Computational Social Choice and Complexity Theory

Ronald de Haan
University of Amsterdam
https://staff.science.uva.nl/r.dehaan/esslli2018
me@ronalddehaan.eu

ESSLLI 2018 – **Day 4**

Recap

- Voting theory
- (Parameterized) complexity theory
- Complexity of winner determination
- Strategic manipulation in voting
- Bribery
- (Nearly) single-peakedness

What we'll do today

- Judgment Aggregation (JA)
- JA Procedures
- Complexity issues
- Modeling Voting in JA

Judgment Aggregation

Example: Discursive Dilemma

- Consider a court case with a jury
- Suppose that they use majority voting

	1	2	3	maj.
k	yes	no	yes	yes
i	no	yes	yes	yes
$k \wedge i$	no	no	yes	no

- k: the defendant killed the victim
- i: it was intentional
- $k \wedge i$: it was murder (*intentional killing*)

Example: Budget Spending

- Consider a city council meeting on budget spending
- Suppose that they use majority voting

	1	2	3	maj.
p_1	no	yes	yes	yes
p_2	yes	no	yes	yes
p_3	yes	yes	no	yes

- p_1, p_2, p_3: projects to fund, each with a cost of 1000 leva
- A total budget of 2000 leva is available

  $(\Gamma = \neg p_1 \vee \neg p_2 \vee \neg p_3)$

Judgment Aggregation

- Agenda: set of propositional formulas and their negations, $\Phi = \{\varphi_1, \ldots, \varphi_n, \neg\varphi_1, \ldots, \neg\varphi_n\}$ – set of issues
  - Pre-agenda: $[\Phi] = \{\varphi_1, \ldots, \varphi_n\}$
- Integrity constraint: propositional formula Γ – logical context
- Judgment set: $J \subseteq \Phi$.
  - complete if for each $i \in [n]$, either $\varphi_i \in J$ or $\neg\varphi_i \in J$
  - consistent if J is logically consistent with Γ
  - feasible opinions: complete and consistent judgment sets
- Profile: sequence $\boldsymbol{J} = (J_1, \ldots, J_m)$ of complete and consistent judgment sets – individual opinions
- Judgment aggregation procedure: a function F that assigns to each profile $\boldsymbol{J}$ a set $F(\boldsymbol{J})$ of judgment sets – (possible) group opinions

The Majority JA Procedure

- Majority is the JA Procedure that outputs the judgment set J containing all issues that are supported by a majority of judgment sets in the profile $\boldsymbol{J}$
  - For an odd number of individuals, Majority($\boldsymbol{J}$) is a complete judgment set
  - But it is not always consistent

Example

- Pre-agenda: $[\Phi] = \{x_1, x_2, x_3\}$

  Agenda: $\Phi = \{x_1, x_2, x_3, \neg x_1, \neg x_2, \neg x_3\}$

- Integrity constraint:

  $$\Gamma = \neg(x_1 \wedge x_2 \wedge x_3)$$

- Profile: $\boldsymbol{J} = (J_1, J_2, J_3)$
  - $J_1 = \{\ x_1, x_2, \boldsymbol{\neg x_3}\ \}$
  - $J_2 = \{\ x_1, \boldsymbol{\neg x_2}, x_3\ \}$
  - $J_3 = \{\ \boldsymbol{\neg x_1}, x_2, x_3\ \}$

  - Majority($\boldsymbol{J}$) $= \{\ x_1, x_2, x_3\ \}$

- Majority($\boldsymbol{J}$) is not consistent

Example

- Pre-agenda: $[\Phi] = \{x_1, x_2, x_3, y_1, y_2, y_3, z\}$

  Agenda: $\Phi = \{x_1, x_2, x_3, y_1, y_2, y_3, z, \neg x_1, \neg x_2, \neg x_3, \neg y_1, \neg y_2, \neg y_3, \neg z\}$

- Integrity constraint:

  $$\Gamma = [\ (x_1 \wedge x_2 \wedge x_3) \to z\ ] \wedge [\ (y_1 \wedge y_2 \wedge y_3) \to z\ ]$$

