Computational Complexity

Lecture 8: Some Sort of Recap

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Recap

What we saw last time..

- Non-uniform complexity
- Circuit complexity
- TMs that take advice
- The Karp-Lipton Theorem: if $\mathsf{NP} \subseteq \mathsf{P}/\mathrm{poly}$, then $\Sigma_2^p = \Pi_2^p$

What will we do today?

- Reflecting on what we've seen before
- Mostly using examples

One-/two-liner overview of complexity classes

- L: deterministic algorithm, logarithmic space (and polynomial time)
- NL: nondeterministic algorithm, logarithmic space (and polynomial time)
- P: solvable in (deterministic) polynomial time
- NP: solutions (*for yes-answers*) can be guessed/checked in polynomial time
- coNP: solutions (*for no-answers*) can be guessed/checked in polynomial time
- Σ_2^p: solutions (*for yes-answers*) have "$\exists\forall$ structure"
- Π_2^p: solutions (*for yes-answers*) have "$\forall\exists$ structure"
- PSPACE: (non)deterministic algorithm, polynomial space (and exponential time)
  OR: unbounded "$\exists\forall\exists\forall\exists\cdots$ structure"
- EXP: solvable in (deterministic) exponential time

Some oracle questions..

- Is it the case that $\mathsf{P}^{\mathsf{P}} = \mathsf{P}$? Yes
- Is it the case that $\mathsf{NP}^{\mathsf{NP}} = \mathsf{NP}$? We don't know..
- Is it the case that $\mathsf{PSPACE}^{\mathsf{PSPACE}} = \mathsf{PSPACE}$? Yes
- Is it the case that $\mathsf{EXP}^{\mathsf{EXP}} = \mathsf{EXP}$? No
- Is it the case that $\mathsf{DTIME}(n^2)^{\mathsf{DTIME}(n^2)} = \mathsf{DTIME}(n^2)$? No

P vs. NP

- Polls on $\mathsf{P} \stackrel{?}{=} \mathsf{NP}$ have been held among computational complexity researchers:
  - In 2002, see: `https://tiny.cc/pnp-poll1`
  - In 2012, see: `https://tiny.cc/pnp-poll2`
  - In 2019, see: `https://tiny.cc/pnp-poll3`
- In these papers, there are some very interesting opinions on the question (and some nerdy jokes)
- Short answer: we have no clue (really), why $\mathsf{P} = \mathsf{NP}$ or $\mathsf{P} \neq \mathsf{NP}$ would be true, but most think that $\mathsf{P} \neq \mathsf{NP}$.

Quiz example #1: checking if a given solution is unique

- What is the complexity of this problem?

- *Input:* A propositional formula φ, and a satisfying truth assignment α for φ.

  Question: Is α the only satisfying assignment for φ?

- This problem is coNP-complete
  - The answer is yes if and only if $\varphi \wedge$ "$\neg\alpha$" is unsatisfiable

Quiz example #2: finding a minimal equivalent DNF formula

- What is the complexity of this problem?

- *Input:* A propositional formula φ, and 1^k for some $k \in \mathbb{N}$.

  Question: Is there a DNF formula ψ of size $\leq k$ such that $\varphi \equiv \psi$?

- This problem is Σ_2^p-complete
  - "$\exists$ part": guess a DNF formula ψ of size $\leq k$
  - "$\forall$ part": check that $\varphi \equiv \psi$

Quiz example #3: equivalence of propositional logic formulas

- What is the complexity of this problem?

- *Input:* Two propositional formulas φ_1, φ_2.

  Question: $\varphi_1 \equiv \varphi_2$?

- This problem is coNP-complete

  - φ is unsatisfiable if and only if $\varphi \equiv (x \wedge \neg x)$

Quiz example #4: 2SAT

- What is the complexity of this problem?
- *Input:* A propositional 2CNF formula φ.

  Question: Is φ satisfiable?

- This problem is NL-complete
  - Reduce to a variant of graph reachability
  - φ is unsatisfiable if and only if there is a path from some x to $\neg x$ to x in the implication graph of φ

Quiz example #5: satisfiability of modal logic K

- What is the complexity of this problem?

- *Input:* A basic modal logic formula φ.

  Question: Is φ satisfiable?

