

Computational Complexity

Lecture 13: Recap and bonus

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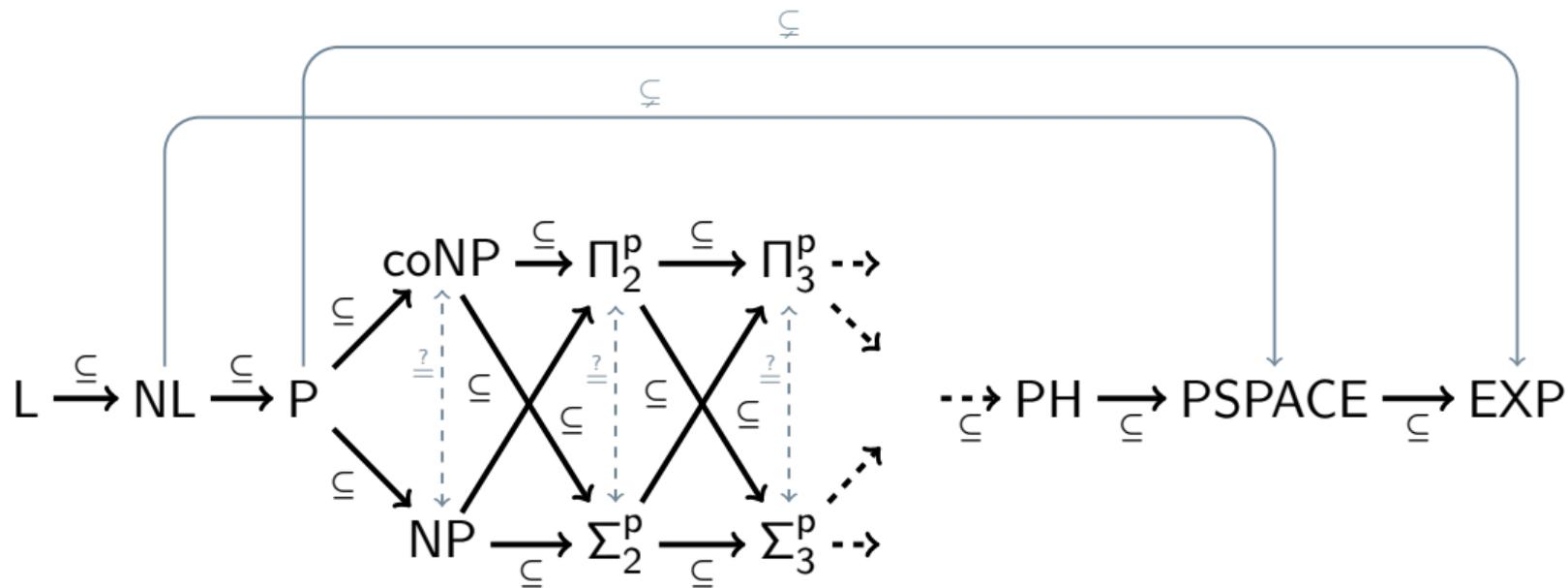
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- A bird's eye overview of what we covered
- (Possible bonus: quick intro into *parameterized complexity theory*)

An overview of complexity classes



The Cook-Levin Theorem

Theorem (Cook 1971, Levin 1969)

3SAT *is* NP-complete.

The Time Hierarchy Theorems

Theorem

If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are time-constructible functions such that $f(n) \log f(n)$ is $o(g(n))$, then $\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n))$.

Theorem

If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are time-constructible functions such that $f(n+1)$ is $o(g(n))$, then $\text{NTIME}(f(n)) \subsetneq \text{NTIME}(g(n))$.