- Profile: $\boldsymbol{J} = (J_1, J_2, J_3)$

  - $J_1 = \{\ x_1, x_2, \boldsymbol{\neg x_3},\ \ y_1, y_2, \boldsymbol{\neg y_3},\ \ \boldsymbol{\neg z}\ \}$
  - $J_2 = \{\ x_1, \boldsymbol{\neg x_2}, x_3,\ \ y_1, \boldsymbol{\neg y_2}, y_3,\ \ \boldsymbol{\neg z}\ \}$
  - $J_3 = \{\ \boldsymbol{\neg x_1}, x_2, x_3,\ \ \boldsymbol{\neg y_1}, y_2, y_3,\ \ \boldsymbol{\neg z}\ \}$

  - $\text{Majority}(\boldsymbol{J}) = \{\ x_1, x_2, x_3,\ \ y_1, y_2, y_3,\ \ \boldsymbol{\neg z}\ \}$

- Majority($\boldsymbol{J}$) is not consistent

General Model

- Very general framework, where you can easily model a wide range of possible 'elections' on interrelated issues
- E.g., pick your favorite combination of:
  - budget constraints
    ("each issue costs c_i, and the total budget is b")
  - dependencies
    ("you can only choose a if you also choose b or c")
  - rankings
    ("rank the issues a, b, c in order of importance")
  - etc.

The Premise-Based Procedure

- One approach to avoiding inconsistency is to use the Premise-Based Procedure (PBP)
- Split the agenda into premises and conclusions, such that:
  - the premises are logically independent of each other
  - any (complete) judgment set over the premises determines the truth value of the conclusions
- Then carry out the Majority rule on the premises, and take the entailed outcome for the conclusions

Example: PBP for the Discursive Dilemma

- Premises: k and i
- Conclusions: $k \wedge i$

	1	2	3	PBP
k	yes	no	yes	yes
i	no	yes	yes	yes
$k \wedge i$	no	no	yes	yes

Consistency

- A Judgment Aggregation procedure F is consistent if for each agenda Φ, each integrity constraint Γ and each profile $\boldsymbol{J}$, each $J \in F(\boldsymbol{J})$ is consistent with Γ

- The Premise-Based Procedure only work for agendas with a particular structure (i.e., agendas with 'premises' and 'conclusions')
  - For our example with the budget constraint, this is not the case

- Are there JA procedures that work for all agendas and that are consistent?

The Kemeny JA Procedure

- The Kemeny JA Procedure is defined as follows
- For a profile $\boldsymbol{J} = (J_1, \ldots, J_p)$ it returns those complete and consistent judgment sets J^* that minimize:

$$\sum_{1 \leq i \leq p} d_{\mathrm{H}}(J_i, J^*)$$

  where $d_{\mathrm{H}}(J, J')$ is the Hamming distance between two judgment sets:

$$d_{\mathrm{H}}(J, J') = \frac{|J \setminus J'| + |J' \setminus J|}{2}$$

  (the number of issues on which J and J' disagree)

- If the majority outcome is consistent, the Kemeny procedure outputs the majority outcome only

Example: Kemeny for Budget Spending

	1	2	3
p_1	no	yes	yes
p_2	yes	no	yes
p_3	yes	yes	no

$$\Gamma = \neg p_1 \vee \neg p_2 \vee \neg p_3$$

- Kemeny($\boldsymbol{J}$) $= \{\{p_1, p_2, \neg p_3\}, \{p_1, \neg p_2, p_3\}, \{\neg p_1, p_2, p_3\}\}$

Modeling Preference Aggregation in JA

- We can model preference aggregation (voting) in judgment aggregation

- Let A be a set of alternatives. Consider the agenda Φ_A and the integrity constraint Γ_A:

$$[\Phi_A] = \{\, p_{a,b} : a, b \in A, a \neq b \,\}$$

$$\Gamma_A = \bigwedge_{\substack{a,b \in A \\ a \neq b}} (p_{a,b} \leftrightarrow \neg p_{b,a}) \wedge \bigwedge_{\substack{a,b,c \in A \\ a \neq b, b \neq c, a \neq c}} ((p_{a,b} \wedge p_{b,c}) \to p_{a,c})$$