- This problem is PSPACE-complete
  - The tableau algorithm runs in polynomial space (or in alternating polynomial time)
  - TQBF can be reduced to this problem

Quiz example #6: satisfiability of modal logic S5

- What is the complexity of this problem?

- *Input:* A modal logic formula φ.

  Question: Is there an S5 Kripke model where φ is true?

- This problem is NP-complete

  - **Theorem:** if there is an S5 Kripke model where φ is true, then there exists an S5 Kripke model with at most $|\varphi|$ states where φ is true.

Quiz example #7: Tiling I

- What is the complexity of this problem?

- *Input:* A set of 4-sided tile types, and 1^n and 1^m for $n, m \in \mathbb{N}$.

  Question: Can we use these tile types to fill an $n \times m$ grid, so that
  (1) the outsides of the grid all have side s_0, and
  (2) neighboring tiles have matching sides?

- This problem is NP-complete

Quiz example #8: Tiling II

- What is the complexity of this problem?

- *Input:* A set of 4-sided tile types, and 1^n for $n \in \mathbb{N}$.

  Question: Can we use these tile types to fill an $n \times m$ grid, for some $m \in \mathbb{N}$, so that

  (1) the outsides of the grid all have side s_0, and
  (2) neighboring tiles have matching sides?

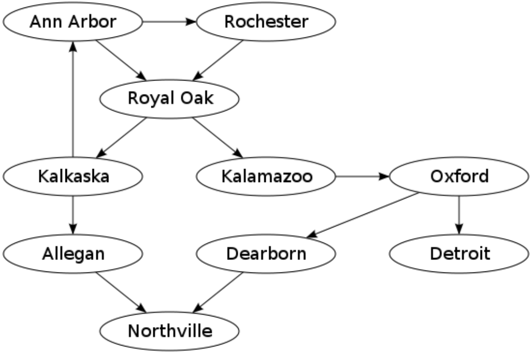
- This problem is PSPACE-complete

Quiz example #9: Generalized Geography

- What is the complexity of this problem? (See: `https://en.wikipedia.org/wiki/Generalized_geography`)

- *Input:* An instance I of *generalized geography*.

  Question: Does Player 1 have a winning strategy?



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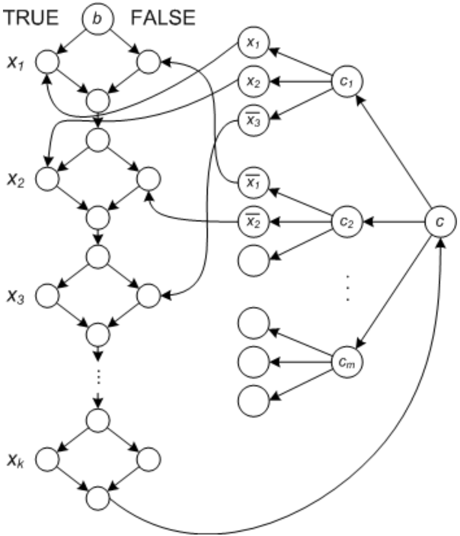

Quiz example #10: reachability in succinctly represented graphs

- What is the complexity of this problem?

- *Input:* A propositional logic formula $\varphi(x_1, \dots, x_n, x'_1, \dots, x'_n)$, and two binary vectors $s, t \in \{0,1\}^n$.

  Question: Consider the directed graph $G = (V, E)$, where: $V = \{0,1\}^n$, and for each $\overline{v}, \overline{w} \in V$, $(\overline{v}, \overline{w}) \in E$ if and only if $\varphi[\overline{u}, \overline{w}]$ is true.

  Is t reachable from s in G?

- This problem is PSPACE-complete

Quiz example #11: 3-colorability for succinctly represented graphs

- What is the complexity of this problem?

- *Input:* A propositional logic formula $\varphi(x_1, \dots, x_n, x'_1, \dots, x'_n)$.

  Question: Consider the undirected graph $G = (V, E)$, where:
  $V = \{0, 1\}^n$, and for each $\overline{v}, \overline{w} \in V$,
  $\{\overline{v}, \overline{w}\} \in E$ if and only if $\varphi[\overline{v}, \overline{w}]$ is true.

  Is the graph G 3-colorable?

- This problem is NEXP-complete

Next time

- Probabilistic algorithms
- Complexity classes BPP, RP, coRP, ZPP