# Computational Complexity

## Lecture 14: Recap and bonus

Ronald de Haan
me@ronalddehaan.eu

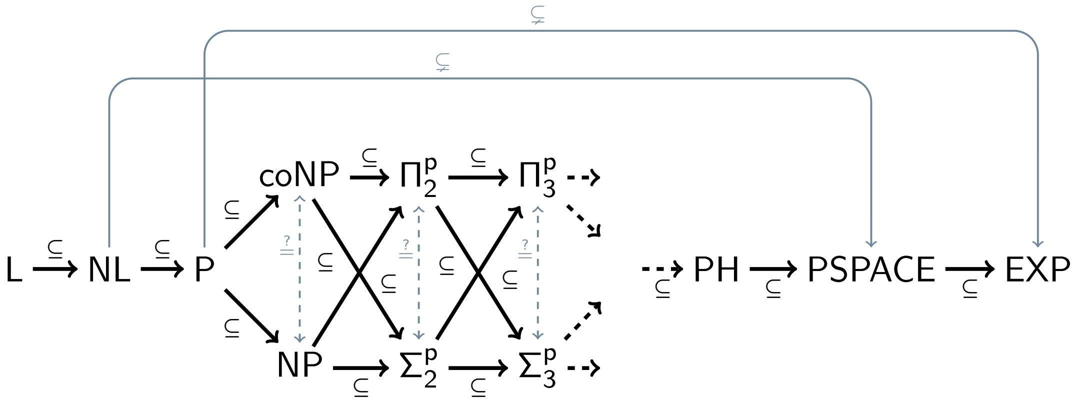
University of Amsterdam

May 26, 2023

# Today

- A bird's eye overview of what we covered
- (Possible bonus: quick intro into *parameterized complexity theory*)

# An overview of complexity classes



# The Cook-Levin Theorem

**Theorem (Cook 1971, Levin 1969)**

3SAT *is* NP*-complete.*

# The Time Hierarchy Theorems

**Theorem**

*If* $f, g : \mathbb{N} \to \mathbb{N}$ *are time-constructible functions such that* $f(n) \log f(n)$ *is* $o(g(n))$*, then* $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$*.*

**Theorem**

*If* $f, g : \mathbb{N} \to \mathbb{N}$ *are time-constructible functions such that* $f(n+1)$ *is* $o(g(n))$*, then* $\mathrm{NTIME}(f(n)) \subsetneq \mathrm{NTIME}(g(n))$*.*

# Space and time

**Theorem**

*If* $S : \mathbb{N} \to \mathbb{N}$ *is a space-constructible function, then:*

$$\mathsf{DTIME}(S(n)) \subseteq \mathsf{SPACE}(S(n)) \subseteq \mathsf{NSPACE}(S(n)) \subseteq \mathsf{DTIME}(2^{O(S(n))}).$$

**Theorem**

*If* $f, g : \mathbb{N} \to \mathbb{N}$ *are space-constructible functions such that* $f(n)$ *is* $o(g(n))$*, then:*

$$\mathsf{SPACE}(f(n)) \subsetneq \mathsf{SPACE}(g(n)) \quad \textit{and} \quad \mathsf{NSPACE}(f(n)) \subsetneq \mathsf{NSPACE}(g(n)).$$

# Oracles and relativizing proofs

**Theorem (Baker, Gill, Solovay 1975)**

*There exist* $A, B \subseteq \{0,1\}^*$ *such that* $\mathsf{P}^A = \mathsf{NP}^A$ *and* $\mathsf{P}^B \neq \mathsf{NP}^B$.

# Quantified Boolean formulas (QBFs)

**Theorem**

TQBF *is* PSPACE*-complete.*

**Theorem**

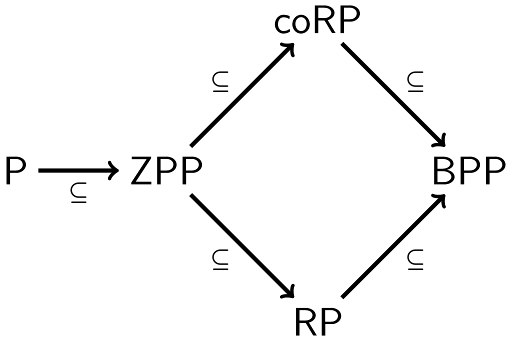
*Let* $i \geq 1$*. Then* $\Sigma_i\mathsf{SAT}$ *is* $\Sigma_i^p$*-complete and* $\Pi_i\mathsf{SAT}$ *is* $\Pi_i^p$*-complete.*

# Circuits and advice

**Theorem (Karp, Lipton 1980)**

*If* NP $\subseteq$ P/poly*, then* $\Sigma_2^p = \Pi_2^p$.