- The linear orders $\succ$ over A are in one-to-one correspondence with the complete and consistent judgment sets $J_\succ$ for Φ_A, Γ_A

- The Kendall-Tau distance between $\succ$ and $\succ'$ is exactly twice the Hamming distance between $J_\succ$ and $J_{\succ'}$

Modeling Preference Aggregation in JA

- For example, take $A = \{a, b, c\}$
- Then $[\Phi_A] = \{p_{a,b}, p_{a,c}, p_{b,c}\}$, and:

$$\begin{aligned}
\Gamma_A = \quad & (p_{a,b} \leftrightarrow \neg p_{b,a}) \wedge \\
& (p_{a,c} \leftrightarrow \neg p_{c,a}) \wedge \\
& (p_{b,c} \leftrightarrow \neg p_{c,b}) \wedge \\
& ((p_{a,b} \wedge p_{b,c}) \rightarrow p_{a,c}) \wedge \\
& ((p_{a,c} \wedge p_{c,b}) \rightarrow p_{a,b}) \wedge \\
& ((p_{b,a} \wedge p_{a,c}) \rightarrow p_{b,c}) \wedge \\
& ((p_{b,c} \wedge p_{c,a}) \rightarrow p_{b,a}) \wedge \\
& ((p_{c,a} \wedge p_{a,b}) \rightarrow p_{c,b}) \wedge \\
& ((p_{c,b} \wedge p_{b,a}) \rightarrow p_{c,a})
\end{aligned}$$

Modeling Preference Aggregation in JA

- The Condorcet paradox
- Take $A = \{a, b, c\}$ and consider Φ_A and Γ_A

P	1	2	3
#1	a	c	b
#2	b	a	c
#3	c	b	a

	1	2	3	maj.
$p_{a,b}$	yes	yes	no	yes
$p_{a,c}$	yes	no	no	no
$p_{b,a}$	no	no	yes	no
$p_{b,c}$	yes	no	yes	yes
$p_{c,a}$	no	yes	yes	yes
$p_{c,b}$	no	yes	no	no

Hardness for the Kemeny JA Procedure

JA-Outcome(F)

Input: an agenda Φ, an integrity constraint Γ a profile $\boldsymbol{J}$, and a partial judgment set J_0.

Output: Is there some $J^* \in F(\boldsymbol{J})$ that agrees with J_0?

Theorem

JA-Outcome(Kemeny) *is* Θ_2^{p}*-complete.*

- *Idea:* we can use the modeling of voting in judgment aggregation as a reduction from WinDet(Kemeny) to JA-Outcome(Kemeny)
  - with $J_0 = \{ p_{a^*,b} : b \in A, b \neq a^* \}$

A Poly-time Consistent JA Procedure

- Does there exist a polynomial-time computable judgment aggregation procedure that is consistent?
- Yes: the Plurality JA procedure
  - Selects as outcomes those judgment sets that appear most often in the profile
- *Note:* this rule can give arguably undesirable outcomes:

	1	2	3	4	5	6	7
p_1	no	no	yes	yes	yes	yes	no
p_2	no	no	yes	yes	yes	no	yes
p_3	no	no	yes	yes	no	yes	yes
p_4	no	no	yes	no	yes	yes	yes
p_5	no	no	no	yes	yes	yes	yes

The Axiomatic Method

- Just like with voting, one can use the axiomatic method to investigate the existence of JA procedurs with certain normatively appealing properties
- A JA procedure is:
  - anonymous if the outcome is always preserved under permuting the individuals in the profile
  - majority preserving if the procedure outputs the majority outcome whenever it is consistent
  - unanimous if whenever all judgment sets in the profile agree on some issue φ, then φ is in all outcomes
  - etc.

An Impossibility Result

Theorem (List, Pettit, 2002)

There is no JA procedure, for agendas $\Phi \supseteq \{p, q, p \wedge q\}$, that is resolute, anonymous, neutral, independent, complete and consistent.

