Definition 1. Consider the representation of graphs where each graph $G = (V, E)$ with $n$ vertices $v_1, \ldots, v_n$ is represented by a binary string $x$ of length $\binom{n}{2}$. The bits in this string indicate whether the edges $\{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_{n-1}, v_n\}$ are present in $E$. (In other words, strings $x$ such that $|x|$ is not equal to $\binom{n}{2}$ for any $n \in \mathbb{N}$ do not represent any graph.) For example, the graph $G_0 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\})$ is represented by the string $x_0 = 101$.

Then consider the following sequence $D = \{D_n\}_{n \in \mathbb{N}}$ of probability distributions over binary strings (representing graphs). Let $p = 1 - 2^{-10} = 1 - \frac{1}{1024} = \frac{1023}{1024}$. For each $n \in \mathbb{N}$, the string $x \in \{0, 1\}^n$ gets assigned the following probability $D_n(x)$ by $D_n$. Let $n_{x, 1}$ be the number of 1’s in $x$ and let $n_{x, 0}$ be the number of 0’s in $x$. Then $D_n(x) = p^{n_{x, 1}} \cdot (1 - p)^{n_{x, 0}}$. In other words, each edge is present with probability $p$ and is absent with probability $(1 - p)$.

Exercise 1. Consider the sequence $D = \{D_n\}_{n \in \mathbb{N}}$ of distributions from Definition 1.

(a) Show that $D$ is $\mathsf{P}$-samplable.

(b) Consider the problem $3\text{COL} \subseteq \{0, 1\}^*$ consisting of all strings $x$ representing a graph that is 3-colorable. Show that $(3\text{COL}, D) \in \text{distP}$.

- Hint: find a polynomial-time computable condition that (i) is true for almost all inputs, and (ii) entails that the graph is not 3-colorable. Consider the algorithm that first checks this property, and if the graph does not have this property, does a brute-force search.

Exercise 2. Consider the variant $D'$ of the sequence $D = \{D_n\}_{n \in \mathbb{N}}$ of distributions from Definition 1 where $p = \frac{1}{2}$ (instead of $\frac{1023}{1024}$). Solve Exercise 1 with $D'$ instead of $D$. 

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