Chapter 4
Piecewise Tensor Product Wavelet Bases by Extensions and Approximation Rates

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Abstract In this chapter, we present some of the major results that have been achieved in the context of the DFG-SPP project "Adaptive Wavelet Frame Methods for Operator Equations: Sparse Grids, Vector-Valued Spaces and Applications to Nonlinear Inverse Problems". This project has been concerned with (nonlinear) elliptic and parabolic operator equations on nontrivial domains as well as with related inverse parameter identification problems. One crucial step has been the design of an efficient forward solver. We employed a spatially adaptive wavelet Rothe scheme. The resulting elliptic subproblems have been solved by adaptive wavelet Galerkin schemes based on generalized tensor wavelets that realize dimension-independent approximation rates. In this chapter, we present the construction of these new tensor bases and discuss some numerical experiments.

4.1 Introduction

The aim of the project "Adaptive Wavelet Frame Methods for Operator Equations" has been the development of optimal convergent adaptive wavelet schemes for elliptic and parabolic operator equations on nontrivial domains. Moreover, we have been concerned with the efficient treatment of related parameter identification problems. For the design of the efficient forward solver, we used variants of the recently developed adaptive wavelet schemes for elliptic operator equations, see., e.g., [4, 12]. (This list is clearly not complete). As usually, the construction of a suitable wavelet basis on the underlying domain is a nontrivial bottleneck. We attacked this problem by generalizing the construction of tensor wavelets on the hypercube to general domains. The resulting fully adaptive solver for the elliptic forward

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problem realizes dimension-independent convergence rates. For the treatment of the related inverse parameter identification problems we used regularization techniques. In particular, we analyzed and developed Tikhonov-regularization schemes with sparsity constraints for such nonlinear inverse problems. As a model problem, we studied the parameter identification problem for a parabolic reaction-diffusion system which describes the gene concentration in embryos at an early state of development (embryogenesis). In this chapter, we will only be concerned with the analysis of the forward problem, and we will concentrate on the construction of the new tensor wavelets and their approximation properties. For the analysis of the inverse problem, in particular concerning the regularity of the associated control-to-state map, we refer to Chap. 3 of this book. Numerical examples of the overall, fully adaptive wavelet scheme can be found in [5].

The approach we will present has partially been inspired by the work of Z. Ciesielski and T. Figiel [3] and of W. Dahmen and R. Schneider [6] who constructed a basis for a range of Sobolev spaces on a domain Ω from corresponding bases on subdomains. The principle idea can be described as follows. Let $\Omega = \cup_{k=0}^{N} \Omega_k \subset \mathbb{R}^n$ be a non-overlapping domain decomposition. By the use of extension operators, we will construct isomorphisms from the Cartesian product of Sobolev spaces on the subdomains, which incorporate suitable boundary conditions, to Sobolev spaces on Ω. By applying such an isomorphism to the union of Riesz bases for the Sobolev spaces on the subdomains, the result is a Riesz basis for the Sobolev space on Ω.

Since the approach can be applied recursively, to understand the construction of such an isomorphism, it is sufficient to consider the case of having two subdomains. For $i \in \{1,2\}$, let R_i be the restriction of functions on Ω to Ω_i, let η_2 be the extension by zero of functions on Ω_2 to functions on Ω, and let E_1 be some extension of functions on Ω_1 to functions on Ω which, for some $m \in \mathbb{N}_0$, is bounded from $H^m(\Omega_1)$ to the target space $H^m(\Omega)$. Then $\begin{bmatrix} R_1 \\ R_2(\mathrm{Id} - E_1 R_1) \end{bmatrix}$: $H^m(\Omega) \to H^m(\Omega_1) \times H^m_{0,\partial\Omega_1 \cap \partial\Omega_2}(\Omega_2)$ is boundedly invertible with inverse $[E_1 \;\; \eta_2]$. ($H^m_{0,\partial\Omega_1 \cap \partial\Omega_2}(\Omega_2)$ is the space of $H^m(\Omega_2)$ functions that vanish up to order $m-1$ at $\partial\Omega_1 \cap \partial\Omega_2$). Consequently, if Ψ_1 is a Riesz basis for $H^m(\Omega_1)$ and Ψ_2 is a Riesz basis for $H^m_{0,\partial\Omega_1 \cap \partial\Omega_2}(\Omega_2)$, then $E_1\Psi_1 \cup \eta_2\Psi_2$ is a Riesz basis for $H^m(\Omega)$.

Our main interest in the construction of a basis from bases on subdomains lies in the use of *piecewise tensor product approximation*. On the hypercube $\square := (0,1)^n$ one can construct a basis for the Sobolev space $H^m(\square)$ (or for a subspace incorporating Dirichlet boundary conditions) by taking an n-fold tensor product of a collection of univariate functions that forms a Riesz basis for $L_2(0,1)$ as well as, properly scaled, for $H^m(0,1)$. Thinking of a univariate wavelet basis of order $d > m$, the advantage of this approach is that the rate of nonlinear best M-term approximation of a sufficiently smooth function u is $d-m$, compared to $\frac{d-m}{n}$ for standard best M-term isotropic (wavelet) approximation of order d on $\square$. The multiplication of the one-dimensional rate $d-m$ by the factor $\frac{1}{n}$ is commonly referred to as the *curse of dimensionality*. However, when it comes to practical applications one should keep in mind that also the constants depend on n – even exponentially in the worst case. This is an intrinsic problem that also holds for other discretizations, e.g., by sparse grids. Nonetheless, tensor wavelets are a tool by which, for moderate space dimensions, the curse of dimensionality is at least diminished.

