Computational Complexity

Exercise Session 5

Note: These exercises are (likely) too much work to solve all during the exercise session.

Exercise 1. A decision problem $L \subseteq \{0,1\}^*$ is *sparse* if there exists a polynomial p such that for every $n \in \mathbb{N}$ it holds that $|L \cap \{0,1\}^n| \leq p(n)$. Show that every sparse decision problem is in $\mathsf{P/poly}$.

Definition 1. $\mathsf{P}^{\mathsf{NP}[\log]}$ is the class of all decision problems $L \subseteq \{0,1\}^*$ for which there exists a polynomial-time deterministic oracle TM $\mathbb{M}$ and an oracle language $O \in \mathsf{NP}$ such that $\mathbb{M}^O$ decides L, and a function $f(n) : \mathbb{N} \to \mathbb{N}$ that is $O(\log n)$ such that for each input $x \in \{0,1\}^*$, $\mathbb{M}^O(x)$ makes at most $f(|x|)$ queries to the oracle O.

Exercise 2. Show that the following problem is in $\mathsf{P}^{\mathsf{NP}[\log]}$:

$$\{\ \varphi \mid \ \varphi \text{ is a propositional logic formula, and the maximum number } m \text{ of variables among } \mathrm{var}(\varphi) \text{ that are set to true in any satisfying truth assignment of } \varphi \text{ is odd.}\ \}$$

Exercise 3. Prove that $\mathsf{RP} \subseteq \mathsf{BPP}$ and that $\mathsf{coRP} \subseteq \mathsf{BPP}$.

Exercise 4. Prove that $\mathsf{BPP} \subseteq \mathsf{PSPACE}$.

Exercise 5. CLIQUE is the problem of deciding, given a graph $G = (V, E)$ and a natural number $k \in \mathbb{N}$, whether there exists a set $C \subseteq V$ such that $|C| = k$ and for all $c_1, c_2 \in C$ with $c_1 \neq c_2$ it holds that $\{c_1, c_2\} \in E$.

For every $\rho < 1$, an algorithm A is called a ρ-approximation algorithm for $\mathsf{MAX\text{-}CLIQUE}$ if for every graph $G = (V, E)$, the algorithms outputs a clique $C \subseteq V$ of G of size at least $\rho \cdot \mu_G$, where μ_G is the maximum size of any clique of G.

Show that for each $\rho < 1$, if there exists a polynomial-time ρ-approximation algorithm for $\mathsf{MAX\text{-}CLIQUE}$, then $\mathsf{P} = \mathsf{NP}$.