \mathcal{L}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ is valid for $s \in \{0,1\}$, meaning that it is valid for $s \in [0,1]$.

Corollary 5.3 gives the existence of a right-inverse div^+ of div that satisfies both $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{-1}(\Omega), L_2(\Omega)^n)$ and, for $s \in [0, \frac{3}{4})$, $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{2s-1}(\Omega), \hat{H}^{2s}(\Omega)^n)$, from which it follows that $I \otimes \operatorname{div}^+ \in \mathcal{L}((\mathscr{P}_\alpha^{1-s})', \mathscr{U}_\alpha^s)$. This implies that for $s \in [0, \frac{3}{4})$, $I \otimes \operatorname{div} \in \mathcal{L}(\mathscr{U}_\alpha^s, (\mathscr{P}_\alpha^{1-s})')$ is surjective, i.e.,

$$\inf_{0 \neq q \in \mathscr{P}_\alpha^{1-s}} \sup_{0 \neq \mathbf{u} \in \mathscr{U}_\alpha^s} \frac{d(\mathbf{u}, q)}{\|\mathbf{u}\|_{\mathscr{U}_\alpha^s} \|q\|_{\mathscr{P}_\alpha^{1-s}}} > 0,$$

and analogously, that for $1 - s \in [0, \frac{3}{4})$, i.e., $s \in (\frac{1}{4}, 1]$,

$$\inf_{0 \neq p \in \mathscr{P}_\beta^s} \sup_{0 \neq \mathbf{v} \in \mathscr{U}_\beta^{1-s}} \frac{d(\mathbf{v}, p)}{\|\mathbf{v}\|_{\mathscr{U}_\beta^{1-s}} \|p\|_{\mathscr{P}_\beta^s}} > 0.$$

We conclude that both these inf-sup conditions are valid for $s \in (\frac{1}{4}, \frac{3}{4})$.

Having established the boundedness of S and both inf-sup conditions, the theory about the well-posedness of saddle-point problems (e.g., [GSS14, sect. 2]) shows that what remains to prove is that

$$(C\mathbf{u})(\mathbf{v}) := c(\mathbf{u}, \mathbf{v}) \tag{5.5}$$

defines an invertible operator between the spaces $\{\mathbf{u} \in \mathscr{U}_\alpha^s \colon d(\mathscr{P}_\alpha^{1-s}, \mathbf{u}) = 0\}$ and $(\{\mathbf{v} \in \mathscr{U}_\beta^{1-s} \colon d(\mathscr{P}_\beta^s, \mathbf{v}) = 0\})'$.

For $(\varsigma, \delta) \in \{(s, \alpha), (1-s, \beta)\}$, using $\mathscr{P}_\delta^{1-\varsigma} \simeq L_2(I; \bar{H}^{1-2\varsigma}(\Omega)) + (\breve{H}_{0,\{\delta\}}^\varsigma)'(I; \bar{H}^1(\Omega))$, we infer that

$$\begin{aligned}
&\{\mathbf{u} \in \mathscr{U}_\delta^\varsigma \colon d(\mathscr{P}_\delta^{1-\varsigma}, \mathbf{u}) = 0\} \\
&\quad = \{\mathbf{u} \in L_2(I; \hat{H}^{2\varsigma}(\Omega)^n) \colon d(L_2(I; \bar{H}^{1-2\varsigma}(\Omega)), \mathbf{u}) = 0\} \\
&\qquad \cap \{\mathbf{u} \in \breve{H}_{0,\{\delta\}}^\varsigma(I; \hat{H}^0(\Omega)^n) \colon d\big((\breve{H}_{0,\{\delta\}}^\varsigma)'(I; \bar{H}^1(\Omega)), \mathbf{u}\big) = 0\} \\
&\quad = \{\mathbf{u} \in L_2(I; \hat{H}^{2\varsigma}(\Omega)^n) \colon (I \otimes \widetilde{\operatorname{div}})\mathbf{u} = 0\} \\
&\qquad \cap \{\mathbf{u} \in \breve{H}_{0,\{\delta\}}^\varsigma(I; \hat{H}^0(\Omega)^n) \colon (I \otimes \widetilde{\operatorname{div}})\mathbf{u} = 0\} \\
&\quad \simeq L_2(I; \hat{H}^{2\varsigma}(\operatorname{div} 0; \Omega)) \cap \breve{H}_{0,\{\delta\}}^\varsigma(I; \hat{H}^0(\operatorname{div} 0; \Omega)) =: \mathscr{U}_\delta^\varsigma(\operatorname{div} 0),
\end{aligned} \tag{5.6}$$

where the last isomorphism is valid for $\varsigma \in [0, \frac{3}{4})$ by virtue of Corollary 5.4.

On the other hand, the analysis from sections 2–3 shows that

$$\begin{aligned}
&C \in \mathcal{L}\mathrm{is}\big(L_2(I; D(A)) \cap H_{0,\{\alpha\}}^1(I; \hat{H}^0(\operatorname{div} 0; \Omega)), L_2(I; \hat{H}^0(\operatorname{div} 0; \Omega))\big), \\
&C \in \mathcal{L}\mathrm{is}\big(L_2(I; \hat{H}^0(\operatorname{div} 0; \Omega)), (L_2(I; D(A)) \cap H_{0,\{\beta\}}^1(I; \hat{H}^0(\operatorname{div} 0; \Omega)))'\big)
\end{aligned}$$

(i.e., the maximal regularity results (3.3) and (3.5) for $\varrho = 1$, $H = \hat{H}^0(\operatorname{div} 0; \Omega)$, and $W = D(A)$), so that by Theorem 3.5,

$$C \in \mathcal{L}\mathrm{is}(\tilde{\mathscr{U}}_\alpha^\varsigma(\operatorname{div} 0), (\tilde{\mathscr{U}}_\beta^{1-\varsigma}(\operatorname{div} 0))') \quad (\varsigma \in [0,1]), \tag{5.7}$$

where

$$\tilde{\mathscr{U}}_\delta^\varsigma(\operatorname{div} 0) := L_2(I; [\hat{H}^0(\operatorname{div} 0; \Omega), D(A)]_\varsigma) \cap \breve{H}_{0,\{\delta\}}^\varsigma(I; \hat{H}^0(\operatorname{div} 0; \Omega)). \tag{5.8}$$

The proof is completed by $\hat{H}^{2\varsigma}(\operatorname{div} 0;\Omega) \simeq [\hat{H}^0(\operatorname{div} 0;\Omega), D(A)]_\varsigma$ for $\varsigma \in [0, \frac{3}{4})$, as shown in Proposition 5.5 using the "moderate elliptic regularity," and thus $\tilde{\mathscr{U}}_\alpha^\varsigma(\operatorname{div} 0) \simeq \mathscr{U}_\alpha^\varsigma(\operatorname{div} 0)$. ∎

Remark 5.7. Adaptive wavelet methods inherit, through the Riesz basis property, "stability" from the underlying infinite-dimensional problem. If the instationary Stokes problem is solved by another numerical method, stability of the finite-dimensional discretized system has to be verified separately. For the presently considered space-time formulation the results from [MSW13] then seem relevant.

