Computational Social Choice and Complexity Theory

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ESSLLI 2018 – **Day 5**

Recap

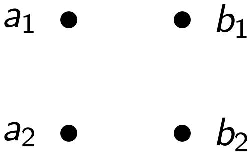
- Voting
- Winner Determination, Manipulation, Bribery
- Domain Restrictions, Single-Peakedness
- Judgment Aggregation

What we'll do today

- Stable Matching
- The Gale-Shapley Algorithm
- Strategizing
- Variants of Matching Problems

Stable Matching

Bipartite Matching of Agents



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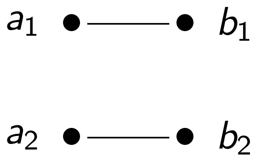

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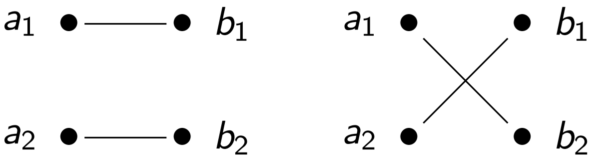

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Stable matching

- Two sets of agents (of same size):
  - $A = \{a_1, \ldots, a_n\}$
  - $B = \{b_1, \ldots, b_n\}$

- Each agent has a preference over all agents from the other side (candidates): a linear order $\succ$
  - All preferences together: preference profile

- A matching is a bijection μ between A and B
  - Blocking pair: $(a, b) \in A \times B$ such that:
    - $(a, b) \notin \mu$,
    - $b \succ_a \mu(a)$, and
    - $a \succ_b \mu(b)$.
  - Matching μ is stable if there exists no blocking pair

Example

$$A = \{a_1, a_2\}, \qquad B = \{b_1, b_2\}$$

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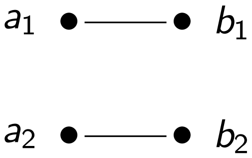

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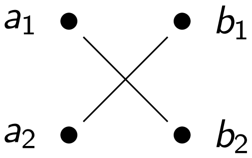
Example

Unstable matching:



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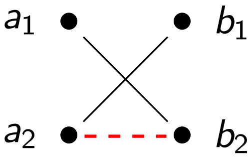

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The Gale-Shapley Algorithm

- Does a stable matching always exist?
- Can we find a stable matching efficiently, if it exists?

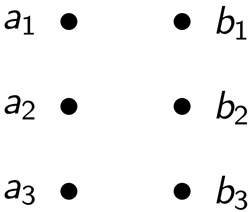
- **Answers:** yes, and yes.
- A stable matching always exists and we can use the Gale-Shapley algorithm to find one.

D. Gale and L.S. Shapley. College Admissions and the Stability of Marriage. American Mathematical Monthly, 69:9–15, 1962.

The Gale-Shapley Algorithm

- We choose one side (say A) as proposing side, and we construct the matching μ in rounds.
- In each round, a currently unmatched agent $a \in A$ proposes to their top ranked agent $b \in B$ that they have not proposed to before
- When some $b \in B$ is proposed to by $a \in A$:
  - if b is currently unmatched, they provisionally accept the match with a
  - if b is currently matched to a', and $a' \succ_b a$, then b rejects a
  - if b is currently matched to a', and $a \succ_b a'$, then b rejects their previous match a' (and a' becomes unmatched again)
- We continue until all agents are matched, and return the constructed matching μ

The Gale-Shapley Algorithm (Example Run 1)



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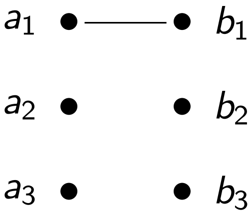

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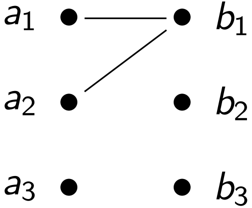

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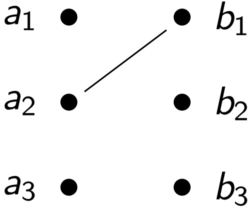

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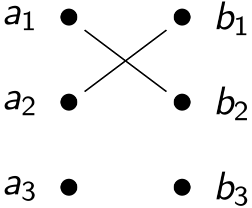

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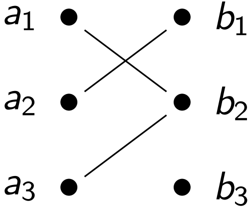

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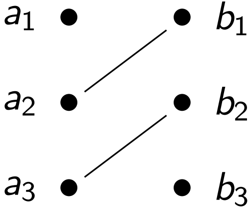

