Computational Complexity

Exercise Session 7

Definition 1. Consider the representation of graphs where each graph $G = (V, E)$ with n vertices $v_1, \dots, v_n$ is represented by a binary string x of length $\binom{n}{2}$. The bits in this string indicate whether the edges $\{v_1, v_2\}, \{v_1, v_3\}, \dots, \{v_{n-1}, v_n\}$ are present in E. (In other words, strings x such that $|x|$ is not equal to $\binom{n}{2}$ for any $n \in \mathbb{N}$ do not represent any graph.)

For example, the graph $G_0 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\})$ is represented by the string $x_0 = 101$.

Then consider the following sequence $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ of probability distributions over binary strings (representing graphs). Let $p = 1 - 2^{-10} = 1 - {}^{1}\!/_{1024} = {}^{1023}\!/_{1024}$. For each $n \in \mathbb{N}$, the string $x \in \{0,1\}^n$ gets assigned the following probability $\mathcal{D}_n(x)$ by $\mathcal{D}_n$. Let $n_{x,1}$ be the number of 1's in x and let $n_{x,0}$ be the number of 0's in x. Then $\mathcal{D}_n(x) = p^{n_{x,1}} \cdot (1-p)^{n_{x,1}}$. In other words, each edge is present with probability p and is absent with probability $(1-p)$.

Exercise 1. Consider the sequence $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ of distributions from Definition 1.

(a) Show that $\mathcal{D}$ is P-samplable.

(b) Consider the problem $\mathsf{3COL} \subseteq \{0,1\}^*$ consisting of all strings x representing a graph that is 3-colorable. Show that $(\mathsf{3COL}, \mathcal{D}) \in \mathsf{distP}$.

- *Hint:* find a polynomial-time computable condition that (i) is true for almost all inputs, and (ii) entails that the graph is not 3-colorable. Consider the algorithm that first checks this property, and if the graph does not have this property, does a brute-force search.

Exercise 2. Consider the variant $\mathcal{D}'$ of the sequence $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ of distributions from Definition 1 where $p = {}^{1}\!/_{2}$ (instead of ${}^{1023}\!/_{1024}$). Solve Exercise 1 with $\mathcal{D}'$ instead of $\mathcal{D}$.