# Probabilistic computation



# PCP and approximation

**Theorem (PCP)**

$\mathrm{NP} = \mathrm{PCP}(\log n, 1)$.

---

*There exists some $\rho < 1$ such that for all $L \in \mathrm{NP}$ there is a polynomial-time reduction $R$ from $L$ to $\mathrm{3SAT}$ where for all $x \in \{0,1\}^*$:*

- *if $x \in L$ then $val(R(x)) = 1$;*
- *if $x \notin L$ then $val(R(x)) < \rho$.*

# ETH

## Definition

Let $\delta_3$ be the infimum of the set of constants $c$ for which there exists an algorithm solving 3SAT in time $O(2^{cn}) \cdot m^{O(1)}$, where $n$ is the number of variables in the $q$-SAT input and $m$ the number of clauses.

The *Exponential-Time Hypothesis (ETH)* states that $\delta_3 > 0$.

## Theorem

*The ETH implies that there is no $2^{o(n)}$-time algorithm for* 3SAT *and that there is no $2^{o(n+m)}$-time algorithm for* 3SAT.

**Definition (distP)**

$\langle L, \mathcal{D} \rangle$ is in the class distP (also called: avgP) if there exists a deterministic TM $\mathbb{M}$ that decides $L$ and a constant $\epsilon > 0$ such that for all $n \in \mathbb{N}$:

$$\mathop{\mathbb{E}}_{x \in_{\mathsf{R}} \mathcal{D}_n} [\ \mathsf{time}_{\mathbb{M}}(x)^{\epsilon}\ ] \text{ is } O(n).$$

# Parameterized complexity: with VC as example

- VC: given a graph $G$ and $u \in \mathbb{N}$, does $G$ have a vertex cover of size $u$?
- This problem is NP-complete, and the best algorithms that we have take exponential time in the worst case.
- This worst-case analysis takes into account **every** possible input.
- Can we take into account additional knowledge about the input that we might have to get more positive worst-case guarantees?

# Parameterized complexity: with VC as example (ct'd)

- Suppose that we are dealing with an application where the value of $u$ is always much smaller than the size of the graph $G$.

- Can we restrict the exponential factor in the running time to just $u$?

- **Answer:** yes!

# Fixed-parameter tractability

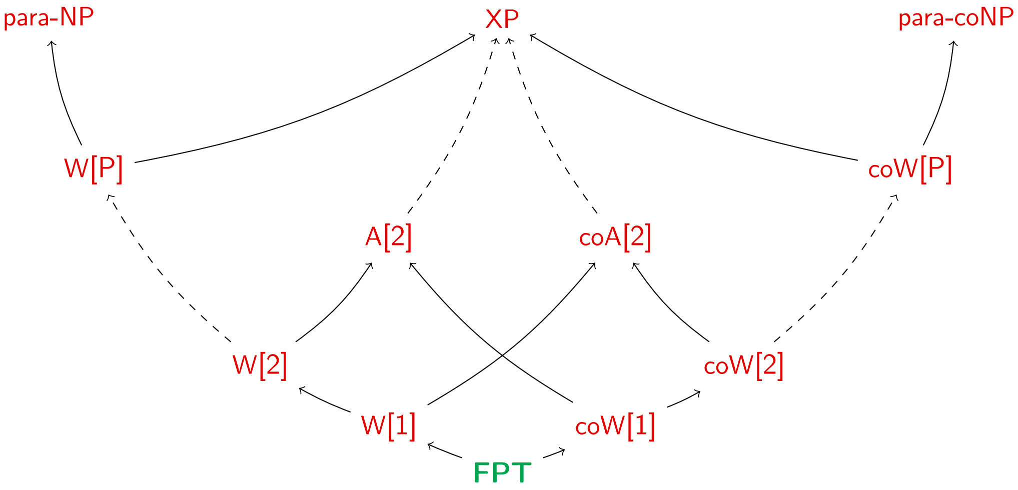
## Definition

A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$ of pairs $(x, k)$, where $x$ is called the *main input* and $k$ is called the *parameter*.

## Definition (FPT)

A parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is *fixed-parameter tractable* if there exist a polynomial $p$, a computable function $f$, and a deterministic TM $\mathbb{M}$ that, when given input $(x, k)$, decides if $(x, k) \in L$ and runs in time $f(k) \cdot p(|x|)$.

# Parameterized complexity landscape



# Parameterized complexity: ‘dialogues’ with your problems

- VC: NP-complete, and no $2^{o(v)}$-time algorithm (assuming ETH)

- With $u$ as parameter? Fixed-parameter tractable
- With $v - u$ as parameter? W[1]-complete
- With the degree $d$ of the graph as parameter? para-NP-complete
- With the treewidth $t$ of the graph as parameter? Fixed-parameter tractable
- Etc.