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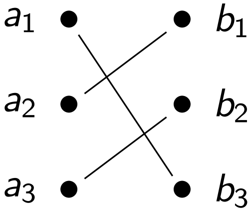

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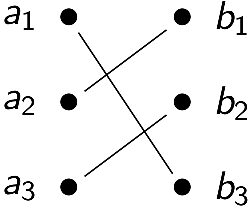

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The Gale-Shapley Algorithm (Example Run 2)

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a_2 •	• b_2
a_3 •	• b_3

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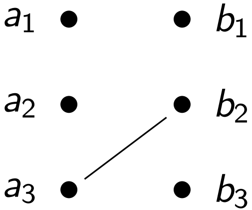

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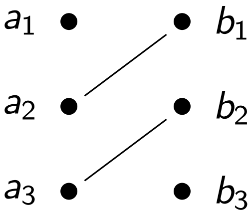

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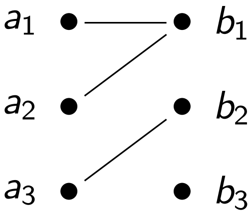

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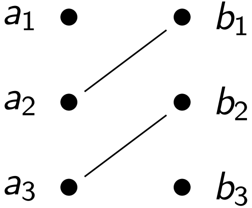

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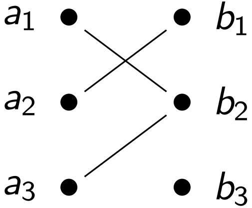

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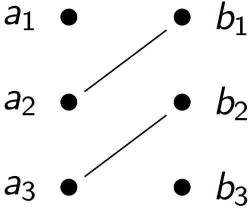

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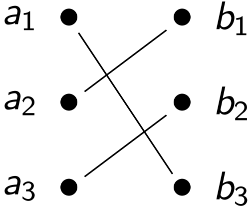

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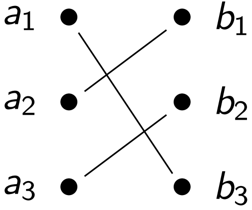

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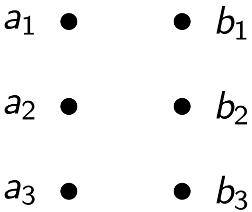
The Gale-Shapley Algorithm (Example Run 3)



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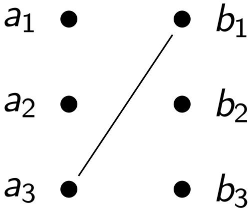
The Gale-Shapley Algorithm (Example Run 3)



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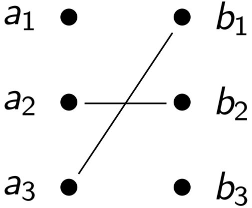

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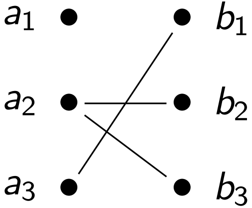

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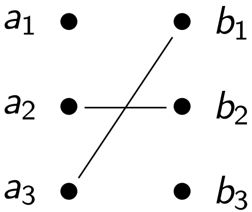

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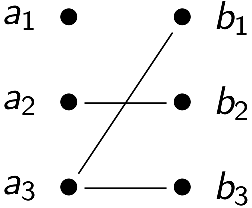

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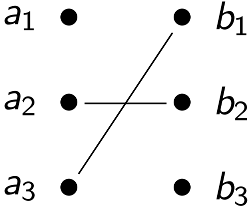

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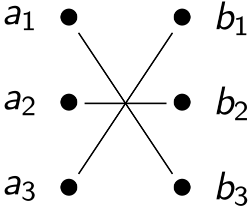

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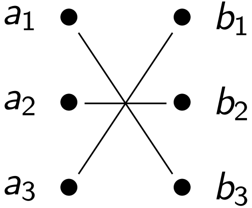

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The Gale-Shapley Algorithm

- **Claim:** the Gale-Shapley algorithm always terminates in $\leq n^2$ rounds
  - In every round, one candidate proposes to one other candidate
  - The proposing agents only go down in their preference list
  - So, every proposal happens at most once
  - Thus, there are at most n^2 rounds

The Gale-Shapley Algorithm

- **Claim:** when the Gale-Shapley algorithm terminates, all agents (on both sides) are matched
  - Every agent on the proposing side is matched
    - Suppose there is some proposing agent c that is not matched
    - Then some non-proposing agent d is also not matched
    - At some point c proposed to d
    - Then d would remain matched to some agent throughout the process
    - Contradiction!
  - Every agent on the non-proposing side is matched
    - By counting: there are equally proposing and non-proposing agents, and matchings are one-to-one

