Computational Complexity

Lecture 1: Introduction

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February 1, 2021

Overview of practical information

- Lecturer: Ronald de Haan (`me@ronalddehaan.eu`)
- TA: Simon Rey (`S.Rey@uva.nl`)
- Course web page: `https://staff.science.uva.nl/r.dehaan/complexity2021/`
- Canvas page: `https://canvas.uva.nl/courses/21489`
- Discourse: `https://talk.computational-complexity.nl/`
- Book: *Computational Complexity: A Modern Approach* (Arora & Barak, 2009)

What will we do today?

- Getting to know each other a bit
- Some explanations about the course and the topic
- Practical things about the course

- Fundamentals of computational complexity:
  Turing machines, big O notation, decision problems, the complexity class P

Some first introductions

- Let me ask you a question (*poll*)

- Get to know some of the other students a bit (*breakout rooms*):

  1. (a) Your name
     (b) Which programme you're in (Logic, AI, CLS, Math, PhD, other)
     (c) Anything else you want to say about yourself

  2. One thing that you expect or want to learn in this course

What is Computational Complexity?

- The study of what you can compute with limited resources
  - E.g.: time, memory space, random bits
    but also: nondeterminism, oracles

- *Computability theory* studies what can be computed **in principle**
- *Computational complexity theory* studies what can be computed **realistically**

What is Computational Complexity? (ct'd)

- Main methodology: distinguish different degrees of difficulty (complexity classes)
  - There is an entire 'zoo' of complexity classes:
    `https://www.complexityzoo.net/`
    (currently listing 545 classes)

- One central question: the **P versus NP problem**
  (one of the $1M *Millennium Prize Problems*)

Relation to other fields

(Or in other words: a bit of marketing)

- Computation plays a role in many areas of society and science
- Therefore, computational complexity is relevant for many areas, e.g.:
  - Computer science, cryptography
  - Economics, game theory
  - Artificial intelligence
  - Biology
  - etc.

Some courses at the UvA that are related

- Recursion Theory *(block 1)*
- Kolmogorov Complexity *(block 5)*
- Knowledge Representation and Reasoning *(block 6)*
- Quantum Computing *(blocks 4+5)*
- Machine Learning Theory *(blocks 4+5)*
- Computational Social Choice *(block 5)*

The time periods in which these courses are taught might change in 2021/2022.

Discourse

- We'll use an online discussion board (using the *Discourse* system): `https://talk.computational-complexity.nl/`
  - Questions about the material
    (If you know the answer to someone else's question, feel free to answer)
  - Reflecting on the material
  - Summarizing the material together
- Feel free to start discussion topics on any of these

Feedback

- During the course:
  - Please give your ideas for improvement, e.g., anonymously on Discourse
  - Especially about logistics (related to online activities)
- After the course:
  - Please fill in the course evaluation questionnaire (for the OC and lecturer)
- *Example of change w.r.t. last year:*
  design homework assignments so that they involve as little as possible "tedious details", which might distract from the main purpose of the homework

Course activities

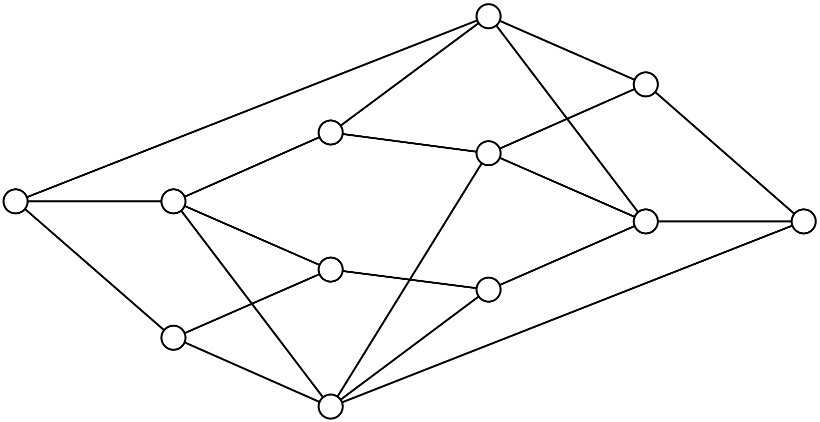
- Lectures:
  - Online, twice 45 minutes, with 15 minute break in between, not recorded
  - Volunteer ‘lecture vigilans’: alert the lecturer when (i) a break is overdue or when (ii) there are unanswered questions in the chat
- Exercise sessions:
  - Online, discuss previous homework assignments, practice with material
- Homework assignments (50% of grade):
  - Five assignments, best four grades count, hand in via Canvas
- Take-home exam (50% of grade):
  - At the end, open book, one week time to complete exam
- Online discussions, question answering

Interactive exercise: graph coloring

- You are given an undirected graph
- The task is to color each node with one of k colors so that **no two connected nodes have the same color**
- *Example application:* nodes are regions with their own radio station, colors are radio frequencies, and two nodes are connected if the regions border each other; assign radio frequencies without conflict (in the border areas)