In view of these results on $\square$, we consider a domain Ω whose closure is the union of subdomains $\alpha_k + \overline{\square}$ for some $\alpha_k \in \mathbb{Z}^n$, or a domain Ω that is a parametric image of such a domain under a piecewise sufficiently smooth, globally C^{m-1} diffeomorphism κ, being a homeomorphism when $m = 1$. We equip $H^m(\Omega)$ (or a subspace incorporating Dirichlet boundary conditions) with a Riesz basis that is constructed using extension operators as discussed before from tensor product wavelet bases of order d on the subdomains, or from push-forwards of such bases. Many topological settings are covered by our approach, i.e., we consider homogeneous Dirichlet boundary conditions on arbitrary Lipschitz domains in two dimensions, see also Example 4.1 below. Our restriction to decompositions of Ω into subdomains from a topological Cartesian partition will allow us to rely on univariate extensions. Indeed, in order to end up with locally supported wavelets, we will apply local, scale-dependent extension operators – i.e., only wavelets that are adapted to the boundary conditions on the interfaces will be extended. We will show the best possible approximation rate $d - m$ for any u that restricted to any of these subdomains has a pull-back that belongs to a weighted Sobolev space.

4.2 Approximation by Tensor Product Wavelets on the Hypercube

We will study non-overlapping domain decompositions, where the subdomains are either unit n-cubes or smooth images of those. Sobolev spaces on these cubes, that appear with the construction of a Riesz basis for a Sobolev space on the domain as a whole, will be equipped with tensor product wavelet bases. From [7], we recall the construction of those bases.

For $t \in [0, \infty) \setminus (\mathbb{N}_0 + \frac{1}{2})$ and $\boldsymbol{\sigma} = (\sigma_\ell, \sigma_r) \in \{0, \ldots, \lfloor t + \frac{1}{2} \rfloor\}^2$, with $\mathscr{I} := (0, 1)$, let

$$H^t_{\boldsymbol{\sigma}}(\mathscr{I}) := \{v \in H^t(\mathscr{I}) : v(0) = \cdots = v^{(\sigma_\ell - 1)}(0) = 0 = v(1) = \cdots = v^{(\sigma_r - 1)}(1)\}.$$

With t and $\boldsymbol{\sigma}$ as above, and for $\tilde{t} \in [0, \infty) \setminus (\mathbb{N}_0 + \frac{1}{2})$ and $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_\ell, \tilde{\sigma}_r) \in \{0, \ldots, \lfloor \tilde{t} + \frac{1}{2} \rfloor\}^2$, we assume that

$$\Psi_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}} := \left\{\psi_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} : \lambda \in \nabla_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}\right\} \subset H^t_{\boldsymbol{\sigma}}(\mathscr{I}), \quad \tilde{\Psi}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}} := \{\tilde{\psi}_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} : \lambda \in \nabla_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}\} \subset H^{\tilde{t}}_{\tilde{\boldsymbol{\sigma}}}(\mathscr{I})$$

are biorthogonal Riesz bases for $L_2(\mathscr{I})$, and, by rescaling, for $H^t_{\sigma}(\mathscr{I})$ and $H^{\tilde{t}}_{\tilde{\sigma}}(\mathscr{I})$, respectively. Furthermore, denoting by $|\lambda| \in \mathbb{N}_0$ the *level* of λ, we assume that for some

$$\mathbb{N} \ni d > t,$$

$\mathscr{W}_1$. $|\langle \tilde{\psi}_\lambda^{(\sigma,\tilde{\sigma})}, u\rangle_{L_2(\mathscr{I})}| \lesssim 2^{-|\lambda|d} \|u\|_{H^d(\operatorname{supp} \tilde{\psi}^{(\sigma,\tilde{\sigma})})} \quad (u \in H^d(\mathscr{I}) \cap H^t_{\sigma}(\mathscr{I}))$,

$\mathscr{W}_2$. $\rho := \sup_{\lambda \in \nabla_{\sigma,\tilde{\sigma}}} 2^{|\lambda|} \max(\operatorname{diam} \operatorname{supp} \tilde{\psi}_\lambda^{(\sigma,\tilde{\sigma})}, \operatorname{diam} \operatorname{supp} \psi_\lambda^{(\sigma,\tilde{\sigma})})$
$\eqsim \inf_{\lambda \in \nabla_{\sigma,\tilde{\sigma}}} 2^{|\lambda|} \max(\operatorname{diam} \operatorname{supp} \tilde{\psi}_\lambda^{(\sigma,\tilde{\sigma})}, \operatorname{diam} \operatorname{supp} \psi_\lambda^{(\sigma,\tilde{\sigma})})$,

$\mathscr{W}_3$. $\sup_{j,k \in \mathbb{N}_0} \#\{|\lambda| = j : [k2^{-j}, (k+1)2^{-j}] \cap (\operatorname{supp} \tilde{\psi}_\lambda^{(\sigma,\tilde{\sigma})} \cup \operatorname{supp} \psi_\lambda^{(\sigma,\tilde{\sigma})}) \neq \emptyset\} < \infty$.

These conditions (as well as ($\mathscr{W}_4$) – ($\mathscr{W}_7$) in Sect. 4.4) are satisfied by following the biorthogonal wavelet constructions on the interval outlined in [8, 11].

It holds that $L_2(\square) = \otimes_{i=1}^n L_2(\mathscr{I})$. Further with

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_i = ((\sigma_i)_\ell, (\sigma_i)_r))_{1 \le i \le n} \in (\{0, \ldots, \lfloor t + \tfrac{1}{2} \rfloor\}^2)^n,$$

we define

$$H^t_{\boldsymbol{\sigma}}(\square) := H^t_{\boldsymbol{\sigma}_1}(\mathscr{I}) \otimes L_2(\mathscr{I}) \otimes \cdots \otimes L_2(\mathscr{I}) \cap \cdots \cap L_2(\mathscr{I}) \otimes \cdots \otimes L_2(\mathscr{I}) \otimes H^t_{\boldsymbol{\sigma}_n}(\mathscr{I}),$$

which is the space of $H^t(\square)$-functions whose normal derivatives of up to orders $(\sigma_i)_\ell$ and $(\sigma_i)_r$ vanish at the facets $\overline{\mathscr{I}}^{i-1} \times \{0\} \times \overline{\mathscr{I}}^{n-i}$ and $\overline{\mathscr{I}}^{i-1} \times \{1\} \times \overline{\mathscr{I}}^{n-i}$, respectively ($1 \le i \le n$) (the proof of this fact given in [7] for $t \in \mathbb{N}_0$ can be generalized to $t \in [0, \infty) \setminus (\mathbb{N}_0 + \frac{1}{2})$).

