Computational Complexity

Handout – Lecture 6

Definition 1. Let $s : \mathbb{N} \to \mathbb{N}$ be a function and let $L \subseteq \{0,1\}^*$ be a language. We say that $L \in \text{SPACE}(s(n))$ if there exists a constant c and a TM $\mathbb{M}$ that for every $x \in \{0,1\}^*$, the machine $\mathbb{M}$ decides whether $x \in L$ and its tape heads visit at most $c \cdot s(n)$ locations of $\mathbb{M}$'s work tapes (so all tapes except the input tape).

Similarly, we say that $L \in \text{NSPACE}(s(n))$ if there exists a constant c and a nondeterministic TM $\mathbb{M}$ that for every $x \in \{0,1\}^*$, the machine $\mathbb{M}$ decides whether $x \in L$ and its tape heads visit at most $c \cdot s(n)$ locations of $\mathbb{M}$'s work tapes—regardless of the nondeterministic choices that it makes.

Definition 2. A *quantified Boolean formula (QBF)* is of the form $Q_1 x_1 Q_2 x_2 \cdots Q_m x_m \ \varphi(x_1, \dots, x_m)$, where each Q_i is one of the two quantifiers $\exists$ or $\forall$, the variables $x_1, \dots, x_m$ range over $\{0,1\}$ and φ is a propositional formula (without quantifiers).[1]

Truth of QBFs is defined recursively. For the base case: a QBF without quantifiers—which is a propositional formula φ where all variables are replaced by (constants representing) truth values—is true if the propositional formula φ is true. For the inductive case: a QBF of the form $\exists x_i \ \psi$ is true if there exists a truth value $b_i \in \{0,1\}$ such that $\psi[x_i \mapsto b_i]$ is true, and a QBF of the form $\forall x_i \ \psi$ is true if for both truth values $b_i \in \{0,1\}$ it holds that $\psi[x_i \mapsto b_i]$ is true.

Definition 3. A function $f : \{0,1\}^* \to \{0,1\}^*$ is *implicitly logspace computable* if:

- f is polynomially bounded, i.e., there exists some c such that $|f(x)| \leq |x|^c$ for every $x \in \{0,1\}^*$,
- and the languages $L_f = \{ \ (x,i) \mid f(x)_i = 1 \ \}$ and $L'_f = \{ \ (x,i) \mid i \leq |f(x)| \ \}$ are in the complexity class L.

A language B is *logspace-reducible* to a language C if there is a function $f : \{0,1\}^* \to \{0,1\}^*$ that is implicitly logspace computable and for each $x \in \{0,1\}^*$ it holds that $x \in B$ if and only if $f(x) \in C$.

[1] Technically, this only defines QBFs in *prenex form*, where all quantifiers appear on the left of the formula. One can also consider QBFs where quantifiers and Boolean connectives can alternate.