Computational Complexity

Homework Sheet 5

Hand in via Canvas before March 16 at 17:00

Exercise 1 (5pt)**.** The problem MAX2SAT consists of all tuples (φ, k) where φ is a 2CNF formula and $k \in \mathbb{N}$ such that there exists a truth assignment $\alpha : \text{var}(\varphi) \to \{0, 1\}$ such that α satisfies at least k clauses of φ. (Note: here we define a 2CNF formula as a CNF formula where each clause contains **at most** 2 literals. Note also: φ might contain several copies of the same clause.)

For every $\rho < 1$, an algorithm A is called a ρ-approximation algorithm for MAX2SAT if for every 2CNF formula ψ with m clauses, $A(\psi)$ outputs a truth assignment satisfying at least $\rho \cdot \mu_\psi$ of ψ's clauses, where μ_ψ is the maximum number of clauses of ψ satisfied by any truth assignment.

Consider the following polynomial-time reduction f from 3SAT to MAX2SAT:

> Let $\varphi = c_1 \wedge \cdots \wedge c_m$ be a 3CNF formula with clauses $c_1, \ldots, c_m$ and containing the propositional variables $p_1, \ldots, p_u$. Then $f(\varphi) = (\psi, k)$ is defined as follows.
>
> - The formula ψ will contain the propositional variables $p_1, \ldots, p_u$, as well as the new variables $q_1, \ldots, q_m$.
> - For each clause $c_j = l_{j,1} \vee l_{j,2} \vee l_{j,3}$ of φ, we add the following 10 clauses to ψ:
>
> $$\begin{aligned}&(l_{j,1}), (l_{j,2}), (l_{j,3}), (q_j),\\ &(\neg l_{j,1} \vee \neg l_{j,2}), (\neg l_{j,1} \vee \neg l_{j,3}), (\neg l_{j,2} \vee \neg l_{j,3}),\\ &(l_{j,1} \vee \neg q_j), (l_{j,2} \vee \neg q_j), (l_{j,3} \vee \neg q_j).\end{aligned}$$
>
> That is ψ consists of the conjunction of the $10m$ resulting clauses.
> - We let $k = 7m$.

(a) Show that this reduction is correct—i.e., that $\varphi \in$ 3SAT if and only if $(\psi, k) \in$ MAX2SAT.

(b) Show that if there is a polynomial-time ρ-approximation algorithm for MAX2SAT for each $\rho < 1$, then P = NP.

- *Hint:* use the PCP Theorem and the function f described above.

(c) Give a polynomial-time $\frac{1}{2}$-approximation algorithm for MAX2SAT.

Exercise 2 (4pt)**.** Consider the following polynomial-time reduction f from 3SAT to 3SAT. Let φ be a 3CNF formula with clauses $c_1, \ldots, c_m$ and containing the propositional variables $p_1, \ldots, p_n$. Then $f(\varphi)$ is defined as the following 3CNF formula:

$$f(\varphi) = \varphi \wedge \left(\bigwedge_{j=1}^{m} q_j \right) \wedge \left(\bigwedge_{j=1}^{m} \bigwedge_{j'=1}^{m} (q_j \vee q_{j'}) \right),$$

where each of the variables q_j is a fresh variable that does not occur in φ. (Note: here we define a 3CNF formula as a CNF formula where each clause contains **at most** 3 literals.)

Let F be the following set of 3CNF formulas:

$$F = \{\ f(\varphi) \mid \varphi \text{ is a 3CNF formula }\},$$

and let FUNNY-3SAT be the following decision problem:

$$\textsc{Funny}\text{-3SAT} = F \cap \text{3SAT}.$$

(a) Show that FUNNY-3SAT is solvable in time $2^{O(\sqrt{|x|})}$, where $|x|$ denotes the input size.

(b) Show that FUNNY-3SAT is not solvable in time $2^{o(\sqrt{|x|})}$, where $|x|$ denotes the input size, assuming that the ETH is true.

Exercise 3 (1pt)**.** Give an example of a decision problem that is not solvable in polynomial time (assuming P $\neq$ NP), yet that is solvable in time $2^{o(|x|)}$, where $|x|$ denotes the input size.