The *tensor product wavelet* collection

$$\boldsymbol{\Psi}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}} := \otimes_{i=1}^n \Psi_{\boldsymbol{\sigma}_i,\tilde{\boldsymbol{\sigma}}_i} = \Big\{ \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} := \otimes_{i=1}^n \psi_{\lambda_i}^{(\boldsymbol{\sigma}_i,\tilde{\boldsymbol{\sigma}}_i)} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}} := \prod_{i=1}^n \nabla_{\boldsymbol{\sigma}_i,\tilde{\boldsymbol{\sigma}}_i} \Big\},$$

and its renormalized version $\big\{\big(\sum_{i=1}^n 4^{t|\lambda_i|}\big)^{-1/2} \boldsymbol{\psi}_{\boldsymbol{\lambda}}^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}\big\}$ are Riesz bases for $L_2(\square)$ and $H^t_{\boldsymbol{\sigma}}(\square)$, respectively. The collection that is dual to $\boldsymbol{\Psi}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}$ reads as

$$\tilde{\boldsymbol{\Psi}}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}} := \otimes_{i=1}^n \tilde{\Psi}_{\boldsymbol{\sigma}_i,\tilde{\boldsymbol{\sigma}}_i} = \Big\{ \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} := \otimes_{i=1}^n \tilde{\psi}_{\lambda_i}^{(\boldsymbol{\sigma}_i,\tilde{\boldsymbol{\sigma}}_i)} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}} \Big\},$$

and its renormalized version $\big\{\big(\sum_{i=1}^n 4^{|\lambda_i|}\big)^{-\tilde{t}/2} \tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda}}^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} : \boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}\big\}$ is a Riesz basis for $H^{\tilde{t}}_{\tilde{\boldsymbol{\sigma}}}(\square)$.

For $\boldsymbol{\lambda} \in \boldsymbol{\nabla}_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}$, we set $|\boldsymbol{\lambda}| := (|\lambda_1|, \ldots, |\lambda_n|)$.

For $\theta \ge 0$, the *weighted Sobolev space* $\mathscr{H}^d_\theta(\mathscr{I})$ is defined as the space of all measurable functions u on $\mathscr{I}$ for which the norm

$$\|u\|_{\mathscr{H}^d_\theta(\mathscr{I})} := \left[\sum_{j=0}^{d} \int_{\mathscr{I}} |x^\theta (1-x)^\theta u^{(j)}(x)|^2 \,\mathrm{d}x\right]^{\frac{1}{2}}$$

is finite. For $m \in \{0, \ldots, \lfloor t \rfloor\}$, we will consider the weighted Sobolev space

$$\mathscr{H}^d_{m,\theta}(\square) := \cap_{p=1}^n \otimes_{i=1}^n \mathscr{H}^d_{\theta-\delta_{ip}\min(m,\theta)}(\mathscr{I}),$$

equipped with a squared norm that is the sum over $p = 1, \ldots, n$ of the squared norms on $\otimes_{i=1}^n \mathscr{H}^d_{\theta-\delta_{ip}\min(m,\theta)}(\mathscr{I})$.

4.3 Construction of Riesz Bases by Extension

Let $\{\square_0, \ldots, \square_N\}$ be a set of hypercubes from $\{\tau + \square : \tau \in \mathbb{Z}^n\}$, and let $\hat{\Omega}$ be a (reference) domain (i.e., open and connected) in $\mathbb{R}^n$ with $\cup_{k=0}^N \square_k \subset \hat{\Omega} \subset (\cup_{k=0}^N \overline{\square}_k)^{\mathrm{int}}$, and such that $\partial\hat{\Omega}$ is the union of (closed) facets of the $\square_k$'s. The case $\hat{\Omega} \subsetneq (\cup_{k=0}^N \overline{\square}_k)^{\mathrm{int}}$ corresponds to the situation that $\hat{\Omega}$ has one or more cracks. We will describe a construction of Riesz bases for Sobolev spaces on $\hat{\Omega}$ from Riesz bases for corresponding Sobolev spaces on the subdomains $\square_k$ using extension operators. We start with giving sufficient conditions ($\mathscr{D}_1$)–($\mathscr{D}_5$) such that suitable extension operators exist.

We assume that there exists a sequence $(\{\hat{\Omega}_k^{(q)} : q \leq k \leq N\})_{0 \leq q \leq N}$ of sets of polytopes, such that $\hat{\Omega}_k^{(0)} = \square_k$ and where each next term in the sequence is created from its predecessor by joining two of its polytopes. More precisely, we assume that for any $1 \leq q \leq N$, there exists a $q \leq \bar{k} = \bar{k}^{(q)} \leq N$ and $q - 1 \leq k_1 = k_1^{(q)} \neq k_2 = k_2^{(q)} \leq N$ such that

$\mathscr{D}_1$. $\hat{\Omega}_{\bar{k}}^{(q)} = \left(\overline{\hat{\Omega}_{k_1}^{(q-1)} \cup \hat{\Omega}_{k_2}^{(q-1)}} \setminus \partial\hat{\Omega}\right)^{\mathrm{int}}$ is connected, and the interface $J := \hat{\Omega}_{\bar{k}}^{(q)} \setminus (\hat{\Omega}_{k_1}^{(q-1)} \cup \hat{\Omega}_{k_2}^{(q-1)})$ is part of a hyperplane,

$\mathscr{D}_2$. $\{\hat{\Omega}_k^{(q)} : q \leq k \leq N, k \neq \bar{k}\} = \{\hat{\Omega}_k^{(q-1)} : q - 1 \leq k \leq N, k \neq \{k_1, k_2\}\}$,

$\mathscr{D}_3$. $\hat{\Omega}_N^{(N)} = \hat{\Omega}$.