- Neutral: all formulas are treated the same (i.e., if φ and φ' are in exactly the same judgment sets in the profile, then either both or none should be in the outcome)
- Independent: whether or not φ is in the outcome only depends on the pattern of individual acceptances of φ

C. List, and P. Pettit. Aggregating Sets of Judgments: An Impossibility Result. Economics and Philosophy, 18(1), 89–110, 2002.

The Axiomatic Method

- *Takeaway:* there is no unique best JA procedure
- Determine what axioms / normative properties are most desirable in the domain where you want to apply judgment aggregation, and try to find a JA procedure that satisfies these axioms

J. Lang, G. Pigozzi, M. Slavkovik, L. van der Torre, and S. Vesic. A partial taxonomy of judgment aggregation rules and their properties. Social Choice and Welfare, 48, 327–356, 2017.

Agenda Safety for the Majority Rule

- For some agendas, the Majority JA procedure will never lead to inconsistent outcomes
- E.g., $\Phi = \{p, q, r, \neg p, \neg q, \neg r\}$, $\Gamma = (p \vee q)$

Theorem (Nehring, Puppe, 2007)

The majority rule is consistent for an agenda Φ and an integrity constraint Γ if and only if Φ has the median property w.r.t. Γ.

- Median property: every inconsistent subset of Φ (w.r.t., Γ) does itself have an inconsistent subset of size at most 2

K. Nehring, and C. Puppe. The Structure of Strategy-Proof Social Choice. Part I: General Characterization and Possibility Results on Median Spaces. Journal of Economic Theory, 135(1), 269–305, 2007.

Agenda Safety for the Majority Rule

- Suppose some agenda Φ has the median property w.r.t. some integrity constraint Γ
- Why is the majority rule consistent for Φ, for an odd number of individuals?
  - Suppose the contrary, i.e., that there is some profile $\boldsymbol{J}$ such that Majority($\boldsymbol{J}$) is inconsistent with Γ
  - Take a minimally inconsistent subset J^* of Majority($\boldsymbol{J}$)
  - Then $|J^*| \leq 2$
  - Since J^* is part of Majority($\boldsymbol{J}$), each $\varphi \in J^*$ is supported by a strict majority
  - Then there must be some individual whose judgment set agrees with J^*
  - Contradiction: then J^* is consistent

Complexity of Agenda Safety

Agenda-Safety

Input: An agenda Φ, and an integrity constraint Γ.

Output: Does Φ have the median property (w.r.t. Γ)—that is, is every minimally inconsistent subset of Φ of size at most 2?

Theorem (Endriss, Grandi, Porello, 2012)

Agenda-Safety is Π_2^p-complete.

U. Endriss, U. Grandi, and D. Porello. Complexity of Judgment Aggregation. Journal of Artificial Intelligence Research (JAIR), 45, 481–514, 2012.

Complexity Issues in the Framework

- Suppose $[\Phi] = \{y, y \to \varphi\}$, where:

$$\begin{aligned}\varphi = \quad & (\neg x_5 \vee \neg x_1 \vee x_7) \wedge (x_4 \vee \neg x_2 \vee x_1) \wedge (\neg x_1 \vee x_6 \vee \neg x_7) \wedge (x_1 \vee \neg x_4 \vee \neg x_2) \wedge \\ & (x_6 \vee \neg x_1 \vee x_5) \wedge (x_3 \vee x_2 \vee x_1) \wedge (x_2 \vee x_1 \vee \neg x_3) \wedge (\neg x_6 \vee \neg x_7 \vee \neg x_1)\end{aligned}$$

- Suppose that an individual reports the judgment set $J = \{y, y \to \varphi\}$
- Is it easy to check whether this is a valid ballot?
  No: this boils down to solving the SAT problem (NP-hard)
- It is polynomial-time solvable to come up with **some** ballot but it could be hard to find a valid ballot that includes some given issue $\varphi \in \Phi$
- This is a computational hurdle for using JA
- Also: could open up a way to control the election

A Different Framework

(Binary Aggregation with Integrity Constraints – BAIC)