Under an additional regularity condition on the *stationary* Stokes operator S_0 defined below, the range of values of s for which Theorem 5.6 is valid can be extended. With

$$(\mathrm{S}_0(\mathbf{u}, p))(\mathbf{v}, q) := a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b(q, \mathbf{u})$$

we have $\mathrm{S}_0 \in \mathcal{L}(\hat{H}^s(\Omega)^n \times \bar{H}^{s-1}(\Omega), (\hat{H}^{2-s}(\Omega)^n \times \bar{H}^{1-s}(\Omega))')$ for $s \in [0, 2]$. The "full" regularity condition imposed in the following theorem is known to be satisfied for $n \in \{2, 3\}$ and $\partial\Omega \in C^2$; see, e.g., [Tem79, Chap. 1, Prop. 2.3].[1]

THEOREM 5.8. *Assuming* $\mathrm{S}_0 \in \mathcal{L}\mathrm{is}(\hat{H}^2(\Omega)^n \times \bar{H}^1(\Omega), \hat{H}^0(\Omega)^n \times \bar{H}^1(\Omega))$*, it holds that*

$$\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$$

for $s \in [0, 1]$.

Proof. Since S_0 is symmetric, we also have $\mathrm{S}_0 \in \mathcal{L}\mathrm{is}(\hat{H}^s(\Omega)^n \times \bar{H}^{s-1}(\Omega), (\hat{H}^{2-s}(\Omega)^n \times \bar{H}^{1-s}(\Omega))')$ for $s = 0$, and so for $s \in [0, 2]$.

Defining $\operatorname{div}^+ := g \mapsto \mathbf{u}$ by $(\mathbf{u}, p) := \mathrm{S}_0^{-1}(0, g)$, we have $\widetilde{\operatorname{div}} \circ \operatorname{div}^+ = I$, and

$$\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{s-1}(\Omega), \hat{H}^s(\Omega)^n) \quad (s \in [0, 2]).$$

Following the proof of Corollary 5.4, replacing $H_0^2(\Omega)$ by $\hat{H}^2(\Omega)$, we infer that

$$\hat{H}^s(\operatorname{div} 0;\Omega) \simeq \{\mathbf{u} \in \hat{H}^s(\Omega)^n \colon \operatorname{div} \mathbf{u} = 0\} \quad (s \in [0, 2]).$$

For the operator A from Proposition 5.5 we find that $D(A) \simeq \hat{H}^2(\operatorname{div} 0;\Omega)$, and so

$$[\hat{H}^0(\operatorname{div} 0;\Omega), D(A)]_{\frac{s}{2}} \simeq \hat{H}^s(\operatorname{div} 0;\Omega) \quad (s \in [0, 2]).$$

Indeed, obviously $D(A) \hookleftarrow \hat{H}^2(\operatorname{div} 0;\Omega)$. To show the reversed embedding, for $\mathbf{f} \in \hat{H}^0(\operatorname{div};\Omega)$ consider $A\mathbf{u} = \mathbf{f}$. After extending $\mathbf{f}$, with preservation of its norm, to $L_2(\Omega)^n$, the solution $\mathbf{u}$ is the first component of the solution $(\mathbf{u}, p)$ of $\mathrm{S}_0(\mathbf{u}, p) = (\mathbf{f}, 0)$, and so $\|\mathbf{u}\|_{H^2(\Omega)^n} \lesssim \|\mathbf{f}\|_{L_2(\Omega)^n}$.

Using these ingredients, by following the proof of Theorem 5.6, the statement is proven. ∎

Remark 5.9. One might think that div^+ constructed by means of the inverse stationary Stokes operator, as employed in the above proof, would also be applicable in the proof of Theorem 5.6. Under the conditions of that theorem, however, such a div^+ is in $\mathcal{L}(\bar{H}^{s-1}(\Omega), \hat{H}^s(\Omega)^n)$ generally for $s \in (\frac{1}{2}, \frac{3}{2})$ only. For the proof of Theorem 5.6, it is needed that also $\operatorname{div}^+ \in \mathcal{L}(\bar{H}^{-1}(\Omega), \hat{H}^0(\Omega)^n)$.

[1] This regularity condition *cannot* be expected to hold for convex $\Omega \subset \mathbb{R}^n$ for $n = 2, 3$ that have an only piecewise smooth boundary as we erroneously suggested in [GSS14]. Such domains will be addressed in Theorem 5.10 and Remark 5.11.

THEOREM 5.10. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, convex polygon. Then for $s \in (0,1)$*

$$\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')\,.$$

Proof. Let

$$\breve{H}^1(\Omega) := \left\{ v \in H^1(\Omega) \colon \|v\|^2_{\breve{H}^1(\Omega)} := |v|^2_{H^1(\Omega)} + \sum_{\mathbf{z}} \int_\Omega \frac{|v(\mathbf{x})|^2}{|\mathbf{x}-\mathbf{z}|^2}\, d\mathbf{x} < \infty \right\},$$

where $\mathbf{z}$ runs over the finite set of corners of the polygon.

Inside the proof of this theorem, we *redefine*

$$\bar{H}^{s-1}(\Omega) := [(\breve{H}^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s}{2}} \quad (s \in [0,2]),$$

and with that, *redefine* the spaces $\mathscr{P}_\delta^s$.

For even any bounded polygon without slits $\Omega \subset \mathbb{R}^2$, in [ASV88, eq. (3.2)] it was shown that $\widetilde{\mathrm{div}} \in \mathcal{L}(\hat{H}^2(\Omega)^2, \breve{H}^1(\Omega)/\mathbb{R})$. From this, one verifies that $\mathrm{S} \in \mathcal{L}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ is valid for $s \in \{0,1\}$, meaning that it is valid for $s \in [0,1]$.

Using that Ω is convex, in [KO76] it was shown that

$$\mathrm{S}_0 \in \mathcal{L}\mathrm{is}\big(\hat{H}^2(\Omega)^n \times H^1(\Omega)/\mathbb{R}, \hat{H}^0(\Omega)^n \times \breve{H}^1(\Omega)/\mathbb{R}\big). \tag{5.9}$$

Now following the steps from the proof of Theorem 5.8, we conclude that $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ for $s \in [0,1]$.

In view of the fact that the definitions of $\mathscr{P}_\beta^s$ and $\mathscr{P}_\alpha^{1-s}$ incorporate the spaces $\bar{H}^{2s-1}(\Omega)$ and $\bar{H}^{1-2s}(\Omega)$, respectively, the proof will be completed once we have shown that

$$[(\breve{H}^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s}{2}} \simeq [(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s}{2}} \quad (s \in (0,2]),$$

i.e.,

$$[(H^1(\Omega)/\mathbb{R})', \breve{H}^1(\Omega)/\mathbb{R}]_{\frac{s}{2}} \simeq [(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s}{2}} \quad (s \in [0,2)).$$

In [ASV88, Thm. 3.1], a right-inverse for $\widetilde{\mathrm{div}}$ was constructed, which we denote here as $\widetilde{\mathrm{div}}^+$, that for $s \in (\frac{1}{2}, 1)$ satisfies $\widetilde{\mathrm{div}}^+ \in \mathcal{L}(H^s(\Omega), \hat{H}^{1+s}(\Omega)^2)$. Using $\widetilde{\mathrm{div}} \in \mathcal{L}(\hat{H}^2(\Omega)^2, \breve{H}^1(\Omega)/\mathbb{R})$ and $\widetilde{\mathrm{div}} \in \mathcal{L}(L_2(\Omega)^2, (H^1(\Omega)/\mathbb{R})')$, we infer that

$$\begin{aligned}\|\widetilde{\mathrm{div}}\,\widetilde{\mathrm{div}}^+ v\|_{[(H^1(\Omega)/\mathbb{R})', \breve{H}^1(\Omega)/\mathbb{R}]_{\frac{s+1}{2}}} &\lesssim \|\widetilde{\mathrm{div}}^+ v\|_{\hat{H}^{1+s}(\Omega)^2} \lesssim \|v\|_{H^s(\Omega)/\mathbb{R}} \\ &\eqsim \|v\|_{[(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}]_{\frac{s+1}{2}}},\end{aligned}$$

with which the proof is completed. ∎

Remark 5.11. In view of the regularity result for the stationary Stokes operator S_0 given in [Dau89, Thm. 9.20], which generalizes (5.9) to the three-dimensional case, it can be envisaged that a result such as Theorem 5.10 also holds on convex polytopes Ω in $\mathbb{R}^3$.

