Computational Complexity

Homework Sheet 6

Hand in via Canvas before March 23 at 17:00

Exercise 1 (2pt)**.** Show that the following problem is NP-complete.

> Red-Blue Dominating Set (RBDS)
>
> *Instance:* An undirected graph $G = (V, E)$, where V is partitioned into R and B, and a positive integer $k \in \mathbb{N}$.
>
> *Question:* Is there a subset $D \subseteq B$ of size at most k such that for all $v \in R$ it holds that v is adjacent in G to some $v' \in D$?

That is, RBDS is the following language:

$$\begin{aligned} \text{RBDS} = \{\ (G, R, B, k) \mid\ & G = (V, E) \text{ is an undirected graph, } V = R \cup B,\ R \cap B = \emptyset,\ k \in \mathbb{N}, \\ & \text{there exists some } D \subseteq B \text{ of size at most } k \text{ such that} \\ & \text{for each } v \in R \text{ there is some } v' \in D \text{ such that } v \text{ and } v' \text{ are adjacent in } G\ \} \end{aligned}$$

Exercise 2 (2pt)**.** Show that P = NP if and only if there exists a polynomial-time algorithm A that, given an undirected graph $G = (V, E)$ and a positive integer $k \in \mathbb{N}$:

- If G has a clique $C \subseteq V$ of size k, then A outputs such a clique C; and
- If G has no clique of size k, then A is allowed to output *anything*.

Definition 1. The complexity class Θ_2^{p} consists of all decision problems L for which there exists a constant c and a polynomial-time oracle Turing machine $\mathbb{M}$ that, when given access to an oracle $O \in \text{NP}$, decides L, and for each input x it queries the oracle O at most $c \cdot \log |x|$ times.

Exercise 3 (3pt)**.** Given a set $\Phi = \{\varphi_1, \dots, \varphi_n\}$ of propositional formulas, we say that a set $\Phi' \subseteq \Phi$ is a *largest satisfiable subset of* Φ if (1) Φ' is satisfiable—i.e., there exists a single truth assignment α that satisfies all $\varphi \in \Phi'$, and (2) all $\Phi'' \subseteq \Phi$ with $|\Phi''| > |\Phi'|$ are not satisfiable.

Show that the problem Max-Sat-Subset-Odd is Θ_2^{p}-complete (under polynomial-time reductions):

> Max-Sat-Subset-Odd
>
> *Instance:* A set $\Phi = \{\varphi_1, \dots, \varphi_n\}$ of propositional formulas.
>
> *Question:* Is the size k of the largest satisfiable subset of Φ odd?

You may use the fact that the following problem Max-Model-Odd is Θ_2^{p}-complete:

> Max-Model-Odd
>
> *Instance:* A propositional formula φ with variables $x_1, \dots, x_n$.
>
> *Question:* Do the truth assignments $\alpha : \{x_1, \dots, x_n\} \to \{0, 1\}$ that set a maximal number of variables among $x_1, \dots, x_n$ to true among those α that satisfy φ, set an odd number of variables among $x_1, \dots, x_n$ to true?

Exercise 4 (3pt)**.** Show that the problem LARGE-MINIMAL-UNSAT-SUBSET is Σ_2^{p}-complete (under polynomial-time reductions):

> LARGE-MINIMAL-UNSAT-SUBSET
>
> *Instance:* A set $\Phi = \{\varphi_1, \dots, \varphi_n\}$ of propositional formulas, and an integer $k \in \mathbb{N}$.
>
> *Question:* Is there some $\Phi' \subseteq \Phi$ of size k that is not satisfiable—that is, for which there exists no truth assignment α that satisfies each formula $\varphi \in \Phi'$—such that all $\Phi'' \subsetneq \Phi'$ are satisfiable?

You may use the fact that the following problem Σ_2SAT(DNF) is Σ_2^{p}-complete:

> Σ_2SAT(DNF)
>
> *Instance:* An instance $\exists \overline{u}_1 \forall \overline{u}_2\, \psi$ of Σ_2SAT, where ψ is in DNF.
>
> *Question:* Is $\exists \overline{u}_1 \forall \overline{u}_2 \psi$ true?