The Gale-Shapley Algorithm

- **Claim:** the matching returned by the Gale-Shapley algorithm is stable
  - Consider a pair (c, d) that is not matched, where c is on the proposing side
  - We distinguish two cases

(1) Either c proposed to d at some point

- Then d prefers their current partner to c

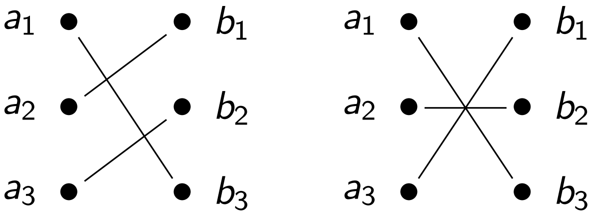
  (since c and d are not matched anymore)

(2) Or c never proposed to d

- Then c prefers their current partner to d

  - In both cases, (c, d) is not a blocking pair

Outcomes of the Gale-Shapley Algorithm



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The Gale-Shapley Algorithm

- What are the properties of the stable matchings returned by the Gale-Shapley algorithm?
  - Does it find all stable matchings?
  - Do the matchings that it finds satisfy certain fairness properties?
  - Can the agents manipulate the outcome by strategically reporting insincere preferences?

A- and B-Optimality

- A stable matching is A-optimal if every agent $a \in A$ is matched to their most preferred agent among the agents b that they are matched with in any stable matching
- A stable matching is B-optimal if every agent $b \in B$ is matched to their most preferred agent among the agents a that they are matched with in any stable matching

Theorem (Gale, Shapley, 1962)

The A-proposing Gale-Shapley algorithm results in the (unique) A-optimal matching. The B-proposing Gale-Shapley algorithm results in the (unique) B-optimal matching.

D. Gale and L.S. Shapley. College Admissions and the Stability of Marriage. American Mathematical Monthly, 69:9–15, 1962.

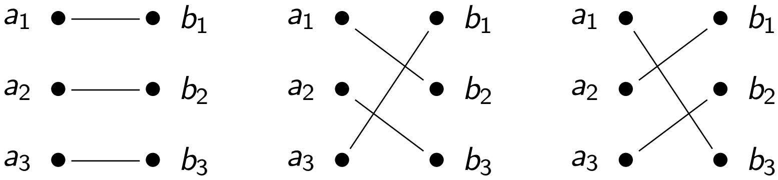
A- and B-Optimality

Theorem (Gale, Shapley, 1962)

The A-proposing Gale-Shapley algorithm results in the (unique) A-optimal matching. The B-proposing Gale-Shapley algorithm results in the (unique) B-optimal matching.

- *Idea:*
  - Suppose some a is matched to b, but there is another stable matching μ' where a is matched to b', and $b' \succ_a b$
  - So a proposed to b' before, but b' rejected
  - Assume that this was the first rejection of a "stable partner"
  - Let a' be the agent that b' chose over a
  - Then a' prefers b' over all "stable partners" (because b' rejecting a was the first rejection of a "stable partner")
  - But then μ' is not stable, as (a', b') is a blocking pair
  - Contradiction!

Examples of Stable Matchings



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- So the Gale-Shapley algorithm does not find all stable matchings

Strategizing

- Strategizing for a mechanism M consists of an agent c reporting an insincere preference order $\succ'_c$ so that the outcome of M for $\succ'_c$ is preferred by c over the outcome of M for $\succ_c$
- Is the Gale-Shapley algorithm strategyproof?

Theorem (Roth, 1982)

The A-proposing Gale-Shapley algorithm is strategyproof for A.
The B-proposing Gale-Shapley algorithm is strategyproof for B.

A.E. Roth. The Economics of Matching: Stability and Incentives. Mathematics of Operations Research, 7:617–628, 1982.

Strategizing

- Is there a stable matching mechanism that is strategyproof for both A and B?

Theorem (Roth, 1982)

There exists no matching mechanism that is stable and strategyproof for both A and B.

Strategizing

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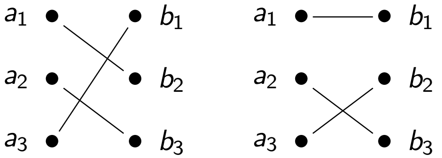

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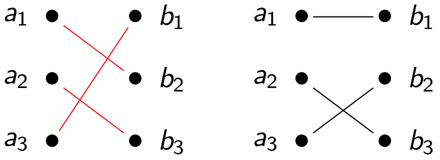

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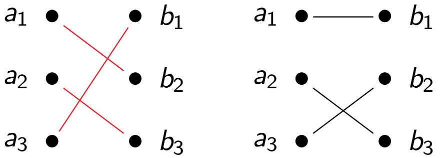

