Computational Complexity

Exercise Session 2

Exercise 1. Show that NP $\subseteq$ EXP.

Exercise 2. Consider the following problem Reverse-3SAT:

> **Instance:** A propositional formula φ in 3CNF—that is, a formula of the form $\varphi = c_1 \wedge \cdots \wedge c_m$, where each c_i is of the form $c_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$, where $l_{i,1}, l_{i,2}, l_{i,3}$ are propositional literals.
>
> **Question:** Is there a truth assignment α to the variables occurring in φ that sets at least one literal in each clause c_i to **false**?

Prove that Reverse-3SAT is NP-complete—that is, prove that is in NP and that it is NP-hard. To show NP-hardness, you may give a reduction from any known NP-complete problem.

- *Hint:* reduce from 3SAT.

Exercise 3. Consider the following problem 3-Coloring:

> **Instance:** An undirected graph $G = (V, E)$.
>
> **Question:** Is there a mapping $\chi : V \to \{1, 2, 3\}$ such that for each $\{v_1, v_2\} \in E$ it holds that $\chi(v_1) \neq \chi(v_2)$?

In this exercise, we will show that 3-Coloring is NP-complete.

(i) Prove that 3-Coloring is in NP.

To show that 3-Coloring is NP-hard, we will give a polynomial-time reduction f from 3SAT to 3-Coloring. We describe this reduction f as follows: for an arbitrary instance φ of 3SAT, we describe what the instance $f(\varphi) = G$ looks like.

Let $\varphi = c_1 \wedge \cdots \wedge c_m$ be an arbitrary 3CNF formula, containing propositional variables $x_1, \ldots, x_n$. The graph $f(\varphi)$ consists of:

- vertices a, b, c,
- edges $\{a, b\}, \{a, c\}, \{b, c\}$,
- vertices $x_i, \overline{x_i}$ for each $1 \leq i \leq n$,
- edges $\{c, x_i\}, \{c, \overline{x_i}\}, \{x_i, \overline{x_i}\}$, for each $1 \leq i \leq n$,
- vertices d_j, e_j, f_j, g_j, h_j, for each $1 \leq j \leq m$,
- edges $\{\ell_{j,1}, d_j\}, \{\ell_{j,2}, e_j\}, \{\ell_{j,3}, f_j\}$, for each clause $c_j = (\ell_{j,1} \vee \ell_{j,2} \vee \ell_{j,3})$ of φ, where $\ell_{j,1}, \ell_{j,2}, \ell_{j,3} \in \{\ x_i, \overline{x_i} \mid 1 \leq i \leq n\ \}$, and
- edges $\{d_j, g_j\}, \{e_j, g_j\}, \{d_j, e_j\}, \{g_j, h_j\}, \{f_j, h_j\}, \{f_j, a\}, \{h_j, a\}$, for each $1 \leq j \leq m$.

Let $\varphi_{\text{ex}} = (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee x_4)$ be an example 3CNF formula.

(ii) Draw the graph $f(\varphi_{\text{ex}})$.

(iii) Show that φ_{ex} is satisfiable. Use a satisfying assignment for φ_{ex} to produce a 3-coloring for $f(\varphi_{\text{ex}})$.

(iv) Prove, for an arbitrary 3CNF formula φ, that φ is satisfiable if and only if $f(\varphi)$ is 3-colorable.

(v) Explain why the function f is polynomial-time computable.