Remark 5.12. In view of the application of the obtained well-posedness results for the instationary Stokes operator S for constructing an adaptive wavelet scheme, we briefly discuss the construction of tensor product wavelet Riesz bases for the spaces $\mathscr{U}_\delta^\varsigma$ and $\mathscr{P}_\delta^{1-\varsigma}$ for $(\varsigma, \delta) \in \{(s,\alpha), (1-s,\beta)\}$.

If Θ (Σ) is a collection of temporal (spatial) wavelets that, when normalized in the corresponding norm, is a Riesz basis for $L_2(I)$ ($\hat{H}^{2\varsigma}(\Omega)^n$) and $\breve{H}^{\varsigma}_{0,\{\delta\}}(I)$ ($L_2(\Omega)^n$), then, properly normalized, the collection $\Theta \otimes \Sigma$ is a Riesz basis for $\mathscr{U}^{\varsigma}_{\delta}$.

Suitable collections Θ are amply available. The same holds true for Σ when Ω is a polytope and $2\varsigma < \frac{3}{2}$, i.e., when $s \in (\frac{1}{4}, \frac{3}{4})$. For those values of s, Σ can be a collection of *continuous* piecewise polynomial wavelets.

The latter means an important step forward compared to our earlier results in [GSS14]. There we established well-posedness as in Theorem 5.6 but for the cases $s \in \{0, 1\}$ only, which require continuously differentiable wavelets whose construction is cumbersome on domains Ω that are not of product type. Moreover, the results in [GSS14] were derived under a "full regularity" condition on the stationary Stokes operator as imposed in Theorem 5.8.

Moving to the construction of a basis for the pressure space, if $\bar{\Theta}$ ($\bar{\Sigma}$) is a collection of temporal (spatial) wavelets that, when normalized in the corresponding norm, is a Riesz basis for $L_2(I)$ ($\bar{H}^{2\varsigma-1}(\Omega)$) and $\breve{H}^{1-\varsigma}_{0,\{\delta\}}(I)'$ ($\bar{H}^1(\Omega)$), then, normalized, the collection $\bar{\Theta} \otimes \bar{\Sigma}$ is a Riesz basis for $\mathscr{P}^{\varsigma}_{\delta}$. For $\varsigma \in [0, 1]$, bases are amply available, for Ω being a general polytope.

Similar to Remark 3.8 for the parabolic problem, thanks to the use of tensor product bases, the instationary Stokes problem can be solved at an asymptotic error versus work rate equal to solving the stationary Stokes problem.

Finally in this section we consider the case of having a possibly *inhomogeneous initial condition* in (5.1). Similarly to Theorem 4.2, we have the next theorem.

THEOREM 5.13. *Let* $s \in [0, \frac{1}{2})$ *be such that* $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}^s_\alpha \times \mathscr{P}^s_\beta, (\mathscr{U}^{1-s}_\beta \times \mathscr{P}^{1-s}_\alpha)')$ *(cf. Theorems* 5.6*,* 5.8*, and* 5.10 *and Remark* 5.11*). Then for* $\mathbf{f} \in (\mathscr{U}^{1-s}_\beta)'$ *and* $g \in (\mathscr{P}^{1-s}_\alpha)'$*, and, for some* $\varepsilon > 0$*,* $u_\alpha \in (\hat{H}^{1-2s-\varepsilon}_0(\Omega)^n)'$*, or even* $u_\alpha \in (H^1_0(\Omega)^n)'$ *when* $s = 0$*, a valid, well-posed weak formulation of the Stokes equations* (2.4) *with initial condition* $\mathbf{u}(\alpha) = \mathbf{u}_\alpha$ *reads as finding* $(\mathbf{u}, p) \in \mathscr{U}^s_\alpha \times \mathscr{P}^s_\beta$ *such that*

$$(\mathrm{S}(\mathbf{u}, p))(\mathbf{v}, q) = \mathbf{f}(\mathbf{v}) - g(q) + \int_\Omega \mathbf{u}_\alpha(\mathbf{x}) \cdot \mathbf{v}(\alpha, \mathbf{x}) d\mathbf{x}$$

$((\mathbf{v}, q) \in \mathscr{U}^{1-s}_\beta \times \mathscr{P}^{1-s}_\alpha)$.

6. Instationary Navier–Stokes. For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a *bounded Lipschitz domain*, and $-\infty < \alpha < \beta \leq \infty$. Given a constant $\nu > 0$, a vector field $\mathbf{f}$ on $I \times \Omega$, and a function g on $I \times \Omega$, we consider the instationary inhomogeneous Navier–Stokes problem with *no-slip* boundary conditions and *homogeneous initial condition* to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \mathbf{f} & \text{on } I \times \Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } I \times \Omega, \\ \mathbf{u} = 0 & \text{on } I \times \partial\Omega, \\ \mathbf{u}(\alpha, \cdot) = 0 & \text{on } \Omega, \\ \int_\Omega p \, d\mathbf{x} = 0. \end{cases} \tag{6.1}$$

With the trilinear form

$$n(\mathbf{y}, \mathbf{z}, \mathbf{v}) := \int_I \int_\Omega \mathbf{y} \cdot \boldsymbol{\nabla}_{\mathbf{x}} \mathbf{z} \cdot \mathbf{v} \, d\mathbf{x} dt \,, \tag{6.2}$$

in variational form (6.1) reads as finding $(\mathbf{u},p)$ in some suitable space H, that incorporates the homogeneous initial/boundary conditions for $\mathbf{u}$ and $\int_\Omega p\,d\mathbf{x}=0$, such that

$$\mathrm{NS}(\mathbf{u},p)(\mathbf{v},q) := (\mathrm{S}(\mathbf{u},p))(\mathbf{v},q) + n(\mathbf{u},\mathbf{u},\mathbf{v}) = \mathbf{f}(\mathbf{v}) - g(q) \tag{6.3}$$

for all $(\mathbf{v},q)$ from another suitable space K.

In this setting of having a nonlinear problem, we call the above space-time variational formulation well-posed when

1. $\mathrm{NS}: H \supset \mathrm{dom}(\mathrm{NS}) \to K'$,
2. there exists a $(\mathbf{u},p)\in H$ such that (6.3) is valid for all $(\mathbf{v},q)\in K$,
3. NS is continuously Frechét differentiable in a neighborhood of $(\mathbf{u},p)$,
4. $D\mathrm{NS}(\mathbf{u},p)\in \mathcal{L}\mathrm{is}(H,K')$.

In view of constructing an efficient numerical solver of the space-time variational problem, we aim also at the situation that

5. both H and K can be conveniently equipped with (wavelet) Riesz bases, or, alternatively, with an infinite nested sequence of finite-dimensional subspaces.

