Computational Complexity

Homework Sheet 5

Hand in via Canvas before March 15, 2021, at 13:00

Exercise 1 (3pt)**.**

(a) Show that $\mathsf{RP} \subseteq \mathsf{NP}$.

(b) Show that $\mathsf{ZPP} = \mathsf{RP} \cap \mathsf{coRP}$.

- *Hint:* use Markov's inequality for showing that $\mathsf{ZPP} \subseteq \mathsf{RP} \cap \mathsf{coRP}$. If X is a non-negative random variable and $a > 0$, then:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Exercise 2 (4pt)**.** The problem MAX2SAT consists of all tuples (φ, k) where φ is a 2CNF formula and $k \in \mathbb{N}$ such that there exists a truth assignment $\alpha : \mathrm{var}(\varphi) \to \{0, 1\}$ such that α satisfies at least k clauses of φ. (Note: here we define a 2CNF formula as a CNF formula where each clause contains **at most** 2 literals. Note also: φ might contain several copies of the same clause.)

For every $\rho < 1$, an algorithm A is called a *ρ-approximation algorithm for* MAX2SAT if for every 2CNF formula ψ with m clauses, $A(\psi)$ outputs a truth assignment satisfying at least $\rho \cdot \mu_\psi$ of ψ's clauses, where μ_ψ is the maximum number of clauses of ψ satisfied by any truth assignment.

Consider the following polynomial-time reduction f from 3SAT to MAX2SAT:

Let $\varphi = c_1 \wedge \cdots \wedge c_m$ be a 3CNF formula with clauses $c_1, \ldots, c_m$ and containing the propositional variables $p_1, \ldots, p_u$. Then $f(\varphi) = (\psi, k)$ is defined as follows.

- The formula ψ will contain the propositional variables $p_1, \ldots, p_u$, as well as the new variables $q_1, \ldots, q_m$.
- For each clause $c_j = l_{j,1} \vee l_{j,2} \vee l_{j,3}$ of φ, we add the following 10 clauses to ψ:

$$\begin{aligned}&(l_{j,1}), (l_{j,2}), (l_{j,3}), (q_j),\\ &(\neg l_{j,1} \vee \neg l_{j,2}), (\neg l_{j,1} \vee \neg l_{j,3}), (\neg l_{j,2} \vee \neg l_{j,3}),\\ &(l_{j,1} \vee \neg q_j), (l_{j,2} \vee \neg q_j), (l_{j,3} \vee \neg q_j).\end{aligned}$$

  That is ψ consists of the conjunction of the $10m$ resulting clauses.
- We let $k = 7m$.

(a) Show that this reduction is correct—i.e., that $\varphi \in \mathsf{3SAT}$ if and only if $(\psi, k) \in \mathsf{MAX2SAT}$.

(b) Show that if there is a polynomial-time ρ-approximation algorithm for MAX2SAT for each $\rho < 1$, then P = NP.

- *Hint:* use the PCP Theorem and the function f described above.

(c) Give a polynomial-time $3/4$-approximation algorithm for MAX2SAT.

- *Note:* your algorithm only needs to work for 2CNF formulas where each clause has length exactly 2.
- *Hint:* compute the expected number of clauses that you will satisfy if you assign each propositional variable (independently) to 1 with probability $1/2$ and to 0 with probability $1/2$. Assign truth values to the propositional variables, using this expected value to guide the choice of truth values.

Exercise 3 (3pt)**.** Consider the following problem Overlapping-Sets. Inputs for this problem consist of a pair $(U, \mathcal{S})$, where U is a finite set (the *universe*) and $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a set of subsets of U, i.e., $S_i \subseteq U$ for all $1 \leq i \leq m$. A subset $\mathcal{C} \subseteq \mathcal{S}$ is a solution if for **each** $S_i, S_j \in \mathcal{C}$ it holds that $S_i \cap S_j \neq \emptyset$.

Let $\rho < 1$. An algorithm A is called a *ρ-approximation algorithm for* Overlapping-Sets if for every input $(U, \mathcal{S})$, the algorithm outputs a solution $\mathcal{C}$ for $(U, \mathcal{S})$ of size at least $\rho \cdot \mu_{(U,\mathcal{S})}$, where $\mu_{(U,\mathcal{S})}$ is the size of the largest solution for $(U, \mathcal{S})$. A probabilistic algorithm A is called a *probabilistic ρ-approximation algorithm for* Overlapping-Sets if for every input $(U, \mathcal{S})$, the probability that the algorithm outputs a solution $\mathcal{C}$ for $(U, \mathcal{S})$ of size at least $\rho \cdot \mu_{(U,\mathcal{S})}$ is at least $2/3$.

(a) Show that the decision variant of Overlapping-Sets is NP-complete—that is, the problem where the inputs consist of $(U, \mathcal{S})$ and $k \in \mathbb{N}$, and the question is to decide if there exists a solution $\mathcal{C}$ for $(U, \mathcal{S})$ of size k.

(b) Show that if there exist a polynomial-time ρ-approximation algorithm for Overlapping-Sets for some $\rho < 1$, then P = NP.

- *Hint:* have a look at Section 11.4 of [1].

(c) Show that if there exist a polynomial-time probabilistic ρ-approximation algorithm[1] for Overlapping-Sets for some $\rho < 1$, then RP = NP.

- *Hint:* show that for any $\rho < 1$, it is NP-hard to decide if $\mu_{(U,\mathcal{S})} = k$, when given $(U, \mathcal{S})$ and k as input, even when one is promised that either $\mu_{(U,\mathcal{S})} = k$ or $\mu_{(U,\mathcal{S})} < \rho \cdot k$.

Remark 1. Answers will be graded on two criteria: they should (1) be correct and intelligent, and also (2) concise and to the point.

Remark 2. If you find a solution to one of the exercises in a paper or book, you can use this to inform your solution. Make sure that you write down the solution in your own words, conveying that you understand what is going on.

Remark 3. You may work in pairs on this homework assignment. If you opt to do this, please register for a group on Canvas (under People > Groups), and both of you should register for the same group. Then, only one submission per group is needed, and both of you will get access to the feedback through Canvas.

References

[1] Sanjeev Arora and Boaz Barak. *Computational Complexity – A Modern Approach.* Cambridge University Press, 2009.

[1] That is, a probabilistic algorithm A for which there exists a polynomial $p(n)$ such that the running time of A is bounded by $p(n)$, regardless of the random choices that it makes.