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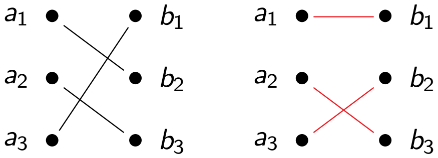

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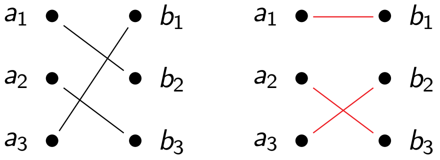

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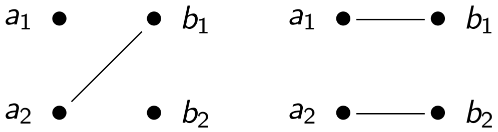
Stable Matching with Incomplete Lists

- Instead of specifying a full linear order over all agents on the other side, agents can specify a set of agents that they find acceptable, and give a linear preference order over them
- An agent prefers remaining unmatched over being matched with an agent they find unacceptable
- A matching is stable in this setting if there is no:
  - blocking pair, and
  - no agent that prefers being unmatched over their current match
- The Gale-Shapley algorithm can be extended to the case with incomplete lists

Stable Matching with Incomplete Lists and Ties

- We can also, in addition, allow agents to specify weak preference orders
  - I.e., allowing ties in their preferences
- In this case, stable matchings can have different size:



preference of a_1: b_1
preference of a_2: $b_1 \succ_{a_2} b_2$

preference of b_1: $a_1 \sim_{b_1} a_2$
preference of b_2: a_2

Stable Matching with Incomplete Lists and Ties

Max-SMTI

Input: two sets A and B of agents, for each agent $c \in A \cup B$ their preferences (with unacceptable agents and ties) over the agents in the other set, and a partial matching μ.

Output: Is there a maximum size stable matching that includes μ?

Theorem (Manlove et al., 2002)

Max-SMTI is NP-complete.

D.F. Manlove, R.W. Irving, K. Iwama, S. Miyazaki, and Y. Morita. Hard Variants of Stable Marriage. Theoretical Computer Science, 276(12):261–279, 2002.

Stable Roommates

- Instead of a bipartite matching scenario with two sets A and B, we have one single set A of agents
- Each agent a specifies a preference over the other agents $A \setminus \{a\}$
- In this setting, a stable matching does not always exist

Theorem (Irving, 1985)

There exists a polynomial-time algorithm to find a stable matching for the stable roommates problem, if it exists.

R.W. Irving. An efficient algorithm for the "stable roommates" problem. Journal of Algorithms, 6(4):577–595, 1985.

Hospital-Residents Matching

- In this variant, A is a set of residents and B is a set of hospitals
  - Each hospital $b \in B$ has a capacity $c_b \in \mathbb{N}$ in addition to a preference over A
  - Both residents and hospitals can specify acceptable matches
- A matching then matches each resident a to ≤ 1 hospital, and each hospital b to $\leq c_b$ residents
- An unmatched pair (a, b) is blocking if a prefers b to their current match (or their currently being unmatched), and if b prefers to be matched with a (i.e., either if b has a free spot, or if b prefers a over one of their current matches)
- The Gale-Shapley algorithm can be extended for HR matching

Further Topics in Matching

- Optimizing for the overall satisfaction of the agents over all stable matchings
  - Average satisfaction
  - Minimizing (maximum) regret
- Different notions of quality for matchings
  - Pareto optimal matchings
  - Popular matchings
  - etc.

Related Story: School Matching in Amsterdam

- (Stable) matching can also be applied to assign students to schools with a limited number of spots
  - In this setting, the "preference" of schools is often based on a priority ranking or a tie-breaking lottery

- The Amsterdam school board decided to change their mechanism in 2015, resulting in a lot of controversy and even a court case!

  This story features:

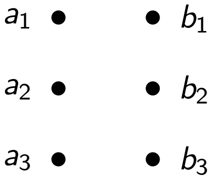
  - The Gale-Shapley (or Deferred Acceptance) algorithm
  - Strategizing
  - Fairness

- Read about it at `https://goo.gl/26915E`

Recap

- Stable Matching
- The Gale-Shapley Algorithm
- Strategizing
- Variants of Matching Problems

Homework exercise



- Let $A = \{a_1, a_2, a_3\}$ and let $B = \{b_1, b_2, b_3\}$.
- Find preferences for a_1, a_2, a_3 (linear orders over B) and preferences for b_1, b_2, b_3 (linear orders over A) such that there is only one stable matching.
- Perform the Gale-Shapley algorithm, both with A proposing and with B proposing.