For some

$$t \in [0, \infty) \setminus (\mathbb{N}_0 + \tfrac{1}{2}),$$

to each of the *closed* facets of all the hypercubes $\square_k$, we associate a number in $\{0, \ldots, \lfloor t + \frac{1}{2} \rfloor\}$ indicating the order of the Dirichlet boundary condition on that facet (where a Dirichlet boundary condition of order 0 means no boundary condition). On facets on the boundary of $\hat{\Omega}$, this number can be chosen at one's convenience (it is dictated by the boundary conditions of the boundary value problem that one wants to solve on $\hat{\Omega}$), and, as will follow from the conditions imposed below, on the other facets it should be either 0 or $\lfloor t+\frac{1}{2}\rfloor$.

By construction, each facet of any $\hat{\Omega}_k^{(q)}$ is a union of some facets of the $\square_{k'}$'s, that will be referred to as subfacets. Letting each of these subfacets inherit the Dirichlet boundary conditions imposed on the $\square_{k'}$'s, we define

$$\mathring{H}^t(\hat{\Omega}_k^{(q)}),$$

and so for $k=q=N$ in particular $\mathring{H}^t(\hat{\Omega})=\mathring{H}^t(\hat{\Omega}_N^{(N)})$, to be the closure in $H^t(\hat{\Omega}_k^{(q)})$ of the smooth functions on $\hat{\Omega}_k^{(q)}$ that satisfy these boundary conditions. Note that for $0\leq k\leq N$, for some $\boldsymbol{\sigma}(k)\in(\{0,\ldots,\lfloor t+\frac{1}{2}\rfloor\}^2)^n$,

$$\mathring{H}^t(\hat{\Omega}_k^{(0)})=\mathring{H}^t(\square_k)=H^t_{\boldsymbol{\sigma}(k)}(\square_k).$$

The boundary conditions on the hypercubes that determine the spaces $\mathring{H}^t(\hat{\Omega}_k^{(q)})$, and the order in which polytopes are joined should be chosen such that

$\mathscr{D}_4$. on the $\hat{\Omega}_{k_1}^{(q-1)}$ and $\hat{\Omega}_{k_2}^{(q-1)}$ sides of J, the boundary conditions are of order 0 and $\lfloor t+\frac{1}{2}\rfloor$, respectively,

and, w.l.o.g. assuming that $J=\{0\}\times\breve{J}$ and $(0,1)\times\breve{J}\subset\Omega_{k_1}^{(q-1)}$,

$\mathscr{D}_5$. for any function in $\mathring{H}^t(\hat{\Omega}_{k_1}^{(q-1)})$ that vanishes near $\{0,1\}\times\breve{J}$, its reflection in $\{0\}\times\mathbb{R}^{n-1}$ (extended with zero, and then restricted to $\hat{\Omega}_{k_2}^{(q-1)}$) is in $\mathring{H}^t(\hat{\Omega}_{k_2}^{(q-1)})$.

The condition ($\mathscr{D}_5$) is a compatibility condition on the subfacets adjacent to the interface, see Fig. 4.1 for an illustration.

Given $1\leq q\leq N$, for $i\in\{1,2\}$, let $R_i^{(q)}$ be the *restriction* of functions on $\hat{\Omega}_{\bar{k}}^{(q)}$ to $\hat{\Omega}_{k_i}^{(q-1)}$, and let $\eta_2^{(q)}$ be the *extension* of functions on $\hat{\Omega}_{k_2}^{(q-1)}$ to $\hat{\Omega}_{\bar{k}}^{(q)}$ by zero. Under the conditions ($\mathscr{D}_1$)–($\mathscr{D}_5$), the extensions $E_1^{(q)}$ of functions on $\hat{\Omega}_{k_1}^{(q-1)}$ to $\hat{\Omega}_{\bar{k}}^{(q)}$ can be constructed (essentially) as tensor products of *univariate extensions* with identity operators in the other Cartesian directions. In the remaining part of this

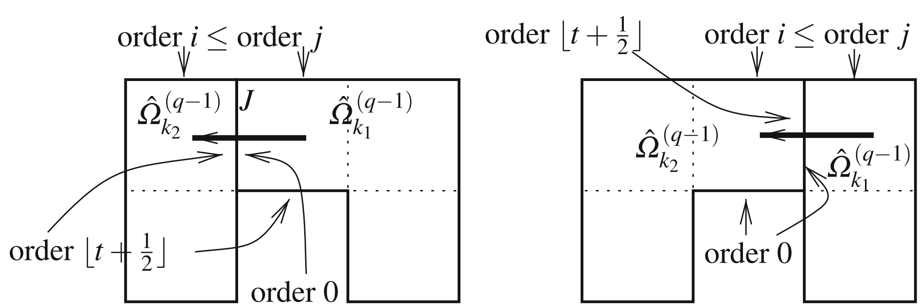

Fig. 4.1 Two illustrations with ($\mathscr{D}_1$)–($\mathscr{D}_5$). The *fat arrow* indicates the action of the extension $E_1^{(q)}$

chapter $[\cdot,\cdot]_{s,2}$ denotes the real interpolation space between two Hilbert spaces. For further information we refer to [1].

Proposition 4.1 ([2, Prop. 4.4.]). *Let G_1 be an extension operator of functions on $(0,1)$ to functions on $(-1,1)$ such that*

$$G_1 \in B(L_2(0,1), L_2(-1,1)), \quad G_1 \in B(H^t(0,1), H^t_{(\lfloor t+\frac{1}{2}\rfloor,0)}(-1,1)).$$

Let $E_1^{(q)}$ be defined by $R_2^{(q)} E_1^{(q)}$ being the composition of the restriction to $(0,1)\times \breve{J}$, followed by an application of

$$G_1 \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id},$$

followed by an extension by 0 to $\hat{\Omega}_{k_2}^{(q-1)} \setminus (-1,0)\times \breve{J}$. Then for $s \in [0,1]$

$$E^{(q)} := [E_1^{(q)} \ \eta_2^{(q)}] \in B\Big(\prod_{i=1}^{2}[L_2(\hat{\Omega}_{k_i}^{(q-1)}), \mathring{H}^t(\hat{\Omega}_{k_i}^{(q-1)})]_{s,2}, [L_2(\hat{\Omega}_{\bar{k}}^{(q)}), \mathring{H}^t(\hat{\Omega}_{\bar{k}}^{(q)})]_{s,2}\Big) \tag{4.1}$$

is boundedly invertible.