We start with deriving upper bounds for the trilinear form n. For $s\geq 0$, let

$$\mathscr{X}^s := L_2(I;H^{2s}(\Omega)^n)\cap H^s(I;L_2(\Omega)^n).$$

For $s\in[0,1]$ and $\delta\in\{\alpha,\beta\}$, obviously $\mathscr{U}_\delta^s \hookrightarrow \mathscr{X}^s$.

Proposition 6.1. *For $s_1,s_2,s_3\geq 0$ with $s_1+s_2+s_3>\frac{n+2}{4}$, it holds that*

$$|n(\mathbf{y},\mathbf{z},\mathbf{v})| \lesssim \|\mathbf{y}\|_{\mathscr{X}^{s_1}}\|\mathbf{z}\|_{\mathscr{X}^{s_2+\frac12}}\|\mathbf{v}\|_{\mathscr{X}^{s_3}} \tag{6.4}$$

($\mathbf{y}\in\mathscr{X}^{s_1}$, $\mathbf{z}\in\mathscr{X}^{s_2+\frac12}$, $\mathbf{v}\in\mathscr{X}^{s_3}$). For $n=2$, (6.4) is also valid for $s_2=0$, $s_1+s_3\geq 1$.

Proof. For $p_i,\,q_i\geq 1$ with $\sum_{i=1}^3\frac{1}{p_i}\leq 1$, $\sum_{i=1}^3\frac{1}{q_i}\leq 1$, Hölder's inequality yields

$$\begin{aligned}\left|\int_I\int_\Omega \mathbf{y}\cdot\nabla_{\mathbf{x}}\mathbf{z}\cdot\mathbf{v}\,d\mathbf{x}dt\right| &\leq \int_I \|\mathbf{y}(t,\cdot)\|_{L_{q_1}(\Omega)^n}\|\mathbf{z}(t,\cdot)\|_{W^1_{q_2}(\Omega)^n}\|\mathbf{v}(t,\cdot)\|_{L_{q_3}(\Omega)^n}dt\\ &\leq \|\mathbf{y}\|_{L_{p_1}(I;L_{q_1}(\Omega)^n)}\|\mathbf{z}\|_{L_{p_2}(I;W^1_{q_2}(\Omega)^n)}\|\mathbf{v}\|_{L_{p_3}(I;L_{q_3}(\Omega)^n)}.\end{aligned}\tag{6.5}$$

From [Ama00, Thm. 5.2], it follows that for $s\geq 0$, $\theta\in[0,1]$, $r<(1-\theta)s$,

$$\mathscr{X}^s\hookrightarrow H^r(I;H^{2s\theta}(\Omega)^n). \tag{6.6}$$

The Sobolev embedding theorem shows that for $p\geq 2$ and $r\geq\frac12-\frac1p$, or $q\geq 2$, $k\in\mathbb{N}_0$, and $t\geq n(\frac12-\frac1q)+k$ (the latter with strict inequality when $q=\infty$ and n is even), it holds that

$$H^r(I)\hookrightarrow L_p(I),\quad H^t(\Omega)\hookrightarrow W^k_q(\Omega), \tag{6.7}$$

respectively. We infer that for $k\in\mathbb{N}_0$, $p,q\geq 2$ and $\frac n2(\frac12-\frac1q)+\frac12-\frac1p<s$,

$$\mathscr{X}^{s+\frac k2}\hookrightarrow L_p(I;W^k_q(\Omega)^n). \tag{6.8}$$

Indeed with $\tilde{s}=s+\frac{k}{2}$, select $r=\frac{1}{2}-\frac{1}{p}$ and θ from the nonempty interval

$$\left(\left(\frac{n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)+\frac{k}{2}\right)/\tilde{s}, 1+\left(\frac{1}{p}-\frac{1}{2}\right)/\tilde{s}\right)\subset(0,1),$$

and apply (6.6) and subsequently (6.7).

We apply (6.8) to (6.5). For some $\varepsilon_i\in[s_i-\frac{n+2}{4},s_i]$, we take $\frac{1}{q_i}=\frac{1}{p_i}:=\frac{1}{2}-\frac{2}{n+2}(s_i-\varepsilon_i)\in[0,\frac{1}{2}]$. We select ε_i such that $\sum_i\varepsilon_i\leq\sum_i s_i-\frac{n+2}{4}$, so that $\sum_{i=1}^3\frac{1}{p_i}=\sum_{i=1}^3\frac{1}{q_i}\leq 1$, and such that $\varepsilon_i\geq 0$ with $\varepsilon_i>0$ whenever $s_i>0$. Then, for $s_i>0$, it holds that $\frac{n}{2}(\frac{1}{2}-\frac{1}{q_i})+\frac{1}{2}-\frac{1}{p_i}=s_i-\varepsilon_i<s_i$, so that for $k\in\mathbb{N}_0$, $\mathscr{X}^{s_i+\frac{k}{2}}\hookrightarrow L_{p_i}(I;W^k_{q_i}(\Omega)^n)$ by (6.8). For $s_i=0$, we have $\varepsilon_i=0$, and so $p_i=q_i=2$, and thus $\mathscr{X}^{s_i+\frac{k}{2}}=\mathscr{X}^{\frac{k}{2}}\hookrightarrow L_2(I;H^k(\Omega)^n)=L_{p_i}(I;W^k_{q_i}(\Omega)^n)$.

It remains to define ε_i that satisfy the above conditions. When for some $i=\imath$, $s_\imath>\frac{n+2}{4}$, then we take $\varepsilon_\imath=s_\imath-\frac{n+2}{4}$, and for $j\neq\imath$, $\varepsilon_j=s_j$. When for all i, $s_i\leq\frac{n+2}{4}$, we take $\varepsilon_i=\min(s_i,(\sum_{j=1}^3 s_j-\frac{n+2}{4})/3)$. With this the proof of the first claim is completed.

To show the second claim, we note that for $s\geq 0$,

$$\mathscr{X}^s\hookrightarrow H^{\frac{s}{2}}(I;H^s(\Omega)^n),$$

which extends (6.6) to the special case $r=\frac{s}{2}$ and $\theta=\frac{1}{2}$. To see this, let $\{\theta\}$ and $\{\sigma\}$ be a Riesz bases for $L_2(I)$ and $L_2(\Omega)^n$, such that $\{\theta/\|\theta\|_{H^s(I)}\}$ and $\{\sigma/\|\sigma\|_{H^{2s}(\Omega)^n}\}$ are Riesz bases for $H^s(I)$ and $H^{2s}(\Omega)^n$, which collections exist. Then

$$\left\{\theta\otimes\sigma/(\|\theta\|_{H^s(I)}\|\sigma\|_{H^{2s}(\Omega)^n})^{\frac{1}{2}}\right\}\text{ and }\left\{\theta\otimes\sigma/\sqrt{\|\theta\|^2_{H^s(I)}+\|\sigma\|^2_{H^{2s}(\Omega)^n}}\right\}$$

are Riesz bases for $H^{\frac{s}{2}}(I;H^s(\Omega)^n)$ and $\mathscr{X}^s$, respectively. Now one infers the statement from $2\|\theta\|_{H^s(I)}\|\sigma\|_{H^{2s}(\Omega)^n}\leq\|\theta\|^2_{H^s(I)}+\|\sigma\|^2_{H^{2s}(\Omega)^n}$.

