Computational Complexity

Exercise Session 2

Exercise 1. Show that $\mathsf{coNP} \subseteq \mathsf{EXP}$.

Exercise 2. Consider the following problem Reverse-3SAT:

> **Instance:** A propositional formula φ in 3CNF—that is, a formula of the form $\varphi = c_1 \wedge \cdots \wedge c_m$, where each c_j is of the form $c_j = \ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3}$, where $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ are propositional literals.
>
> **Question:** Is there a truth assignment α to the variables occurring in φ that sets at least one literal in each clause c_j to **false**?

Prove that Reverse-3SAT is NP-complete—that is, prove that is in NP and that it is NP-hard. To show NP-hardness, you may give a reduction from any known NP-complete problem.

- *Hint:* reduce from 3SAT.

Exercise 3. Consider the following problem CLIQUE:

> **Instance:** An undirected graph $G = (V, E)$, and a positive integer $k \in \mathbb{N}$.
>
> **Question:** Does G contain a clique of size k—that is, is there a set $C \subseteq V$ of vertices with $|C| = k$ such that for each $v, v' \in C$ with $v \neq v'$ it holds that $\{v, v'\} \in E$?

In this exercise, we will show that CLIQUE is NP-complete.

(i) Prove that CLIQUE is in NP.

To show that CLIQUE is NP-hard, we will give a polynomial-time reduction f from 3SAT to CLIQUE. We describe this reduction f as follows: for an arbitrary instance φ of 3SAT, we describe what the instance $f(\varphi) = (G, k)$ looks like.

Let $\varphi = c_1 \wedge \cdots \wedge c_m$ be an arbitrary 3CNF formula, containing propositional variables $x_1, \ldots, x_n$, where $c_j = \ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3}$ for each $1 \leq j \leq m$. Then we construct the graph $f(\varphi)$ as follows.

- We introduce vertices $v_{j,1}, v_{j,2}, v_{j,3}$, for each $1 \leq j \leq m$. That is, for each clause c_j we add three vertices—one for each literal occurring in the clause.
- Two vertices $v_{j,l}$ and $v_{j',l'}$ are connected with an edge if and only if $j \neq j'$ and the literals $\ell_{j,l}$ and $\ell_{j',l'}$ are not each other's negation.

Finally, we set $k = m$.

Let $\varphi_{\text{ex}} = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_1})$ be an example 3CNF formula.

(ii) Let $f(\varphi_{\text{ex}}) = (G_{\text{ex}}, k_{\text{ex}})$. Compute k_{ex} and draw the graph G_{ex}.

(iii) Show that φ_{ex} is satisfiable. Use a satisfying assignment for φ_{ex} to produce a clique of size k_{ex} for G_{ex}.

(iv) Prove, for an arbitrary 3CNF formula φ, that φ is satisfiable if and only if $f(\varphi) = (G, k) \in$ CLIQUE.

(v) Explain why the function f is polynomial-time computable.

Exercise 4 (reduction from HamCycle to HamPath)**.** Consider the following problem HamPath:

> **Instance:** An undirected graph $G = (V, E)$, and two vertices $s, t \in V$ such that $s \neq t$.
>
> **Question:** Is there a Hamiltonian path in G from s to t—in other words, a path from s to t that visits each vertex exactly once?

Consider also the following problem HamCycle:

> **Instance:** An undirected graph $G = (V, E)$.
>
> **Question:** Is there a Hamiltonian cycle in G—in other words, a cycle that visits each vertex exactly once?

Give a polynomial-time reduction from HamCycle to HamPath.

Exercise 5 (self-reducibility of 3SAT)**.** Suppose that you have a polynomial-time algorithm A for (the decision problem) 3SAT. Show that you can use A to construct a polynomial-time algorithm B that, when given as input a 3CNF formula φ, outputs a satisfying assignment α for φ if such an assignment exists, and that outputs 0 otherwise.