A Riesz basis on $\hat{\Omega}$ can now be constructed as follows.

Corollary 4.1 ([2, Cor. 4.6]). *For $0 \le k \le N$, let $\boldsymbol{\Psi}_k$ be a Riesz basis for $L_2(\square_k)$, that renormalized in $H^t(\square_k)$, is a Riesz basis for $\mathring{H}^t(\square_k) = H^t_{\boldsymbol{\sigma}(k)}(\square)$. Let E be the composition for $q = 1,\ldots,N$ of the mappings $E^{(q)}$ defined in (4.1), trivially extended with identity operators in coordinates $k \in \{q-1,\ldots,N\} \setminus \{k_1^{(q)}, k_2^{(q)}\}$. Then it holds that*

$$E \in B\Big(\prod_{k=0}^{n}[L_2(\square_k), \mathring{H}^t(\square_k)]_{s,2}, [L_2(\hat{\Omega}), \mathring{H}^t(\hat{\Omega})]_{s,2}\Big) \tag{4.2}$$

is boundedly invertible. Further, for $s \in [0,1]$, the collection $E(\prod_{k=0}^N \boldsymbol{\Psi}_k)$, normalized in the corresponding norm, is a Riesz basis for $[L_2(\hat{\Omega}), \mathring{H}^t(\hat{\Omega})]_{s,2}$.

For the dual basis $E^{-*}(\prod_{k=0}^N \tilde{\boldsymbol{\Psi}}_k)$ a similar result holds. In particular, for $s \in [0,1]$, it is, properly scaled, a Riesz basis for $[L_2(\hat{\Omega}), \mathring{H}^{\tilde{t}}(\hat{\Omega})]_{s,2}$. We refer to [2] for a detailed presentation.

The construction of Riesz bases on the reference domain $\hat{\Omega}$ extends to more general domains in a standard fashion. Let Ω be the image of $\hat{\Omega}$ under a homeomorphism κ. We define the *pull-back* κ^* by $\kappa^* w = w \circ \kappa$, and so its inverse κ^{-*}, known as the *push-forward*, satisfies $\kappa^{-*} v = v \circ \kappa^{-1}$.

Proposition 4.2 ([2, Prop. 4.11.]). *Let κ^* be boundedly invertible as a mapping both from $L_2(\Omega)$ to $L_2(\hat{\Omega})$ and from $H^t(\Omega)$ to $H^t(\hat{\Omega})$. Setting $\mathring{H}^t(\Omega) := \Im \kappa^{-*}|_{\mathring{H}^t(\hat{\Omega})}$, we have that $\kappa^{-*} \in B([L_2(\hat{\Omega}), \mathring{H}^t(\hat{\Omega})]_{s,2}, [L_2(\Omega), \mathring{H}^t(\Omega)]_{s,2})$ is*

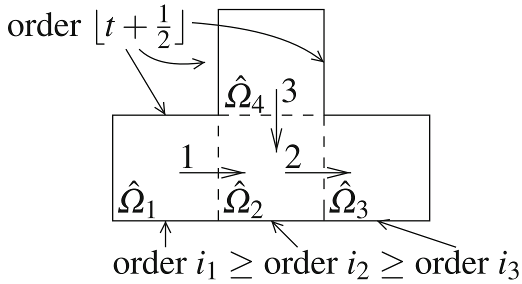

Fig. 4.2 Example extension directions and compatible boundary conditions

boundedly invertible ($s \in [0,1]$). *So if* $\boldsymbol{\Psi}$ *is a Riesz basis for* $L_2(\hat{\Omega})$ *and, properly scaled, for* $\overset{\circ}{H}{}^t(\hat{\Omega})$*, then for* $s \in [0,1]$, $\kappa^{-*}\boldsymbol{\Psi}$ *is, properly scaled, a Riesz basis for* $[L_2(\Omega), \overset{\circ}{H}{}^t(\Omega)]_{s,2}$. *If* $\tilde{\boldsymbol{\Psi}}$ *is the collection dual to* $\boldsymbol{\Psi}$*, then* $|\mathrm{det} D\kappa^{-1}(\cdot)|\kappa^{-*}\tilde{\boldsymbol{\Psi}}$ *is the collection dual to* $\kappa^{-*}\boldsymbol{\Psi}$.

We conclude this section by discussing some of the topological aspects of the construction.

Example 4.1. Consider a T-shaped domain decomposed into four subcubes as depicted in Fig. 4.2. In such a setting it is not possible to arrange the subdomains in a linear fashion. Further, when constructing a basis on such a domain, the ordering and directions of the extension operators are not unique. However, both aspects influence the boundary conditions that may be imposed. When proceeding as depicted, in the first step wavelets on $\hat{\Omega}_1$ are extended to $\hat{\Omega}_2$. Then the resulting basis is extended to $\hat{\Omega}_3$. Finally wavelets on $\hat{\Omega}_4$ are extended along the bottom interface. This set of extensions and its ordering is compatible with all boundary conditions that satisfy the restrictions depicted in Fig. 4.2, e.g., with homogeneous Dirichlet boundary conditions. In the second step of the construction only tensor wavelets on $\hat{\Omega}_2$ are extended. Consequently interchanging the ordering of the first two extensions does not change the resulting basis.