Furthermore, for $n=2$, we have $H^{\frac{s}{2}}(I;H^s(\Omega)^2)\hookrightarrow L_{\frac{2}{1-s}}(I;L_{\frac{2}{1-s}}(\Omega)^2)$, and obviously $\mathscr{X}^{\frac{1}{2}}\hookrightarrow L_2(I;H^1(\Omega)^n)$. Taking $p_i=q_i=\frac{2}{1-s_i}$ for $i=1,3$, and $p_2=q_2=2$, one has $\sum_{i=1}^3\frac{1}{p_i}=\sum_{i=1}^3\frac{1}{q_i}\leq 1$ when $s_1+s_3\geq 1$, which completes the proof of (6.4) for the special case. ∎

THEOREM 6.2. *Let $n=3$ and $s\in(\frac{3}{4},1]$, or $n=2$ and $s\in[\frac{1}{2},1]$. Then*

(i) $\mathrm{NS}:\mathscr{U}^s_\alpha\times\mathscr{P}^s_\beta\to(\mathscr{U}^{1-s}_\beta\times\mathscr{P}^{1-s}_\alpha)'$.

(ii) *For $(\bar{\mathbf{u}},\bar{p})\in\mathscr{U}^s_\alpha\times\mathscr{P}^s_\beta$, its Fréchet derivative is given by*

$$\begin{aligned}D\mathrm{NS}(\bar{\mathbf{u}},\bar{p})=D\mathrm{NS}(\bar{\mathbf{u}})\colon(\mathbf{u},p)\mapsto((\mathbf{v},q)\mapsto\mathrm{S}(\mathbf{u},p)(\mathbf{v},q)\\+n(\mathbf{u},\bar{\mathbf{u}},\mathbf{v})+n(\bar{\mathbf{u}},\mathbf{u},\mathbf{v}))\end{aligned}$$

and satisfies

$$\bar{\mathbf{u}}\mapsto D\mathrm{NS}(\bar{\mathbf{u}})-S\in\mathcal{L}\left(\mathscr{U}^s_\alpha,\mathcal{L}(\mathscr{U}^s_\alpha\times\mathscr{P}^s_\beta,(\mathscr{U}^{1-s}_\beta\times\mathscr{P}^{1-s}_\alpha)')\right).\tag{6.9}$$

Let, additionally, s be such that $\mathrm{S}\in\mathcal{L}\mathrm{is}(\mathscr{U}^s_\alpha\times\mathscr{P}^s_\beta,(\mathscr{U}^{1-s}_\beta\times\mathscr{P}^{1-s}_\alpha)')$ (cf. Theorems 5.6, 5.8, *and* 5.10 *and Remark* 5.11*). Then*

(iii) *for $(\mathbf{f},g)\in(\mathscr{U}^{1-s}_\beta\times\mathscr{P}^{1-s}_\alpha)'$ sufficiently small,*

$$\mathrm{NS}(\mathbf{u},p)(\mathbf{v},q)=\mathbf{f}(\mathbf{v})-g(q)\quad((\mathbf{v},q)\in\mathscr{U}^{1-s}_\beta\times\mathscr{P}^{1-s}_\alpha),\tag{6.10}$$

has a unique solution $(\mathbf{u},p)$ in some ball in $\mathscr{U}^s_\alpha\times\mathscr{P}^s_\beta$ around the origin, and $\|\mathbf{u}\|_{\mathscr{U}^s_\alpha}+\|p\|_{\mathscr{P}^s_\beta}\lesssim\|\mathbf{f}\|_{(\mathscr{U}^{1-s}_\beta)'}+\|g\|_{(\mathscr{P}^{1-s}_\alpha)'}$.

(iv) $D\mathrm{NS}(\mathbf{u}) \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ *for sufficiently small* $\mathbf{u} \in \mathscr{U}_\alpha^s$.

Proof. Recall that $\mathrm{S} \in \mathcal{L}(\mathscr{U}_\alpha^\varsigma \times \mathscr{P}_\beta^\varsigma, (\mathscr{U}_\beta^{1-\varsigma} \times \mathscr{P}_\alpha^{1-\varsigma})')$ for $\varsigma \in [0,1]$. An application of Proposition 6.1 with $s_1 = s$, $s_2 = s - \frac{1}{2}$, $s_3 = 1 - s$ shows that

$$|n(\mathbf{u}, \mathbf{w}, \mathbf{v})| \lesssim \|\mathbf{u}\|_{\mathscr{U}_\alpha^s} \|\mathbf{w}\|_{\mathscr{U}_\alpha^s} \|\mathbf{v}\|_{\mathscr{U}_\beta^{1-s}}, \tag{6.11}$$

and so in particular $|n(\mathbf{u}, \mathbf{u}, \mathbf{v})| \lesssim \|\mathbf{u}\|^2_{\mathscr{U}_\alpha^s} \|\mathbf{v}\|_{\mathscr{U}_\beta^{1-s}}$, which implies (i).

From

$$n(\mathbf{u}+\mathbf{h}, \mathbf{u}+\mathbf{h}, \cdot) - n(\mathbf{u}, \mathbf{u}, \cdot) = n(\mathbf{h}, \mathbf{u}, \cdot) + n(\mathbf{u}, \mathbf{h}, \cdot) + n(\mathbf{h}, \mathbf{h}, \cdot),$$

together with an application of (6.11), we arrive at the claimed expression for the Fréchet derivative in (ii)

Also the second statement in (ii) is an easy consequence of (6.11).

To show (iii), we observe that from $n(\mathbf{u}, \mathbf{u}, \cdot) - n(\mathbf{w}, \mathbf{w}, \cdot) = n(\mathbf{u}-\mathbf{w}, \mathbf{u}, \cdot) + n(\mathbf{w}, \mathbf{u}-\mathbf{w}, \cdot)$ and (6.11), the nonlinearity is locally Lipschitz: With $N(\mathbf{u})(\mathbf{v}) := n(\mathbf{u}, \mathbf{u}, \mathbf{v})$ we find

$$\|N(\mathbf{u}) - N(\mathbf{w})\|_{(\mathscr{U}_\beta^{1-s})'} \lesssim (\|\mathbf{u}\|_{\mathscr{U}_\alpha^s} + \|\mathbf{w}\|_{\mathscr{U}_\alpha^s}) \|\mathbf{u} - \mathbf{w}\|_{\mathscr{U}_\alpha^s}\ . \tag{6.12}$$

Now using the additional assumption, the statement about the solvability follows by an application of Banach's fixed point theorem, e.g., see [Tem79] or [GSS14, Lem. 5.1].

Assertion (iv) follows from (6.9) for sufficiently small $\mathbf{u}$ in $\mathscr{U}_\alpha^s$. ∎

Considering our desirata (1)–(5), we conclude that (1), (3) are satisfied, and (2), (4) are valid under a small data assumption. For $n = 2$, (5) is satisfied, but not for $n = 3$. The condition $s > \frac{3}{4}$ imposes piecewise smoothness on trial spaces, and in addition global C^1 regularity.

