Computational Complexity

Homework Sheet 1

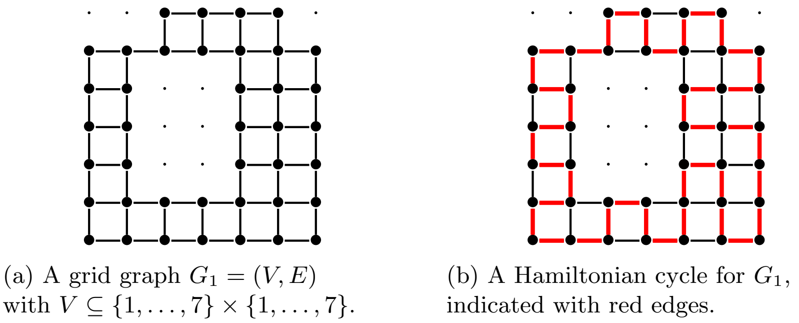
Hand in via Canvas before April 19, 2023, at 23:59
`https://canvas.uva.nl/courses/36220/assignments/409540`

Definition 1. A *grid graph* is an undirected graph $G = (V, E)$ where $V \subseteq \{1, \dots, w\} \times \{1, \dots, h\}$ for some $w, h \in \mathbb{N}$, and where $\{(i_1, j_1), (i_2, j_2)\} \in E$ if and only if either (i) $|i_1 - i_2| = 1$ and $j_1 = j_2$, or (ii) $i_1 = i_2$ and $|j_1 - j_2| = 1$. In other words, the nodes of a grid graph are a subset of points on a w by h grid, and all nodes that are directly adjacent to each other on this grid are connected by an edge.

An example of a grid graph is depicted in Figure 1a.

Definition 2. A *Hamiltonian cycle* C of an undirected graph $G = (V, E)$ is a cycle in G that visits each vertex $v \in V$ exactly once. In other words, $C \subseteq E$ is a subset of edges such that (i) for each $v \in V$, there are exactly two edges in C that are adjacent to v, and (ii) C forms a single connected cycle.

An example of a Hamiltonian cycle is depicted in Figure 1b.



(a) A grid graph $G_1 = (V, E)$ with $V \subseteq \{1, \dots, 7\} \times \{1, \dots, 7\}$.

(b) A Hamiltonian cycle for G_1, indicated with red edges.

Figure 1: Hamiltonian cycles of grid graphs.

Proposition 1 ([1])**.** Let GGHC (for *Grid Graph Hamiltonian Cycle*) be the problem where the input is a grid graph G, and the question is to decide if G has a Hamiltonian cycle. GGHC is NP-complete.

Definition 3. *Slitherlink*[1] is a (fun) puzzle played on a rectangular grid (and on the edges between the the corners of the grid cells).

A *Slitherlink instance* I consists of a rectangular grid of r by s cells, for some $r, s \in \mathbb{N}$. These cells are filled with either a number in $\{0, 1, 2, 3\}$ or are left blank. The numbers in the cells indicate how many of the four lines next to each cell must be used in a solution. An example of a Slitherlink instance is shown in Figure 2a.

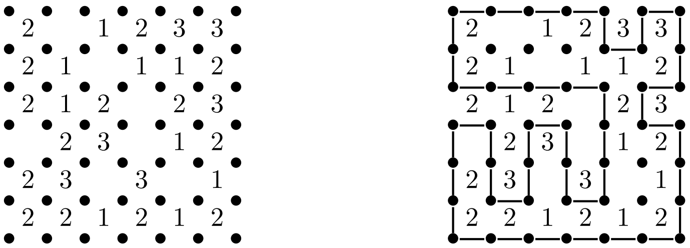
A *solution* for an instance I consists of a subset of the lines that are adjacent to the cells in the grid that (i) forms a single cycle that does not cross itself, and that (ii) for each cell numbered with a number ℓ contains exactly ℓ of the lines adjacent to this cell. An example of a solution for a Slitherlink instance is shown in Figure 2b.