4.4 Approximation by – Piecewise – Tensor Product Wavelets

In the setting of Corollary 4.1 we select the bases on the subdomains $\square_k = \square + \alpha_k$, $\alpha_k \in \mathbb{Z}^n$, to be $\boldsymbol{\Psi}_{\boldsymbol{\sigma}(k),\tilde{\boldsymbol{\sigma}}(k)}(\cdot - \alpha_k)$, $\tilde{\boldsymbol{\Psi}}_{\boldsymbol{\sigma}(k),\tilde{\boldsymbol{\sigma}}(k)}(\cdot - \alpha_k)$, as constructed in Sect. 4.2. In this setting, for $m \in \{0,\ldots,\lfloor t \rfloor\}$ we study the approximation of functions $u \in \overset{\circ}{H}{}^m(\Omega) := [L_2(\Omega), \overset{\circ}{H}{}^t(\Omega)]_{m/t,2}$, that also satisfy

$$u \in \kappa^{-*}(\prod_{k=0}^{N} \mathscr{H}^d_{m,\theta}(\square_k)) := \{v : \Omega \to \mathbb{R} : v \circ \kappa \in \prod_{k=0}^{N} \mathscr{H}^d_{m,\theta}(\square_k)\}, \tag{4.3}$$

by $\kappa^{-*}E\big(\prod_{k=0}^{N} \boldsymbol{\Psi}_{\boldsymbol{\sigma}(k),\tilde{\boldsymbol{\sigma}}(k)}(\cdot - \alpha_k)\big)$ in the $H^m(\Omega)$-norm. Since, as is assumed in Proposition 4.2, $\kappa^* \in B(\overset{\circ}{H}{}^m(\Omega), \overset{\circ}{H}{}^m(\hat{\Omega}))$ is boundedly invertible, it is sufficient to study this issue for the case that $\kappa = \mathrm{Id}$ and so $\Omega = \hat{\Omega}$.

We will apply extension operators $E_1^{(q)}$ that are built from univariate extension operators. The latter will be chosen such that the resulting primal and dual wavelets on $\hat{\Omega}$, restricted to each $\square_k \subset \hat{\Omega}$, are tensor products of collections of univariate functions. We make the following additional assumptions on the univariate wavelets. For $\boldsymbol{\sigma} = (\sigma_\ell, \sigma_r) \in \{0, \ldots, \lfloor t + \frac{1}{2} \rfloor\}^2$, $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_\ell, \tilde{\sigma}_r) \in \{0, \ldots, \lfloor \tilde{t} + \frac{1}{2} \rfloor\}^2$, and with $\mathbf{0} := (0, 0)$,

$\mathscr{W}_4$. $V_j^{(\boldsymbol{\sigma})} := \operatorname{span}\{\psi_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} : \lambda \in \nabla_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}, |\lambda| \leq j\}$ is independent of $\tilde{\boldsymbol{\sigma}}$, and $V_j^{(\boldsymbol{\sigma})} = V_j^{(\mathbf{0})} \cap H^t_{\boldsymbol{\sigma}}(\mathscr{I})$,

$\mathscr{W}_5$. $\nabla_{\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}}}$ is the disjoint union of $\nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell}$, $\nabla^{(I)}$, $\nabla^{(r)}_{\sigma_r,\tilde{\sigma}_r}$ such that

i. $\sup_{\lambda \in \nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell},\, x \in \operatorname{supp} \psi_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}} 2^{|\lambda|}|x| \lesssim \rho, \qquad \sup_{\lambda \in \nabla^{(r)}_{\sigma_r,\tilde{\sigma}_r},\, x \in \operatorname{supp} \psi_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}} 2^{|\lambda|}|1 - x| \lesssim \rho,$

ii. For $\lambda \in \nabla^{(I)}$, $\psi_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} = \psi_\lambda^{(\mathbf{0},\mathbf{0})}$, $\tilde{\psi}_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} = \tilde{\psi}_\lambda^{(\mathbf{0},\mathbf{0})}$, and the extensions of $\psi_\lambda^{(\mathbf{0},\mathbf{0})}$ and $\tilde{\psi}_\lambda^{(\mathbf{0},\mathbf{0})}$ by zero are in $H^t(\mathbb{R})$ and $H^{\tilde{t}}(\mathbb{R})$, respectively,

$$\mathscr{W}_6. \begin{cases} \operatorname{span}\{\psi_\lambda^{(\mathbf{0},\mathbf{0})}(1 - \cdot) : \lambda \in \nabla^{(I)},\, |\lambda| = j\} = \operatorname{span}\{\psi_\lambda^{(\mathbf{0},\mathbf{0})} : \lambda \in \nabla^{(I)},\, |\lambda| = j\}, \\ \operatorname{span}\{\psi_\lambda^{(\sigma_\ell,\sigma_r),(\tilde{\sigma}_\ell,\tilde{\sigma}_r)}(1 - \cdot) : \lambda \in \nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell},\, |\lambda| = j\} = \\ \qquad \operatorname{span}\{\psi_\lambda^{(\sigma_r,\sigma_\ell),(\tilde{\sigma}_r,\tilde{\sigma}_\ell)} : \lambda \in \nabla^{(r)}_{\sigma_r,\tilde{\sigma}_r},\, |\lambda| = j\}, \end{cases}$$

$$\mathscr{W}_7. \begin{cases} \psi_\lambda^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}(2^l \cdot) \in \operatorname{span}\{\psi_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} : \mu \in \nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell}\} \quad (l \in \mathbb{N}_0,\ \lambda \in \nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell}), \\ \psi_\lambda^{(\mathbf{0},\mathbf{0})}(2^l \cdot) \in \operatorname{span}\{\psi_\mu^{(\mathbf{0},\mathbf{0})} : \mu \in \nabla^{(I)}\} \quad (l \in \mathbb{N}_0,\ \lambda \in \nabla^{(I)}). \end{cases}$$

In the setting of Proposition 4.1 we choose the univariate extension operator to be a Hestenes extension [6, 9, 10], that is,

$$\breve{G}_1 v(-x) = \sum_{l=0}^{L} \gamma_l (\zeta v)(\beta_l x) \quad (v \in L_2(\mathscr{I}),\ x \in \mathscr{I}), \tag{4.4}$$

(and, being an extension, $\breve{G}_1 v(x) = v(x)$ for $x \in \mathscr{I}$), where $\gamma_l \in \mathbb{R}$, $\beta_l > 0$, and $\zeta : [0, \infty) \to [0, \infty)$ is some smooth cut-off function with $\zeta \equiv 1$ in a neighborhood of 0, and $\operatorname{supp} \zeta \subset [0, \min_l(\beta_l, \beta_l^{-1})]$.