For existence results, i.e., (2), for large data, we refer to the extensive literature on this topic. Since usually these results concern only the velocities in a divergence-free setting, we note the following: Let $\bar{\mathbf{u}} \in \{\mathbf{w} \in \mathscr{U}_\alpha^s \colon I \otimes \widetilde{\operatorname{div}}\, \mathbf{w} = 0\}$ be such that

$$\mathrm{NS}(\bar{\mathbf{u}}, 0)(\mathbf{v}, 0) = \mathbf{f}(\mathbf{v}) \quad (\mathbf{v} \in \{\mathbf{w} \in \mathscr{U}_\beta^{1-s} \colon I \otimes \widetilde{\operatorname{div}}\, \mathbf{w} = 0\}), \tag{6.13}$$

with $s \in [0,1]$ being such that $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$. The last property implies that

$$\inf_{0 \neq p \in \mathscr{P}_\beta^s} \sup_{0 \neq \mathbf{v} \in \mathscr{U}_\beta^{1-s}} \frac{d(\mathbf{v}, p)}{\|\mathbf{v}\|_{\mathscr{U}_\beta^{1-s}} \|p\|_{\mathscr{P}_\beta^s}} > 0,$$

which together with (6.13) implies the existence of a (unique) $\bar{p} \in \mathscr{P}_\beta^s$ with $d(\mathbf{v}, \bar{p}) = \mathbf{f}(\mathbf{v}) - \mathrm{NS}(\bar{\mathbf{u}}, 0)(\mathbf{v}, 0)$ for all $\mathbf{v} \in \mathscr{U}_\beta^{1-s}$, so that $(\bar{\mathbf{u}}, \bar{p})$ is a solution of (6.10).

In the next theorem, we show that under a (moderate) regularity condition on $\bar{\mathbf{u}}$, but *without* a smallness assumption, it holds that $D\mathrm{NS}(\bar{\mathbf{u}}) \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$, i.e., (4) is valid.[2]

[2]We are embarrassed to admit that Theorem 5.3, Remark 5.4, and Theorem 5.7 from [GSS14] are not correct as stated. Viewing $\mathbf{u} \mapsto \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \bar{\mathbf{u}}$ as a first order perturbation of the spatial differential operator $\mathbf{u} \mapsto -\nu \mathbf{\Delta}_{\mathbf{x}} \mathbf{u}$ is valid only under the provision of substantial extra regularity $\bar{\mathbf{u}} \in L_\infty(I; W^1_\infty(\Omega)^n)$.

THEOREM 6.3. *Let $|I| < \infty$, and let $s \in [\frac{1}{2}, 1]$ be such that* $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ *and, for $\varsigma \in [0, s]$, $D(A^{\frac{\varsigma}{2}}) \simeq \{\mathbf{u} \in \hat{H}^\varsigma(\Omega)^n \colon \widetilde{\mathrm{div}}\, \mathbf{u} = 0\}$, with A as defined in Proposition* 5.5. *For some $\bar{s} > \frac{n}{4}$, let $\bar{\mathbf{u}} \in \mathscr{U}_\alpha^{\bar{s}}$. Then $D\mathrm{NS}(\bar{\mathbf{u}}) \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$.*

Remark 6.4. In the cases that we verified $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$, we did this under assumptions that guarantee $D(A^{\frac{\varsigma}{2}}) \simeq \{\mathbf{u} \in \hat{H}^\varsigma(\Omega)^n \colon \widetilde{\mathrm{div}}\, \mathbf{u} = 0\}$ for $\varsigma \in [0, s]$.

Remark 6.5. Let $|I| < \infty$, and let $s \in (\frac{n}{4}, 1]$ be such that $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ and, for $\varsigma \in [0, s]$, $D(A^{\frac{\varsigma}{2}}) \simeq \{\mathbf{u} \in \hat{H}^\varsigma(\Omega)^n \colon \widetilde{\mathrm{div}}\, \mathbf{u} = 0\}$. Then any solution $(\mathbf{u}, p) \in \mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s$ of (6.10) is locally unique. Indeed, by Theorem 6.3, we have $D\mathrm{NS}(\mathbf{u}) \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$, and (6.9) shows that $\mathbf{u} \mapsto D\mathrm{NS}(\mathbf{u})$ is continuous, so that the statement is a consequence of the implicit function theorem. This statement about local uniqueness extends to $n = 2$ and $s = \frac{1}{2}$ at solutions $(\mathbf{u}, p)$ with $\mathbf{u} \in \mathscr{U}_\alpha^{\bar{s}}$ for some $\bar{s} > \frac{1}{2}$.

Proof. From section 5, in particular, (5.2), (5.5), and (5.8), recall the definitions of $c(\ ,\ )$, C, and $\tilde{\mathscr{U}}_\delta^s(\mathrm{div}\, 0)$. We define $(\delta C(\bar{\mathbf{u}})\mathbf{u})(\mathbf{v}) := n(\bar{\mathbf{u}}, \mathbf{u}, \mathbf{v}) + n(\mathbf{u}, \bar{\mathbf{u}}, \mathbf{v})$.

The general theory about the well-posedness of saddle-point problems shows that $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ is equivalent to $\mathrm{S} \in \mathcal{L}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$, $C \in \mathcal{L}\mathrm{is}(\{\mathbf{w} \in \mathscr{U}_\alpha^s \colon (I \otimes \widetilde{\mathrm{div}})\mathbf{w} = 0\}, (\{\mathbf{w} \in \mathscr{U}_\beta^{1-s} \colon (I \otimes \widetilde{\mathrm{div}})\mathbf{w} = 0\})')$, and two inf-sup conditions (cf. proof of Theorem 5.6). So the only thing to verify is whether the condition involving C is satisfied with C reading as $C + \delta C(\bar{\mathbf{u}})$.

The assumption that $D(A^{\frac{\varsigma}{2}}) \simeq \{\mathbf{u} \in \hat{H}^s(\Omega)^n \colon \widetilde{\mathrm{div}}\, \mathbf{u} = 0\}$ for $\varsigma \in [0, s]$ implies that

$$\{\mathbf{w} \in \mathscr{U}_\delta^\varsigma \colon (I \otimes \widetilde{\mathrm{div}})\mathbf{w} = 0\} \simeq \tilde{\mathscr{U}}_\delta^\varsigma(\mathrm{div}\, 0) \tag{6.14}$$

(cf., e.g., [DS10, Thm. 3.2]), so that it suffices to prove

$$C + \delta C(\bar{\mathbf{u}}) \in \mathcal{L}\mathrm{is}(\tilde{\mathscr{U}}_\alpha^s(\mathrm{div}\, 0), \tilde{\mathscr{U}}_\beta^{1-s}(\mathrm{div}\, 0)')\ . \tag{6.15}$$

For $\kappa \in \mathbb{R}$, let h_κ denote the operator of multiplication by the exponential function $t \mapsto e^{\kappa t}$. Due to the assumption $|I| < \infty$, for $\gamma \in \{\alpha, \beta\}$ and for every $\kappa \in \mathbb{R}$ it holds that $h_{\pm\kappa} \in \mathcal{L}\mathrm{is}(\tilde{\mathscr{U}}_\gamma^\varsigma(\mathrm{div}\, 0), \tilde{\mathscr{U}}_\gamma^\varsigma(\mathrm{div}\, 0))$ ($\varsigma \in [0, 1]$). Also $(\delta C(\bar{\mathbf{u}})(h_\kappa \mathbf{u}))(h_{-\kappa}\mathbf{v}) = (\delta C(\bar{\mathbf{u}})\mathbf{u})(\mathbf{v})$, and, with $C_\kappa := C + \kappa I$, there holds $(C(h_\kappa \mathbf{u}))(h_{-\kappa}\mathbf{v}) = (C_\kappa \mathbf{u})(\mathbf{v})$, so that

$$(C + \delta C(\bar{\mathbf{u}}))(\mathbf{u})(\mathbf{v}) = (C_\kappa + \delta C(\bar{\mathbf{u}}))(h_{-\kappa}\mathbf{u})(h_\kappa \mathbf{v}) \quad (\mathbf{u} \in \tilde{\mathscr{U}}_\alpha^s(\mathrm{div}\, 0), \mathbf{v} \in \tilde{\mathscr{U}}_\beta^{1-s}(\mathrm{div}\, 0))\ .$$

