Exercise 1 (5pt). The problem MAX2SAT consists of all tuples \( \langle \varphi, k \rangle \) where \( \varphi \) is a 2CNF formula and \( k \in \mathbb{N} \) such that there exists a truth assignment \( \alpha : \text{var}(\varphi) \to \{0,1\} \) such that \( \alpha \) satisfies at least \( k \) clauses of \( \varphi \). (Note: here we define a 2CNF formula as a CNF formula where each clause contains at most 2 literals. Note also: \( \varphi \) might contain several copies of the same clause.)

For every \( \rho < 1 \), an algorithm \( A \) is called a \( \rho \)-approximation algorithm for MAX2SAT if for every 2CNF formula \( \psi \) with \( m \) clauses, \( A(\psi) \) outputs a truth assignment satisfying at least \( \rho \cdot \mu_\psi \) of \( \psi \)'s clauses, where \( \mu_\psi \) is the maximum number of clauses of \( \psi \) satisfied by any truth assignment.

Consider the following polynomial-time reduction \( f \) from 3SAT to MAX2SAT:

Let \( \varphi = c_1 \land \ldots \land c_m \) be a 3CNF formula with clauses \( c_1, \ldots, c_m \) and containing the propositional variables \( p_1, \ldots, p_u \). Then \( f(\varphi) = (\psi, k) \) is defined as follows.

- The formula \( \psi \) will contain the propositional variables \( p_1, \ldots, p_u \), as well as the new variables \( q_1, \ldots, q_u \).
- For each clause \( c_j = l_{j,1} \lor l_{j,2} \lor l_{j,3} \) of \( \varphi \), we add the following 10 clauses to \( \psi \):
  - \((l_{j,1}), (l_{j,2}), (l_{j,3}), (q_j)\),
  - \((-l_{j,1} \lor -l_{j,2}), (-l_{j,1} \lor -l_{j,3}), (-l_{j,2} \lor -l_{j,3})\),
  - \((l_{j,1} \lor -q_j), (l_{j,2} \lor -q_j), (l_{j,3} \lor -q_j)\).

That is \( \psi \) consists of the conjunction of the 10\( m \) resulting clauses.

- We let \( k = 7m \).

(a) Show that this reduction is correct—i.e., that \( \varphi \in 3\text{SAT} \) if and only if \( \langle \psi, k \rangle \in \text{MAX2SAT} \).

(b) Show that if there is a polynomial-time \( \rho \)-approximation algorithm for MAX2SAT for each \( \rho < 1 \), then \( P = NP \).
   
   - Hint: use the PCP Theorem and the function \( f \) described above.

(c) Give a polynomial-time \( \frac{1}{2} \)-approximation algorithm for MAX2SAT.

Exercise 2 (4pt). Consider the following polynomial-time reduction \( f \) from 3SAT to 3SAT. Let \( \varphi \) be a 3CNF formula with clauses \( c_1, \ldots, c_m \) and containing the propositional variables \( p_1, \ldots, p_n \). Then \( f(\varphi) \) is defined as the following 3CNF formula:

\[
f(\varphi) = \varphi \land \left( \bigwedge_{j=1}^{m} q_j \right) \land \left( \bigwedge_{j=1}^{m} \bigwedge_{j'=1}^{m} (q_j \lor q_{j'}) \right),
\]

where each of the variables \( q_j \) is a fresh variable that does not occur in \( \varphi \). (Note: here we define a 3CNF formula as a CNF formula where each clause contains at most 3 literals.)

Let \( F \) be the following set of 3CNF formulas:

\[
F = \{ f(\varphi) \mid \varphi \text{ is a 3CNF formula } \},
\]

and let \( \text{FUNNY-3SAT} \) be the following decision problem:

\[
\text{FUNNY-3SAT} = F \cap 3\text{SAT}.
\]
(a) Show that FUNNY-3SAT is solvable in time \(2^{O(\sqrt{|x|})}\), where \(|x|\) denotes the input size.

(b) Show that FUNNY-3SAT is not solvable in time \(2^{o(\sqrt{|x|})}\), where \(|x|\) denotes the input size, assuming that the ETH is true.

Exercise 3 (1pt). Give an example of a decision problem that is not solvable in polynomial time (assuming \(P \neq NP\)), yet that is solvable in time \(2^{o(|x|)}\), where \(|x|\) denotes the input size.