Let SLITH be the decision problem where the input is (a description of) a Slitherlink instance I, and the question is to decide whether there exists a solution for I.

Exercise 1 (4pt)**.** Prove that SLITH is NP-complete.

- Please clearly show both membership in NP and NP-hardness.
- *Hint:* to show NP-hardness, use Proposition 1 and give a polynomial-time reduction from GGHC.
- *Hint:* use the gadgets shown in Figure 3.

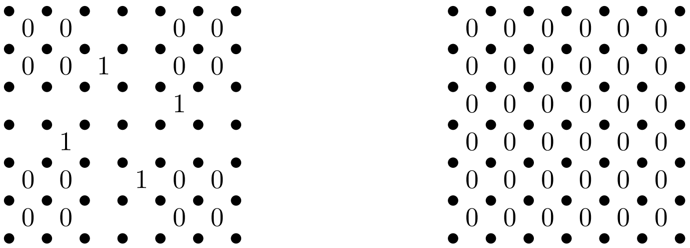
[1] See, e.g., `https://www.puzzle-loop.com/`.



(a) A Slitherlink instance I_1.

(b) A solution for I_1.

Figure 2: An example of a Slitherlink instance and a solution.



(a) Gadget A.

(b) Gadget B.

Figure 3: Slitherlink gadgets.

Exercise 2 (3pt)**.** Let $A \subseteq \{0,1\}^*$ be an NP-complete language. Let p be a polynomial and let $\mathbb{M}_A$ be a polynomial-time Turing machine such that, for all $x \in \{0,1\}^*$:

$$x \in A \text{ if and only if there exists some } u \in \{0,1\}^{p(|x|)} \text{ such that } \mathbb{M}_A(x,u) = 1.$$

(a) Let $B = \{\ (x,z) \mid \text{there exists } z' \in \{0,1\}^* \text{ such that } |zz'| = p(|x|) \text{ and } \mathbb{M}_A(x,zz') = 1\ \}$. Prove that B is in NP.

(b) Suppose that we have access to A as an oracle. Basically this means that we have a subroutine that, given a string y, tells in a single step whether $y \in A$. (See Definition 3.4 of Arora & Barak, 2009.) Construct a polynomial-time Turing machine $\mathbb{M}_{\text{search}}$ (with access to an A-oracle) that, given $x \in \{0,1\}^*$, if $x \in A$ outputs a string u such that $\mathbb{M}_A(x,u) = 1$ and if $x \notin A$ outputs 0. (Describe how $\mathbb{M}_{\text{search}}$ works at a high level.) Use **(a)**.

- *Hint:* use the fact that A is NP-complete.

Exercise 3 (3pt)**.** Let $A \subseteq \{0,1\}^*$ be a language. When a Turing machine $\mathbb{M}$ has access to an A-oracle, we write $\mathbb{M}^A$. We say that A is *auto-reducible* if there is a polynomial-time Turing machine $\mathbb{M}^A$ with oracle access to A such that for all $x \in \{0,1\}^*$:

$$x \in A \text{ if and only if } \mathbb{M}^A(x) = 1,$$

with the special requirement that on input x the Turing machine $\mathbb{M}^A$ is not allowed to query the oracle A for x.

Suppose that A is NP-complete. Prove that A is auto-reducible. Use **Exercise 2**.

Remark 1. Answers will be graded on two criteria: they should (1) be correct and intelligent, and also (2) concise and to the point.

Remark 2. If you find a solution to one of the exercises in a paper or book, you can use this to inform your solution. Make sure that you write down the solution in your own words, conveying that you understand what is going on.

References

[1] Alon Itai, Christos H Papadimitriou, and Jayme Luiz Szwarcfiter. Hamilton paths in grid graphs. *SIAM J. Comput.*, 11(4):676–686, 1982.