

Computational Complexity

Exercise Session 2

Exercise 1. Show that $\text{NP} \subseteq \text{EXP}$.

Exercise 2. Consider the following problem **Reverse-3SAT**:

Instance: A propositional formula φ in 3CNF—that is, a formula of the form $\varphi = c_1 \wedge \cdots \wedge c_m$, where each c_j is of the form $c_j = \ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3}$, where $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ are propositional literals.

Question: Is there a truth assignment α to the variables occurring in φ that sets at least one literal in each clause c_j to **false**?

Prove that **Reverse-3SAT** is NP-complete—that is, prove that it is in NP and that it is NP-hard. To show NP-hardness, you may give a reduction from any known NP-complete problem.

- *Hint:* reduce from 3SAT.

Exercise 3. Consider the following problem **CLIQUE**:

Instance: An undirected graph $G = (V, E)$, and a positive integer $k \in \mathbb{N}$.

Question: Does G contain a clique of size k —that is, is there a set $C \subseteq V$ of vertices with $|C| = k$ such that for each $v, v' \in C$ with $v \neq v'$ it holds that $\{v, v'\} \in E$?

In this exercise, we will show that **CLIQUE** is NP-complete.

- (i) Prove that **CLIQUE** is in NP.

To show that **CLIQUE** is NP-hard, we will give a polynomial-time reduction f from 3SAT to **CLIQUE**. We describe this reduction f as follows: for an arbitrary instance φ of 3SAT, we describe what the instance $f(\varphi) = (G, k)$ looks like.

Let $\varphi = c_1 \wedge \cdots \wedge c_m$ be an arbitrary 3CNF formula, containing propositional variables x_1, \dots, x_n , where $c_j = \ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3}$ for each $1 \leq j \leq m$. Then we construct the graph $f(\varphi)$ as follows.

- We introduce vertices $v_{j,1}, v_{j,2}, v_{j,3}$, for each $1 \leq j \leq m$. That is, for each clause c_j we add three vertices—one for each literal occurring in the clause.
- Two vertices $v_{j,l}$ and $v_{j',l'}$ are connected with an edge if and only if $j \neq j'$ and the literals $\ell_{j,l}$ and $\ell_{j',l'}$ are not each other's negation.

Finally, we set $k = m$.

Let $\varphi_{\text{ex}} = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_1})$ be an example 3CNF formula.

- (ii) Let $f(\varphi_{\text{ex}}) = (G_{\text{ex}}, k_{\text{ex}})$. Compute k_{ex} and draw the graph G_{ex} .
- (iii) Show that φ_{ex} is satisfiable. Use a satisfying assignment for φ_{ex} to produce a clique of size k_{ex} for G_{ex} .
- (iv) Prove, for an arbitrary 3CNF formula φ , that φ is satisfiable if and only if $f(\varphi) = (G, k) \in \text{CLIQUE}$.
- (v) Explain why the function f is polynomial-time computable.

Exercise 4 (self-reducibility of 3SAT). Suppose that you have a polynomial-time algorithm A for (the decision problem) 3SAT. Show that you can use A to construct a polynomial-time algorithm B that, when given as input a 3CNF formula φ , outputs a satisfying assignment α for φ if such an assignment exists, and that outputs 0 otherwise.