- Issues: a set $\mathcal{I} = \{x_1, \ldots, x_n\}$ of propositional variables
- Integrity constraint: propositional formula Γ over the variables $x_1, \ldots, x_n$ – logical context
- Ballot: $(b_1, \ldots, b_n) \in \{0, 1\}^n$
  - corresponds to a truth assignment α to the variables $x_1, \ldots, x_n$ – namely, $\alpha(x_i) = b_i$
  - consistent if $\alpha \models \Gamma$
  - feasible opinions: consistent ballots
- Profile: sequence $\boldsymbol{r} = (r_1, \ldots, r_m)$ of consistent ballots – individual opinions
- Judgment aggregation procedure: a function F that assigns to each profile $\boldsymbol{r}$ a set $F(\boldsymbol{r})$ of ballots – (possible) group opinions

Example

- Issues: $\mathcal{I} = \{x_1, x_2, x_3\}$
- Integrity constraint:

  $$\Gamma = \neg(x_1 \wedge x_2 \wedge x_3)$$

- Profile: $\boldsymbol{r} = (r_1, r_2, r_3)$
  - $r_1 = \{\ 1, 1, \mathbf{0}\ \}$
  - $r_2 = \{\ 1, \mathbf{0}, 1\ \}$
  - $r_3 = \{\ \mathbf{0}, 1, 1\ \}$
  - Majority($\boldsymbol{r}$) $= \{\ 1, 1, 1\ \}$

- Majority($\boldsymbol{r}$) is not consistent

Complexity Issues in the Different Framework

- Suppose:

$$\begin{aligned}\Gamma = \quad & (\neg x_5 \vee \neg x_1 \vee x_7) \wedge (x_4 \vee \neg x_2 \vee x_1) \wedge (\neg x_1 \vee x_6 \vee \neg x_7) \wedge (x_1 \vee \neg x_4 \vee \neg x_2) \wedge \\ & (x_6 \vee \neg x_1 \vee x_5) \wedge (x_3 \vee x_2 \vee x_1) \wedge (x_2 \vee x_1 \vee \neg x_3) \wedge (\neg x_6 \vee \neg x_7 \vee \neg x_1)\end{aligned}$$

- It is polynomial-time solvable to check if a ballot is valid
- Is it easy to come up with a valid ballot?

  No: this boils down to solving the SAT problem (NP-hard)

- This is an unreasonable computational burden to put on participants in the election
- Also: could open up a way to control the election

Using Logic Fragments (for the BAIC framework)

- We would like to use a logic fragment

  (a subset $\mathcal{C}$ of propositional formulas)

  for the integrity constraint Γ, with the property that:

  - Given a formula $\varphi \in \mathcal{C}$ and a partial truth assignment $\alpha : \mathsf{Var}(\varphi) \to \{0, 1\}$,

    it is polynomial-time solvable to find a truth assignment $\alpha' : \mathsf{Var}(\varphi) \to \{0, 1\}$ that extends α and that satisfies φ

    (if it exists)

- This way, the basic operations for participating in and administering the election are efficient

The 2CNF Fragment

- A 2CNF formula is a propositional formula φ in conjunctive normal form (CNF), where each clause has at most 2 literals
  - In other words, conjunctions of clauses of the following form:

$$(x_1 \vee x_2) \quad \text{or} \quad (\neg x_1 \vee x_2) \quad \text{or} \quad (x_1 \vee \neg x_2) \quad \text{or} \quad (\neg x_1 \vee \neg x_2)$$

- Given a 2CNF formula φ and a partial truth assignment $\alpha : \text{Var}(\varphi) \to \{0, 1\}$, it is polynomial-time solvable to find a truth assignment $\alpha' : \text{Var}(\varphi) \to \{0, 1\}$ that extends α and that satisfies φ (if it exists)

- If we take the second judgment aggregation framework (BAIC) and restrict the constraints to be 2CNF formulas, none of the complexity issues for basic operations arise

Kemeny for the 2CNF Fragment

Theorem

JA-Outcome(Kemeny) *for the BAIC framework is polynomial-time solvable, when the constraint is restricted to be a 2CNF formula.*

- *Idea:*
  - If $\Gamma \in$ 2CNF, then the agenda satisfies the median property
  - So for every profile $\boldsymbol{J}$, Majority($\boldsymbol{J}$) is consistent, and the Kemeny procedure selects the majority outcome