With such an extension operator at hand the obvious approach is to define $E_1^{(q)}$ according to Proposition 4.1 with $G_1 = \breve{G}_1$. A problem with the choice $G_1 = \breve{G}_1$ is that generally the desirable property $\operatorname{diam}(\operatorname{supp} G_1 u) \lesssim \operatorname{diam}(\operatorname{supp} u)$ is not implied. Indeed, think of the application of a Hestenes extension to a u with a small support that is not located near the interface.

To solve this and the corresponding problem for the adjoint extension, following [6] we will apply our construction using the modified, *scale-dependent* univariate extension operator

$$G_1 : u \mapsto \sum_{\lambda \in \nabla^{(\ell)}_{0,0}} \langle u, \tilde{\psi}_\lambda^{(\mathbf{0},\mathbf{0})} \rangle_{L_2(\mathscr{I})} \breve{G}_1 \psi_\lambda^{(\mathbf{0},\mathbf{0})} + \sum_{\lambda \in \nabla^{(I)} \cup \nabla^{(r)}_{0,0}} \langle u, \tilde{\psi}_\lambda^{(\mathbf{0},\mathbf{0})} \rangle_{L_2(\mathscr{I})} \eta_1 \psi_\lambda^{(\mathbf{0},\mathbf{0})}. \tag{4.5}$$

We focus on univariate extension operators with $\beta_l = 2^l$. This, together with ($\mathscr{W}_7$) ensures that the extended wavelets are locally (weighted sums of) univariate wavelets. Consequently most properties, like the locality on the primal and dual side in the sense of ($\mathscr{W}_2$), and ($\mathscr{W}_3$), as well as piecewise Sobolev smoothness are inherited by the extended wavelets. Further, by the symmetry assumption ($\mathscr{W}_6$), the extended part of a wavelet belongs to the span of boundary adapted wavelets. Therefore, together with ($\mathscr{W}_4$), we derive the technically useful property that extended wavelets $G_1\psi_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}$ belong piecewise to spaces $V_j^{(\mathbf{0})}$ with additionally $j \leq |\mu| + 2L$. This property, together with the locality of the primal and dual wavelets, is key for our central approximation result in Theorem 4.1.

Proposition 4.3 ([2, Prop. 5.2]). *Assuming ρ to be sufficiently small, the scale-dependent extension G_1 from (4.5) satisfies, for $\boldsymbol{\sigma} \in \{0,\dots,\lfloor t+\frac{1}{2}\rfloor\}^2$, $\tilde{\boldsymbol{\sigma}} \in \{0,\dots,\lfloor \tilde{t}+\frac{1}{2}\rfloor\}^2$*

$$G_1\psi_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} = \begin{cases} \eta_1\psi_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} & \text{when } \mu \in \nabla^{(I)} \cup \nabla^{(r)}_{\sigma_r,\tilde{\sigma}_r}, \\ \breve{G}_1\psi_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})} & \text{when } \mu \in \nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell}. \end{cases} \tag{4.6}$$

Assuming, additionally, $\breve{G}_1$ to be a Hestenes extension *with $\beta_l = 2^l$, the resulting adjoint extension $G_2 := (\mathrm{Id} - \eta_1 G_1^*)\eta_2$ satisfies*

$$G_2(\tilde{\psi}_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}(1+\cdot)) = \begin{cases} \eta_2(\tilde{\psi}_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}(1+\cdot)) & \text{when } \mu \in \nabla^{(I)} \cup \nabla^{(\ell)}_{\sigma_\ell,\tilde{\sigma}_\ell}, \\ \breve{G}_2(\tilde{\psi}_\mu^{(\boldsymbol{\sigma},\tilde{\boldsymbol{\sigma}})}(1+\cdot)) & \text{when } \mu \in \nabla^{(r)}_{\sigma_r,\tilde{\sigma}_r}. \end{cases} \tag{4.7}$$

We have $G_1 \in B(L_2(0,1), L_2(-1,1))$, $G_1 \in B(H^t(0,1), H^t(-1,1))$, and further $G_1^* \in B(H^{\tilde{t}}(-1,1), H^{\tilde{t}}_{(\lfloor \tilde{t}+\frac{1}{2}\rfloor,0)}(0,1))$. Finally, G_1 and G_2 are local in the following sense

$$\begin{cases} \operatorname{diam}(\operatorname{supp} R_2 G_1 u) \lesssim \operatorname{diam}(\operatorname{supp} u) & (u \in L_2(0,1)), \\ \operatorname{diam}(\operatorname{supp} R_1 G_2 u) \lesssim \operatorname{diam}(\operatorname{supp} u) & (u \in L_2(-1,0)). \end{cases} \tag{4.8}$$

A typical example of a Hestenes extension with $\beta_l = 2^l$ is the reflection, i.e., $L = 0$, $\gamma_0 = 1$.

Remark 4.1. Although implicitly claimed otherwise in [6, (4.3.12)], we note that (4.7), and so the second property in (4.8), *cannot* be expected for $\breve{G}_1$ being a general Hestenes extension as given by (4.4), without assuming that $\beta_l = 2^l$.

We may now formulate the central approximation result. Recall that by utilizing the scale-dependent extension operator in the construction presented in Sect. 4.3, we end up with a pair of biorthogonal wavelet Riesz bases

$$\Big(E\big(\prod_{k=0}^{N}\boldsymbol{\Psi}_k\big),\quad E^{-*}\big(\prod_{k=0}^{N}\tilde{\boldsymbol{\Psi}}_k\big)\Big) = \Big(\{\boldsymbol{\psi}_{\boldsymbol{\lambda},p} : (\boldsymbol{\lambda},p) \in \nabla(\hat{\Omega})\}, \{\tilde{\boldsymbol{\psi}}_{\boldsymbol{\lambda},p} : (\boldsymbol{\lambda},p) \in \nabla(\hat{\Omega})\}\Big)$$

for $L_2(\hat{\Omega})$, that is for $s \in [0,1]$ and properly scaled a pair of Riesz bases for $[L_2(\hat{\Omega}), \mathring{H}^t(\hat{\Omega})]_{s,2}$ and $[L_2(\hat{\Omega}), \mathring{H}^{\tilde{t}}(\hat{\Omega})]_{s,2}$, respectively. In particular the index set is given by $\nabla(\hat{\Omega}) = \bigcup_{k=0}^{N} \nabla_{\sigma(k),\tilde{\sigma}(k)} \times \{k\}$.

