Exercise 1 (3pt). Define:

\[ P_{\log} = \bigcup_{c,d \in \mathbb{N}} \text{DTIME}(n^c)/(d \log n). \]

That is, \( P_{\log} \) is the class of all languages that can be decided in polynomial time with \( O(\log n) \) bits of advice. Prove that \( \text{SAT} \not\in P_{\log} \), unless \( P = \text{NP} \).

- *Hint:* iterate over all possible advice strings of length \( O(\log n) \).

- *Hint:* you may assume that for any string \( x \) that represents a propositional formula \( \varphi \) and any truth assignment \( \alpha \) to (some of) the variables of \( \varphi \), one can in polynomial-time encode the formula \( \varphi[\alpha] \) as a string \( x' \) that is of the same length as \( x \)—where \( \varphi[\alpha] \) is obtained from \( \varphi \) by instantiating each variable \( z \) in the domain of \( \alpha \) by \( \alpha(z) \).

Exercise 2 (2pt). Let \( P_1 \) be the class of languages that can be decided in polynomial time with a single bit of advice (for each input size). That is \( P_1 = \bigcup_{c \in \mathbb{N}} \text{DTIME}(n^c)/1 \). Prove that \( P \subseteq P_1 \).

- *Hint:* use the following undecidable language UHALT:

\[ \text{UHALT} = \{ 1^n \mid n's\ binary\ expansion\ encodes\ a\ pair\ \langle M, x \rangle\ such\ that\ M\ is\ a\ Turing\ machine\ that\ halts\ on\ input\ x \}. \]

Exercise 3 (2pt). Show that \( \text{RP} \subseteq \text{NP} \).

Exercise 4 (3pt). Show that \( \text{ZPP} = \text{RP} \cap \text{coRP} \).

- *Hint:* use Markov’s inequality for showing that \( \text{ZPP} \subseteq \text{RP} \cap \text{coRP} \). If \( X \) is a non-negative random variable and \( a > 0 \), then:

\[ \mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}. \]