The claim (6.15) then follows if for sufficiently large $\kappa > 0$ it holds that

$$C_\kappa + \delta C(\bar{\mathbf{u}}) \in \mathcal{L}\mathrm{is}(\tilde{\mathscr{U}}_\alpha^s(\mathrm{div}\, 0), \tilde{\mathscr{U}}_\beta^{1-s}(\mathrm{div}\, 0)'). \tag{6.16}$$

To prove (6.16), we first consider the case that $s \in [\frac{1}{2}, 1)$. The case $s = 1$ will be discussed separately. Taking $\varepsilon \in (0, \bar{s} - \frac{n}{4})$ and $\varepsilon \leq 1 - s$, thanks to $\bar{s}, s \geq \frac{1}{2}$, $1 - s - \varepsilon \geq 0$, and $\bar{s} > \frac{n}{4}$, two applications of Proposition 6.1, together with (6.14), show

$$|(\delta C(\bar{\mathbf{u}})\mathbf{u})(\mathbf{v})| \lesssim \|\bar{\mathbf{u}}\|_{\mathscr{U}_\alpha^{\bar{s}}} \|\mathbf{u}\|_{\tilde{\mathscr{U}}_\alpha^s(\mathrm{div}\, 0)} \|\mathbf{v}\|_{\tilde{\mathscr{U}}_\beta^{1-s-\varepsilon}(\mathrm{div}\, 0)},$$

i.e.,

$$\delta C(\bar{\mathbf{u}}) \in \mathcal{L}(\tilde{\mathscr{U}}_\alpha^s(\operatorname{div} 0)), \tilde{\mathscr{U}}_\beta^{1-s-\varepsilon}(\operatorname{div} 0)'). \tag{6.17}$$

In other words, $\delta C(\bar{\mathbf{u}})$ is a compact perturbation of C_κ.

Similarly to (5.7), for $\kappa \geq 0$ and $\varsigma \in [0,1]$, $C_\kappa \in \mathcal{L}\mathrm{is}(\tilde{\mathscr{U}}_\alpha^\varsigma(\operatorname{div} 0), \tilde{\mathscr{U}}_\beta^{1-\varsigma}(\operatorname{div} 0)')$. For $\mathbf{f} \in L_2(I; \hat{H}^0(\operatorname{div} 0; \Omega)) \simeq \tilde{\mathscr{U}}_\beta^0(\operatorname{div} 0)'$, $\mathbf{u} := C_\kappa^{-1}\mathbf{f} \in \tilde{\mathscr{U}}_\alpha^1(\operatorname{div} 0)$ satisfies

$$\int_I\int_\Omega \nu\nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\mathbf{v} + \kappa\mathbf{u}\cdot\mathbf{v}\, d\mathbf{x}dt = \int_I\int_\Omega \mathbf{f}\cdot\mathbf{v} - \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{v}\, d\mathbf{x}dt \quad \Big(\mathbf{v} \in L_2(I; \hat{H}^0(\operatorname{div} 0; \Omega))\Big)\ .$$

Substituting $\mathbf{v} = \mathbf{u}$, and using that $-\int_I\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot \mathbf{u}\, d\mathbf{x}dt = -\frac{1}{2}\int_I \frac{d}{dt}\|\mathbf{u}\|^2_{L_2(\Omega)^n} dt \leq \frac{1}{2}\|\mathbf{u}(\alpha)\|^2_{L_2(\Omega)^n} = 0$, we find that $\kappa\|\mathbf{u}\|^2_{L_2(I;L_2(\Omega)^n)} \leq \|\mathbf{f}\|_{L_2(I;L_2(\Omega)^n)}\|\mathbf{u}\|_{L_2(I;L_2(\Omega)^n)}$ or, for $\kappa > 0$,

$$\|C_\kappa^{-1}\|_{\mathcal{L}(L_2(I;\hat{H}^0(\operatorname{div} 0;\Omega)), L_2(I;\hat{H}^0(\operatorname{div} 0;\Omega)))} \leq \kappa^{-1}. \tag{6.18}$$

From $C_\kappa^{-1} = C_0^{-1} - C_0^{-1}\kappa I C_\kappa^{-1}$, we infer $\sup_{\kappa>0}\|C_\kappa^{-1}\|_{\mathcal{L}(\tilde{\mathscr{U}}_\beta^0(\operatorname{div} 0)', \tilde{\mathscr{U}}_\alpha^1(\operatorname{div} 0))} < \infty$. Similarly, $C_\kappa^{-1} = C_0^{-1} - C_\kappa^{-1}\kappa I C_0^{-1}$ shows $\sup_{\kappa>0}\|C_\kappa^{-1}\|_{\mathcal{L}(\tilde{\mathscr{U}}_\beta^1(\operatorname{div} 0)', \tilde{\mathscr{U}}_\alpha^0(\operatorname{div} 0))} < \infty$. By an interpolation argument, we arrive at

$$\sup_{\kappa>0}\sup_{\varsigma\in[0,1]}\|C_\kappa^{-1}\|_{\mathcal{L}(\tilde{\mathscr{U}}_\beta^{1-\varsigma}(\operatorname{div} 0)', \tilde{\mathscr{U}}_\alpha^\varsigma(\operatorname{div} 0))} < \infty. \tag{6.19}$$

Recalling that $\varepsilon \leq 1-s$, we have $\frac{s}{1-\varepsilon} \in [0,1]$. From $s+\varepsilon-1 = \varepsilon\cdot 0 + (1-\varepsilon)(\frac{s}{1-\varepsilon}-1)$ and $s = \varepsilon\cdot 0 + (1-\varepsilon)\frac{s}{1-\varepsilon}$, an application of the Riesz–Thorin theorem shows that

$$\begin{aligned}\|C_\kappa^{-1}\|_{\mathcal{L}(\tilde{\mathscr{U}}_\beta^{1-\varepsilon-s}(\operatorname{div} 0)', \tilde{\mathscr{U}}_\alpha^s(\operatorname{div} 0))} \leq{}& \|C_\kappa^{-1}\|^\varepsilon_{\mathcal{L}(L_2(I;\hat{H}^0(\operatorname{div} 0;\Omega)), L_2(I;\hat{H}^0(\operatorname{div} 0;\Omega)))} \\ &\times \|C_\kappa^{-1}\|^{1-\varepsilon}_{\mathcal{L}(\tilde{\mathscr{U}}_\beta^{1-\frac{s}{1-\varepsilon}}(\operatorname{div} 0)', \tilde{\mathscr{U}}_\alpha^{\frac{s}{1-\varepsilon}}(\operatorname{div} 0))}.\end{aligned} \tag{6.20}$$

From (6.18) and (6.19) we infer that the right-hand side can be made arbitrarily small by taking κ large. Now writing $C_\kappa + \delta C(\bar{\mathbf{u}}) = C_\kappa(I + C_\kappa^{-1}\delta C(\bar{\mathbf{u}}))$, and combining the latter result with (6.17) and (6.19), the proof of (6.16) for the case $s \in [\frac{1}{2}, 1)$ is completed.