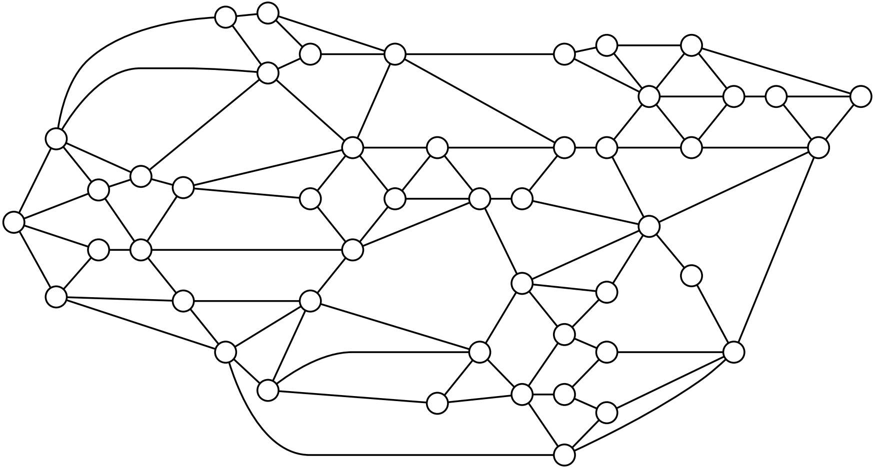
Color this graph with 2 colors

https://tiny.cc/2col



Color this graph with 3 colors

https://tiny.cc/3col



On your mark, get set, go!

- Coloring in breakout rooms (5 minutes)
- After that, we'll discuss your successes, frustrations, etc. ;-)

Quadratic vs. Exponential

- Important difference between
  algorithms that run in time, say, n^2
  vs. algorithms that run in time, say, 2^n

- Illustration (time needed for 10^{10} steps per second):

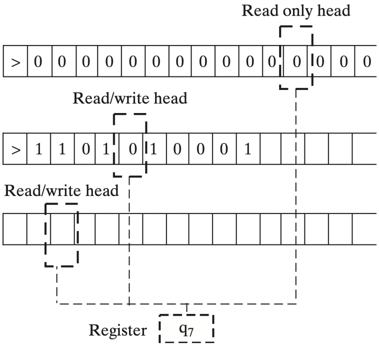
n	n^2 steps	2^n steps
2	0.00000002 msec	0.00000002 msec
5	0.00000015 msec	0.00000019 msec
10	0.00001 msec	0.0001 msec
20	0.00004 msec	0.10 msec
50	0.00025 msec	31.3 hours
100	0.001 msec	9.4×10^{11} years
1000	0.100 msec	7.9×10^{282} years

- # of atoms in universe $\approx 10^{80}$

Definition (Turing machines; TMs)

A *Turing machine* $\mathbb{M}$ is a tuple (Γ, Q, δ), where:

- Γ is the *alphabet*: a finite set of symbols, including 0, 1, $\square$ (the blank symbol), and $\triangleright$ (the start symbol)
- Q is a finite set of *states*, including a designated start state q_{start} and a designated halting state q_{halt}
- $\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{\mathsf{L}, \mathsf{R}, \mathsf{S}\}^k$ is a *transition function*, for some $k \geq 2$ (the number of tapes of the machine)



Model of computation (ct'd)

Definition (TM computing a function)

A TM $\mathbb{M}$ computes the following (partial) function f, where for each $x \in \Sigma^*$:

- $f(x) = y$ if $\mathbb{M}$ halts on input x with output y,
- $f(x) =$ undefined if $\mathbb{M}$ does not halt on input x

Definition (running time)

Let $\mathbb{M}$ be a TM and $g : \mathbb{N} \to \mathbb{N}$ be a function. Then $\mathbb{M}$ runs in time $g(n)$ if for each input $x \in \Sigma^n$ of length n, the machine $\mathbb{M}$ halts after (at most) $g(n)$ steps.

- **Note:** we will switch (often implicitly) between
  the conceptual level ("algorithms")
  and the fully formal level ("Turing machines")

- Typically, we are interested in how (roughly) the running time scales, not in all the details
- We use what is called asymptotic analysis

Definition (Big O)

Let $f, g : \mathbb{N} \to \mathbb{N}$. We say that f is $O(g)$ if there exists a constant $c \in \mathbb{N}$ and an $n_0 \in \mathbb{N}$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

- **Note:** in addition to "f is $O(g)$", the following are also used: "$f = O(g)$", "$f \in O(g)$", "$f(n)$ is $O(g(n))$", etc.

For example,
$4n^2 + 3n + 10$ is $O(n^2)$

Take $c = 8$
and $n_0 = 4$

Decision problems

- To simplify the theory, we restrict our attention to yes/no questions

Definition (Decision problems)

A *decision problem* is a function $f : \Sigma^* \to \{0, 1\}$ where for each input $x \in \Sigma^*$ the correct output $f(x)$ is either 0 or 1.

Alternatively: a formal language $L \subseteq \Sigma^*$ where $x \in L$ if and only if $f(x) = 1$.

- For decision problems, we typically look at TMs that have two halting states:
  q_{acc} (for *accept*: $f(x) = 1$)
  and q_{rej} (for *reject*: $f(x) = 0$)

The complexity class P

Definition (polynomial-time computability)

A function $f : \Sigma^* \to \Sigma^*$ is *polynomial-time computable* (or *computable in polynomial time*)
if there exist a TM $\mathbb{M}$ and a constant $c \in \mathbb{N}$ such that:

- $\mathbb{M}$ computes f
- $\mathbb{M}$ runs in time $O(|x|^c)$

Definition (the complexity class P)

Pis the class (set) consisting of all decision problems $L \subseteq \Sigma^*$ that are computable in polynomial time.

Recap

- 2-coloring vs. 3-coloring
- n^2 vs. 2^n
- Turing machines
- Decision problems
- Polynomial time and the class P

Next time

- The universal Turing machine
- Nondeterministic Turing machines
- More complexity classes: NP and coNP
- Polynomial-time reductions