- **Downside:** the fragment of 2CNF has limited expressivity
  - E.g., the example of the discursive dilemma and the example of budget spending that we say cannot be expressed using a 2CNF constraint Γ

The Horn Fragment

- A Horn formula is a propositional formula φ in conjunctive normal form (CNF), where each clause has at most 1 positive literal
  - In other words, conjunctions of clauses of the following form:

$$(\neg x_1 \vee \cdots \vee \neg x_u) \quad \text{or} \quad (y \vee \neg x_1 \vee \cdots \vee \neg x_u)$$

- Given a Horn formula φ and a partial truth assignment $\alpha : \mathrm{Var}(\varphi) \to \{0,1\}$, it is polynomial-time solvable to find a truth assignment $\alpha' : \mathrm{Var}(\varphi) \to \{0,1\}$ that extends α and that satisfies φ (if it exists)

- If we take the second judgment aggregation framework (BAIC) and restrict the constraints to be Horn formulas, none of the complexity issues for basic operations arise

Kemeny for the Horn Fragment

Theorem (De Haan, 2018)

JA-Outcome(Kemeny) *for the BAIC framework is* Θ_2^p*-hard, even when the constraint is restricted to be a Horn formula.*

- So restricting the setting to avoid complexity issues in the basic operations of the election does not automatically buy us efficient elections

R. de Haan. Hunting for Tractable Languages for Judgment Aggregation. Proceedings of the 16th International Conference on Principles of Knowledge Representation and Reasoning (KR), 2018.

Further Fragments

- There are fragments that strike a certain balance between
  (1) expressivity and compactness, and
  (2) efficiency of computing outcomes for various judgment aggregation procedures.
  - Boolean circuits in Decomposable Negation Normal Form (DNNF)
  - These offer an efficient encoding of budget constraints, for example

Theorem (De Haan, 2018)

JA-Outcome(Kemeny) *for the BAIC framework is polynomial-time solvable, even when the constraint is restricted to be a DNNF Circuit.*

Relation Between the Frameworks

- For every agenda Φ and constraint Γ in the original judgment aggregation framework one can find an equivalent set $\mathcal{I}$ of issues and constraint Γ' in the BAIC framework,

  and vice versa

- Equivalent translations might be hard to find and be of exponential size

U. Endriss, U. Grandi, R. de Haan, and J. Lang, Succinctness of Languages for Judgment Aggregation. In Proceedings of the 15th International Conference on Principles of Knowledge Representation and Reasoning (KR), 2016.

Rationality and Feasibility Constraints

- Another variant on the judgment aggregation framework:

  have two integrity constraints Γ_1, Γ_2

  - Γ_1 used as constraint for the individual opinions
  - Γ_2 used as constraint for the collective opinion

U. Endriss. Judgment Aggregation with Rationality and Feasibility Constraints. In Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), 2018.

Modeling Borda Voting

- With two integrity constraints, we can model Borda voting:
  - Constraint Γ_1 for the individual opinions:

$$\Gamma_1 = \bigwedge_{\substack{a,b \in A \\ a \neq b}} (p_{a,b} \leftrightarrow \neg p_{b,a}) \wedge \bigwedge_{\substack{a,b,c \in A \\ a \neq b, b \neq c, a \neq c}} ((p_{a,b} \wedge p_{b,c}) \rightarrow p_{a,c})$$

  - Constraint Γ_2 for the collective opinion:

$$\Gamma_2 = \bigvee_{a \in A} \left(\bigwedge_{\substack{b \in A \\ a \neq b}} (p_{a,b} \wedge \neg p_{b,a}) \wedge \bigwedge_{\substack{b,c \in A \\ a \neq b, a \neq c, b \neq c}} (p_{b,c} \wedge p_{c,b}) \right)$$

Theorem (Endriss, 2018)

When encoding a preference profile P using Γ_1, the outcomes of the Kemeny judgment procedure w.r.t. Γ_2 correspond exactly to the Borda winners of P

Recap

- Judgment Aggregation (JA)
- Insufficiency of the majority procedure
- Other JA Procedures
- Complexity issues
- Different JA frameworks
- Modeling Voting in JA