Theorem

If $S : \mathbb{N} \rightarrow \mathbb{N}$ is a space-constructible function, then:

$$\text{DTIME}(S(n)) \subseteq \text{SPACE}(S(n)) \subseteq \text{NSPACE}(S(n)) \subseteq \text{DTIME}(2^{O(S(n))}).$$

Theorem

If $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are space-constructible functions such that $f(n)$ is $o(g(n))$, then:

$$\text{SPACE}(f(n)) \subsetneq \text{SPACE}(g(n)) \quad \text{and} \quad \text{NSPACE}(f(n)) \subsetneq \text{NSPACE}(g(n)).$$

Theorem (Baker, Gill, Solovay 1975)

There exist $A, B \subseteq \{0, 1\}^$ such that $P^A = NP^A$ and $P^B \neq NP^B$.*

Theorem

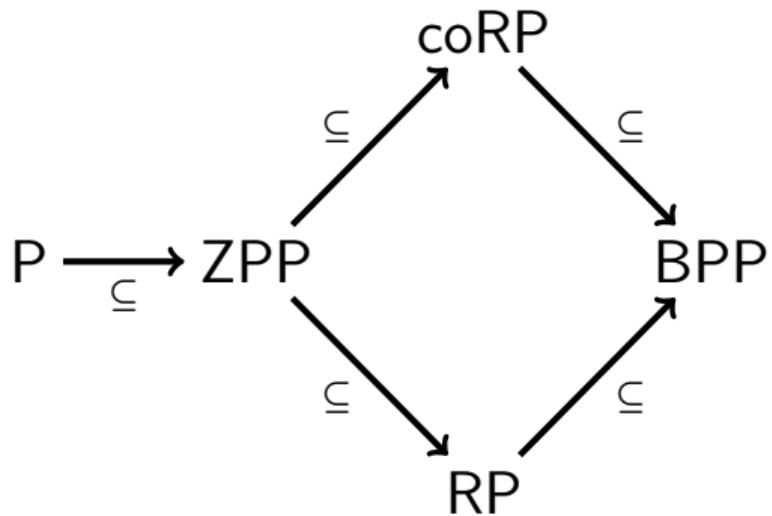
TQBF is PSPACE-complete.

Theorem

Let $i \geq 1$. Then $\Sigma_i\text{SAT}$ is Σ_i^P -complete and $\Pi_i\text{SAT}$ is Π_i^P -complete.

Theorem (Karp, Lipton 1980)

If $\text{NP} \subseteq \text{P/poly}$, then $\Sigma_2^P = \Pi_2^P$.



Theorem (PCP)

$\text{NP} = \text{PCP}(\log n, 1)$.

There exists some $\rho < 1$ such that for all $L \in \text{NP}$ there is a polynomial-time reduction R from L to 3SAT where for all $x \in \{0, 1\}^$:*

- *if $x \in L$ then $\text{val}(R(x)) = 1$;*
- *if $x \notin L$ then $\text{val}(R(x)) < \rho$.*

Definition

Let δ_3 be the infimum of the set of constants c for which there exists an algorithm solving 3SAT in time $O(2^{cn}) \cdot m^{O(1)}$, where n is the number of variables in the q -SAT input and m the number of clauses.

The *Exponential-Time Hypothesis (ETH)* states that $\delta_3 > 0$.

Theorem

The ETH implies that there is no $2^{o(n)}$ -time algorithm for 3SAT and that there is no $2^{o(n+m)}$ -time algorithm for 3SAT.

Definition (distP)

$\langle L, \mathcal{D} \rangle$ is in the class distP (also called: avgP) if there exists a deterministic TM M that decides L and a constant $\epsilon > 0$ such that for all $n \in \mathbb{N}$:

$$\mathbb{E}_{x \in_R \mathcal{D}_n} [\text{time}_M(x)^\epsilon] \text{ is } O(n).$$

- VC: given a graph G and $u \in \mathbb{N}$, does G have a vertex cover of size u ?
- This problem is NP-complete, and the best algorithms that we have take exponential time in the worst case.
- This worst-case analysis takes into account **every** possible input.

- Can we take into account additional knowledge about the input that we might have to get more positive worst-case guarantees?