Finally, for the case $s = 1$, we write $C_\kappa + \delta C(\bar{\mathbf{u}}) = (I + \delta C(\bar{\mathbf{u}})C_\kappa^{-1})C_\kappa$. Taking $\varepsilon \in (0, \bar{s} - \frac{n}{4})$, two applications of Proposition 6.1, together with (6.14), show that

$$|(\delta C(\bar{\mathbf{u}})\mathbf{u})(\mathbf{v})| \lesssim \|\bar{\mathbf{u}}\|_{\mathscr{U}_\alpha^{\bar{s}}}\|\mathbf{u}\|_{\tilde{\mathscr{U}}_\alpha^{1-\varepsilon}(\operatorname{div} 0)}\|\mathbf{v}\|_{\tilde{\mathscr{U}}_\beta^0(\operatorname{div} 0)},$$

i.e.,

$$\delta C(\bar{\mathbf{u}}) \in \mathcal{L}(\tilde{\mathscr{U}}_\alpha^{1-\varepsilon}(\operatorname{div} 0)), \tilde{\mathscr{U}}_\beta^0(\operatorname{div} 0)).$$

Similarly to (6.20), one infers that $\|C_\kappa^{-1}\|_{\mathcal{L}((\tilde{\mathscr{U}}_\beta^0(\operatorname{div} 0)), \tilde{\mathscr{U}}_\alpha^{1-\varepsilon}(\operatorname{div} 0))}$ can be made arbitrarily small by taking κ large, from which the proof for this case follows. ∎

In Theorem 6.3 we imposed $\bar{\mathbf{u}} \in \mathscr{U}_\alpha^{\bar{s}}$ for some $\bar{s} > \frac{n}{4}$ to ensure that $\delta C(\bar{\mathbf{u}})$ is a perturbation of strictly lower order to the instationary Stokes operator. The arguments employed in [Fon10] indicate that for $n = 2$ and I being the half-line

(α, ∞), (6.13) defines a diffeomorphism from $\{\mathbf{w} \in \mathscr{U}_\alpha^{\frac{1}{2}} : I \otimes \widetilde{\operatorname{div}}\, \mathbf{w} = 0\}$ to $\{\mathbf{w} \in \mathscr{U}_\beta^{\frac{1}{2}} : I \otimes \widetilde{\operatorname{div}}\, \mathbf{w} = 0\}'$ (where here the homogeneous condition at time $t = \beta$ is void since $\beta = \infty$). In other words, it seems that in space dimension $n = 2$ the additional regularity condition $\bar{\mathbf{u}} \in \mathscr{U}_\alpha^{\bar{s}}$ for some $\bar{s}$ strictly larger than $\frac{1}{2}$ can be avoided.

So far we considered the instationary Navier–Stokes equations with homogeneous *initial conditions*. The approach of appending inhomogeneous conditions as natural boundary conditions as employed for the Stokes problem in Theorem 5.13 is not applicable because it requires searching the solution $(\mathbf{u}, p) \in \mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s$ for some $s < \frac{1}{2}$, whereas we established in Theorem 6.2 that $\mathrm{NS} : \mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s \to (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})'$ for $s \geq \frac{1}{2}$ if $n = 2$ and for $s > \frac{3}{4}$ if $n = 3$.

Therefore, let $\bar{\mathbf{u}}_0$ denote a lifting of the given, inhomogeneous initial datum $\mathbf{u}_0$ to the space-time cylinder $I \times \Omega$. Then, writing the solution in the form $(\mathbf{u} + \bar{\mathbf{u}}_0, p)$, $\mathbf{u}$ satisfies homogeneous initial conditions. One infers that with

$$\begin{aligned} \tilde{\mathrm{S}}(\mathbf{u}, p)(\mathbf{v}, q) &:= \mathrm{S}(\mathbf{u}, p)(\mathbf{v}, q) + n(\bar{\mathbf{u}}_0, \mathbf{u}, \mathbf{v}) + n(\mathbf{u}, \bar{\mathbf{u}}_0, \mathbf{v}), \\ \widetilde{\mathrm{NS}}(\mathbf{u}, p)(\mathbf{v}, q) &:= \tilde{\mathrm{S}}(\mathbf{u}, p)(\mathbf{v}, q) + n(\mathbf{u}, \mathbf{u}, \mathbf{v}), \end{aligned}$$

the pair $(\mathbf{u}, p)$ solves formally

$$\widetilde{\mathrm{NS}}(\mathbf{u}, p)(\mathbf{v}, q) = \mathbf{f}(\mathbf{v}) - g(q) - c(\bar{\mathbf{u}}_0, \mathbf{v}) - d(q, \bar{\mathbf{u}}_0) - n(\bar{\mathbf{u}}-, \bar{\mathbf{u}}_0, \mathbf{v}) \tag{6.21}$$

for all test functions $(\mathbf{v}, q)$.

The same proof that showed Theorem 6.3 also shows the following.

Proposition 6.6. *Let $|I| < \infty$, and let $s \in [\frac{1}{2}, 1]$ be such that $\mathrm{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$ and, for $\varsigma \in [0, s]$, $D(A^{\frac{\varsigma}{2}}) \simeq \{\mathbf{u} \in \hat{H}^\varsigma(\Omega)^n : \widetilde{\operatorname{div}}\, \mathbf{u} = 0\}$, with A as defined in Proposition* 5.5. *For some $\bar{s} > \frac{n}{4}$, let $\bar{\mathbf{u}}_0 \in \mathscr{U}_\alpha^{\bar{s}}$. Then $\tilde{S} \in \mathcal{L}\mathrm{is}(\mathscr{U}_\alpha^s \times \mathscr{P}_\beta^s, (\mathscr{U}_\beta^{1-s} \times \mathscr{P}_\alpha^{1-s})')$.*

Again, the arguments applied in [Fon10] indicate that for $n = 2$ and $I = (\alpha, \infty)$ the condition $\bar{\mathbf{u}}_0 \in \mathscr{U}_\alpha^{\bar{s}}$ for some $\bar{s} > \frac{n}{4}$ can be omitted.

Remark 6.7. Alternatively, by applying the technique of Theorem 6.2, the condition $|I| < \infty$ can be replaced by the condition of $\bar{\mathbf{u}}_0 \in \mathscr{U}_\alpha^{\bar{s}}$ being sufficiently small. This setting allows $\bar{s} = s = \frac{1}{2}$ when $n = 2$.

With these results at hand, the analysis for the operator NS given in Theorems 6.2 and 6.3 can be repeated for $\widetilde{\mathrm{NS}}$. In applications, the construction of the required lifting of $\mathbf{u}_0$ to $\bar{\mathbf{u}}_0 \in \mathscr{U}_\alpha^{\bar{s}}$ can be nontrivial.

7. Conclusion. We proposed well-posed space-time variational saddle-point formulations of instationary incompressible Stokes and Navier–Stokes equations, in scales of fractional Bochner–Sobolev spaces. A novel aspect is that the formulations do not require a full-regularity condition on the stationary Stokes operator and therefore apply on general bounded Lipschitz spatial domains.

The variational formulations can be the basis of space-time adaptive numerical solution methods. In particular, for Stokes and, when $n = 2$, Navier–Stokes equations, all arising temporal and spatial Sobolev spaces can be conveniently equipped with bases of continuous piecewise polynomial wavelets. By equipping the arising Bochner spaces with the resulting tensor product bases, the whole time evolution problem can be solved by an adaptive wavelet method at the best possible convergence rate, and for Stokes, at linear cost. Under mild (Besov) smoothness conditions, this rate is equal as when solving one instance of the corresponding stationary problem.

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