Computational Complexity

Practice Exam

This practice exam consists of five questions worth 10 points in total. The actual exam will also consist of several questions worth 10 points in total. This practice exam is intended to show what type of questions are likely to show up on the exam. The topics of the questions on the exam can be any of the topics we discussed in the course (not just the topics of the questions on the practice exam).

For the exam, you are allowed to use the book and any paper notes you have brought. Using electronic devices is not allowed.

You are allowed to use results that you have proved in the homework. If you do, please mention it, i.e., "Here we use ... as was shown in the homework." When using a result from the book, please mention the theorem number or page number.

Definition 1. Let L be a language. We say that L is *length-decreasing self-reducible* if there is a polynomial-time oracle TM M such that for each $x \in \{0,1\}^*$:

$$x \in L \quad \text{if and only if} \quad M^L(x) = 1,$$

and the computation of $M^L(x)$ only queries L on strings of length strictly less than $|x|$.

Question 1 (2pt)**.** Let L be a language such that $L \subseteq \{1\}^*$. Prove that L is in P if and only if L is length-decreasing self-reducible.

Question 2 (1pt)**.** Consider the following language K, that consists of all strings x encoding a propositional formula φ, containing the propositional variables $p_1, \dots, p_m$, such that there is a truth assignment $\alpha : \{p_1, \dots, p_m\} \to \{0,1\}$ for which holds:

- there exist i, j such that $\alpha(p_i) \neq \alpha(p_j)$, and
- α satisfies φ.

Prove that K is NP-complete.

Definition 2. Let $L_1, L_2 \subseteq \{0,1\}^*$ be languages. We define the *concatenation* $L_1 \circ L_2$ of L_1 and L_2 as follows:

$$L_1 \circ L_2 = \{\ x_1 x_2 \mid x_1 \in L_1, x_2 \in L_2\ \}.$$

That is, $L_1 \circ L_2$ contains all strings x that can be split into two strings x_1 and x_2 such that $x_1 \in L_1$ and $x_2 \in L_2$.

Question 3 (3pt)**.** Prove that:

(a) NP is closed under union, i.e., if $L_1, L_2 \in \text{NP}$ then $L_1 \cup L_2 \in \text{NP}$.

(b) NP is closed under intersection, i.e., if $L_1, L_2 \in \text{NP}$ then $L_1 \cap L_2 \in \text{NP}$.

(b) NP is closed under concatenation, i.e., if $L_1, L_2 \in \text{NP}$ then $L_1 \circ L_2 \in \text{NP}$.

Definition 3. A set $S \subseteq \{0,1\}^*$ is called *sparse* if it has polynomial density, i.e., if there exists a constant c such that for each $n \in \mathbb{N}$:

$$|S \cap \{0,1\}^n| \leq n^c + c.$$

We use SPARSE to denote the class of all sparse sets.

Question 4 (2pt)**.** Let $L \subseteq \{0,1\}^*$ be a language. Prove that $L \in \text{P/poly}$ if and only if $L \in \text{P}^{\text{SPARSE}}$.

- *Hint:* Use a sparse set S to encode the advice, and vice versa. Remember to prove the implication clearly in both directions.

Question 5 (2pt)**.** We use $\mathcal{P}(\{0,1\}^*)$ to denote the set of all languages $L \subseteq \{0,1\}^*$.

(a) Give an example of a decision problem $L \subseteq \{0,1\}^*$ that is solvable in time $O(1)$.

(b) Does there exist a complexity class $\text{K} \subseteq \mathcal{P}(\{0,1\}^*)$ that has both of the following properties (i) and (ii)? If so, give a description of such a class K. If not, prove that no such class K exists.

  (i) $\text{EXP} \subsetneq \text{K}$.

  (ii) $\text{P/poly} \subseteq \text{K}$.

(c) Does there exist a complexity class $\text{K} \subseteq \mathcal{P}(\{0,1\}^*)$ that has both of the following properties (i) and (ii)? If so, give a description of such a class K. If not, prove that no such class K exists.

  (i) $\text{EXP} \cap \text{K} = \emptyset$.

  (ii) $\text{P} \subseteq \text{K}$.