Parameterized complexity: with VC as example (ct'd)

- Suppose that we are dealing with an application where the value of u is always much smaller than the size of the graph G .
- Can we restrict the exponential factor in the running time to just u ?
- Answer: **yes!**

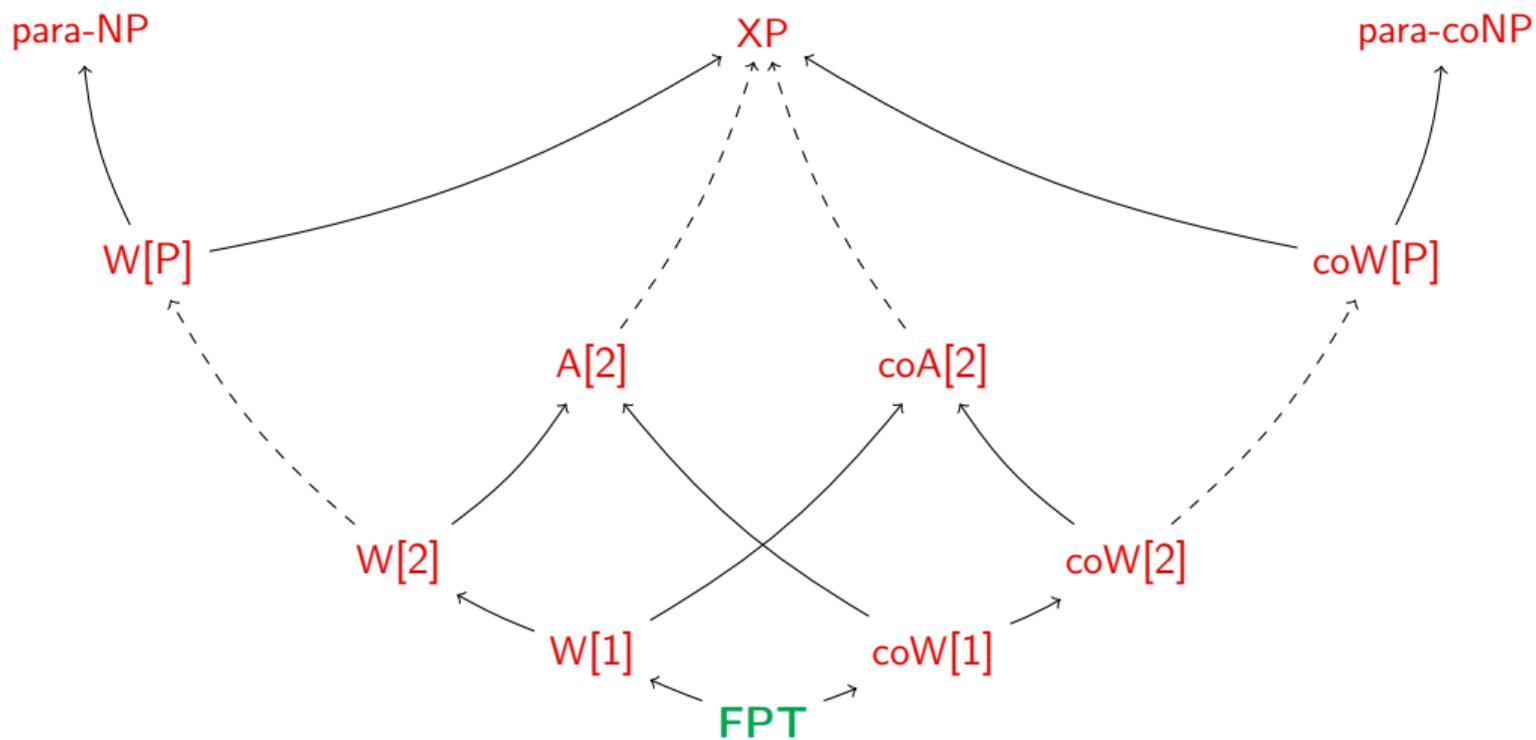
Definition

A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$ of pairs (x, k) , where x is called the *main input* and k is called the *parameter*.

Definition (FPT)

A parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is *fixed-parameter tractable* if there exist a polynomial p , a computable function f , and a deterministic TM \mathbb{M} that, when given input (x, k) , decides if $(x, k) \in L$ and runs in time $f(k) \cdot p(|x|)$.

Parameterized complexity landscape



- VC: NP-complete, and no $2^{o(v)}$ -time algorithm (assuming ETH)
- With u as parameter? Fixed-parameter tractable
- With $v - u$ as parameter? W[1]-complete
- With the degree d of the graph as parameter? para-NP-complete
- With the treewidth t of the graph as parameter? Fixed-parameter tractable
- Etc.