Theorem 4.1 ([2, Thm. 5.6]). *Let the $E_1^{(q)}$ be defined using the scale-dependent extension operators as in Proposition 4.3. Then for any $\theta \in [0,d)$, there exists a (nested) sequence $(\nabla_M)_{M\in\mathbb{N}} \subset \nabla(\hat{\Omega})$ with $\#\nabla_M \eqsim M$, such that*

$$\inf_{v\in \operatorname{span}\{\psi_{\lambda,p}:(\lambda,p)\in\nabla_M\}} \|u-v\|_{H^m(\hat{\Omega})} \lesssim M^{-(d-m)} \sqrt{\sum_{k=0}^{N} \|u\|^2_{\mathscr{H}^d_{m,\theta}(\square_k)}}, \tag{4.9}$$

for any $u \in \mathring{H}^m(\hat{\Omega})$ for which the right-hand side is finite, i.e., that satisfies (4.3) (with $\kappa = \mathrm{Id}$). For $m = 0$, the factor $M^{-(d-m)}$ in (4.9) has to be read as $(\log M)^{(n-1)(\frac{1}{2}+d)} M^{-d}$.

The issue whether we may expect (4.3) for u to hold is nontrivial. Fortunately, in [2], we were able to prove that this property holds for the solutions of a large class of boundary value problems over polygonal or polyhedral domains.

4.5 Numerical Results

As domains, we consider the *slit domain* $\Omega = (0,2)^2 \setminus \{1\}\times[1,2)$, the 3-dimensional *L-shaped domain* $\Omega = (0,2)^2 \times (0,1) \setminus [1,2)^2 \times (0,1)$, and the *Fichera corner domain* $\Omega = (0,2)^3 \setminus [1,2)^3$. The corresponding domain decompositions and the directions in which the extension operator is applied are illustrated in Fig. 4.3.

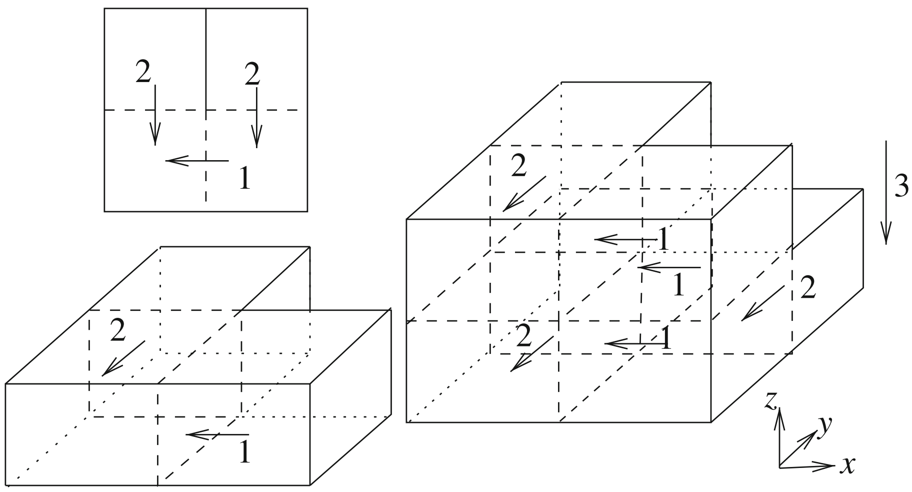

Fig. 4.3 The direction and ordering of the extensions

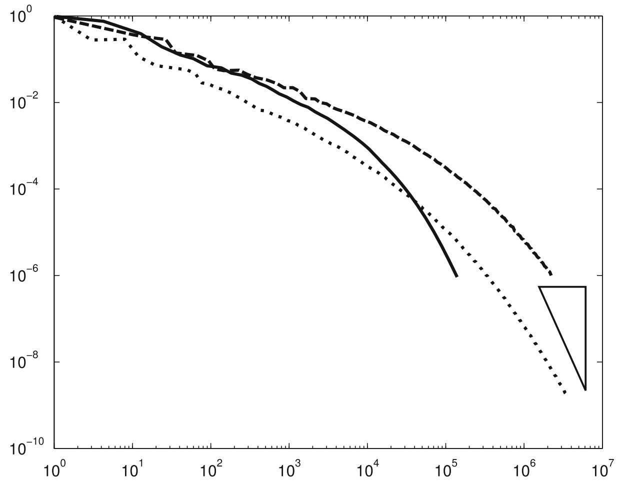

Fig. 4.4 Support length vs. relative residual on the slit domain (*line*), L-shaped domain (*dotted*) and Fichera corner domain (*dashed*)

As extension operator, we apply the reflection suited for $\frac{1}{2} < t < \frac{3}{2}$, $0 < \tilde{t} < \frac{1}{2}$, which is sufficient for our aim of constructing a Riesz basis for $H_0^1(\Omega)$.

Using piecewise tensor product bases, we solved the problem of finding $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}x = f(v) \quad (v \in H_0^1(\Omega)) \tag{4.10}$$

by applying the *adaptive wavelet-Galerkin method* [4, 12]. We choose the forcing vector $f = 1$.

As the univariate bases for the tensor wavelet construction we choose C^1, piecewise quartic (d = 5) (multi-) wavelets. The chosen solver is known to produce a sequence of approximations that converges in the $H^1(\Omega)$-norm with the same rate as best M-term wavelet approximation. We therefore expect the approximation rate $d - m = 5 - 1 = 4$.

The numerical results are presented in Fig. 4.4.

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