Computational Complexity

Lecture 8: Some Sort of Recap

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Recap

*What we saw last time..*

- Non-uniform complexity
- Circuit complexity
- TMs that take advice
- The Karp-Lipton Theorem: if $\text{NP} \subseteq \text{P/poly}$, then $\Sigma^p_2 = \Pi^p_2$
What will we do today?

- Reflecting on what we’ve seen before
- Mostly using examples
One-/two-liner overview of complexity classes

- **L**: deterministic algorithm, logarithmic space (and polynomial time)
- **NL**: nondeterministic algorithm, logarithmic space (and polynomial time)
- **P**: solvable in (deterministic) polynomial time
- **NP**: solutions (for yes-answers) can be guessed/checked in polynomial time
- **coNP**: solutions (for no-answers) can be guessed/checked in polynomial time
- **Σ²_p**: solutions (for yes-answers) have “∃∀ structure”
- **Π²_p**: solutions (for yes-answers) have “∀∃ structure”
- **PSPACE**: (non)deterministic algorithm, polynomial space (and exponential time)  
  OR: unbounded “∃∀∃∀∃⋯ structure”
- **EXP**: solvable in (deterministic) exponential time
Some oracle questions..

- Is it the case that $\mathsf{P}^\mathsf{P} = \mathsf{P}$? Yes
- Is it the case that $\mathsf{NP}^\mathsf{NP} = \mathsf{NP}$? We don’t know.
- Is it the case that $\mathsf{PSPACE}^{\mathsf{PSPACE}} = \mathsf{PSPACE}$? Yes
- Is it the case that $\mathsf{EXP}^{\mathsf{EXP}} = \mathsf{EXP}$? No
- Is it the case that $\mathsf{DTIME}(n^2)^{\mathsf{DTIME}(n^2)} = \mathsf{DTIME}(n^2)$? No
Polls on $P \overset{?}{=} NP$ have been held among computational complexity researchers:
  
  - In 2002, see: https://tiny.cc/pnp-poll1
  - In 2012, see: https://tiny.cc/pnp-poll2
  - In 2019, see: https://tiny.cc/pnp-poll3

In these papers, there are some very interesting opinions on the question (and some nerdy jokes)

Short answer: we have no clue (really), why $P = NP$ or $P \neq NP$ would be true, but most think that $P \neq NP$. 
Quiz example #1: checking if a given solution is unique

- What is the complexity of this problem?

- **Input:** A propositional formula \( \varphi \), and a satisfying truth assignment \( \alpha \) for \( \varphi \).
  
  **Question:** Is \( \alpha \) the only satisfying assignment for \( \varphi \)?

- This problem is **coNP-complete**
  
  - The answer is yes if and only if \( \varphi \land \neg \alpha \) is unsatisfiable
Quiz example #2: finding a minimal equivalent DNF formula

What is the complexity of this problem?

Input: A propositional formula $\varphi$, and $1^k$ for some $k \in \mathbb{N}$.

Question: Is there a DNF formula $\psi$ of size $\leq k$ such that $\varphi \equiv \psi$?

This problem is $\Sigma^p_2$-complete

- "$\exists$ part": guess a DNF formula $\psi$ of size $\leq k$
- "$\forall$ part": check that $\varphi \equiv \psi$
What is the complexity of this problem?

Input: Two propositional formulas $\varphi_1, \varphi_2$.

Question: $\varphi_1 \equiv \varphi_2$?

This problem is coNP-complete

$\varphi$ is unsatisfiable if and only if $\varphi \equiv (x \land \neg x)$
What is the complexity of this problem?

**Input:** A propositional 2CNF formula \( \varphi \).

**Question:** Is \( \varphi \) satisfiable?

This problem is **NL-complete**

- Reduce to a variant of graph reachability

- \( \varphi \) is unsatisfiable if and only if there is a path from some \( x \) to \( \neg x \) to \( x \) in the implication graph of \( \varphi \)
Quiz example #5: satisfiability of modal logic K

What is the complexity of this problem?

*Input:* A basic modal logic formula $\varphi$.

*Question:* Is $\varphi$ satisfiable?

This problem is **PSPACE-complete**

- The tableau algorithm runs in polynomial space (or in alternating polynomial time)
- TQBF can be reduced to this problem
Quiz example #6: satisfiability of modal logic S5

What is the complexity of this problem?

Input: A modal logic formula $\varphi$.

Question: Is there an S5 Kripke model where $\varphi$ is true?

This problem is NP-complete

Theorem: If there is an S5 Kripke model where $\varphi$ is true, then there exists an S5 Kripke model with at most $|\varphi|$ states where $\varphi$ is true.
What is the complexity of this problem?

**Input:** A set of 4-sided tile types, and $1^n$ and $1^m$ for $n, m \in \mathbb{N}$.

**Question:** Can we use these tile types to fill an $n \times m$ grid, so that (1) the outsides of the grid all have side $s_0$, and (2) neighboring tiles have matching sides?

This problem is **NP-complete**
What is the complexity of this problem?

**Input:** A set of 4-sided tile types, and $1^n$ for $n \in \mathbb{N}$.

**Question:** Can we use these tile types to fill an $n \times m$ grid, for some $m \in \mathbb{N}$, so that

1. the outsides of the grid all have side $s_0$, and
2. neighboring tiles have matching sides?

This problem is **PSPACE-complete**
Quiz example #9: Generalized Geography

- What is the complexity of this problem? (See: https://en.wikipedia.org/wiki/Generalized_geography)

- **Input:** An instance $I$ of *generalized geography*.

- **Question:** Does Player 1 have a winning strategy?
What is the complexity of this problem? (See: https://en.wikipedia.org/wiki/Generalized_geography)

**Input:** An instance $I$ of generalized geography.

**Question:** Does Player 1 have a winning strategy?

This problem is PSPACE-complete.
What is the complexity of this problem?

**Input:** A propositional logic formula $\varphi(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$, and two binary vectors $s, t \in \{0, 1\}^n$.

**Question:** Consider the directed graph $G = (V, E)$, where:
- $V = \{0, 1\}^n$, and for each $\overline{v}, \overline{w} \in V$,
- $(\overline{v}, \overline{w}) \in E$ if and only if $\varphi[\overline{u}, \overline{w}]$ is true.

Is $t$ reachable from $s$ in $G$?

This problem is **PSPACE-complete**
What is the complexity of this problem?

**Input:** A propositional logic formula $\varphi(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$.

**Question:** Consider the undirected graph $G = (V, E)$, where:

- $V = \{0, 1\}^n$, and for each $\bar{v}, \bar{w} \in V$,
- $\{\bar{v}, \bar{w}\} \in E$ if and only if $\varphi[\bar{v}, \bar{w}]$ is true.

Is the graph $G$ 3-colorable?

This problem is **NEXP-complete**
Next time

- Probabilistic algorithms
- Complexity classes BPP, RP, coRP, ZPP