Computational Complexity

Handout – Lecture 8

Definition 1 (Circuits)**.** An *n-input single-output Boolean circuit C* is a directed acyclic graph with:

- n sources (nodes with no incoming edges), labelled 1 to n, and
- one sink (a node with no outgoing edges).

All non-source vertices are called *gates*, and are labelled with $\wedge$, $\vee$, or $\neg$:

- $\wedge$-gates and $\vee$-gates have in-degree 2 (exactly two incoming edges),
- $\neg$-gates have in-degree 1 (exactly one incoming edge).

If C is an n-input single-output Boolean circuit and $x \in \{0,1\}^n$ is a string, then the output $C(x)$ of C on x is defined by plugging in x in the source nodes and applying the operators of the gates, and taking for $C(x)$ the resulting value in $\{0,1\}$ of the sink gate.

Definition 2 (Circuit families)**.** Let $t : \mathbb{N} \to \mathbb{N}$ be a function. A *$t(n)$-size circuit family* is a sequence $\{C_n\}_{n\in\mathbb{N}}$ of Boolean circuits, where each C_n has n inputs and a single output, and $|C_n| \leq t(n)$ for each $n \in \mathbb{N}$.

Definition 3 (SIZE($t(n)$))**.** Let $t : \mathbb{N} \to \mathbb{N}$ be a function. A language $L \subseteq \{0,1\}^*$ is in SIZE($t(n)$) if there exists a constant $c \in \mathbb{N}$ and a $(c \cdot t(n))$-size circuit family $\{C_n\}_{n\in\mathbb{N}}$ such that for each $x \in \{0,1\}^*$:

$$x \in L \quad \text{if and only if} \quad C_n(x) = 1, \text{ where } n = |x|.$$

Definition 4 (TIME($t(n)$)/$a(n)$)**.** Let $t, a : \mathbb{N} \to \mathbb{N}$ be functions. The class DTIME($t(n)$)/$a(n)$ of languages decidable by *$O(t(n))$-time Turing machines with $a(n)$ bits of advice* contains every language $L \subseteq \{0,1\}^*$ such that there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\alpha_n \in \{0,1\}^{a(n)}$ for each $n \in \mathbb{N}$ and an $O(t(n))$-time deterministic Turing machine $\mathbb{M}$ such that for for each $x \in \{0,1\}^*$:

$$x \in L \quad \text{if and only if} \quad \mathbb{M}(x, \alpha_n) = 1, \text{ where } n = |x|.$$