Computational Complexity

Homework Sheet 5

Hand in before March 14, 23:59
Preferably by email to J.M.Czajkowski@cwi.nl

Exercise 1 (2pt). Show that RP ⊆ NP.

Exercise 2 (4pt). Show that ZPP = RP ∩ coRP.

• Hint: use Markov’s inequality for showing that ZPP ⊆ RP ∩ coRP. If X is a non-negative random variable and a > 0, then:

\[ \mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}. \]

Exercise 3 (4pt). The problem MAX2SAT consists of all tuples \( \langle \varphi, k \rangle \) where \( \varphi \) is a 2CNF formula and \( k \in \mathbb{N} \) such that there exists a truth assignment \( \alpha : \text{var}(\varphi) \rightarrow \{0, 1\} \) such that \( \alpha \) satisfies at least \( k \) clauses of \( \varphi \). (Note: here we define a 2CNF formula as a CNF formula where each clause contains at most 2 literals. Note also: \( \varphi \) might contain several copies of the same clause.)

For every \( \rho \geq 1 \), an algorithm \( A \) is called a \( \rho \)-approximation algorithm for MAX2SAT if for every 2CNF formula \( \varphi \) with \( m \) clauses, \( A(\varphi) \) outputs a truth assignment satisfying at least \( \rho \cdot \mu_\varphi \) of \( \varphi \)'s clauses, where \( \mu_\varphi \) is the maximum number of clauses of \( \varphi \) satisfied by any truth assignment.

Consider the following polynomial-time reduction \( f \) from 3SAT to MAX2SAT:

Let \( \varphi = c_1 \land \cdots \land c_m \) be a 3CNF formula with clauses \( c_1, \ldots, c_m \) and containing the propositional variables \( p_1, \ldots, p_u \). Then \( f(\varphi) = \langle \psi, k \rangle \) is defined as follows.

• The formula \( \psi \) will contain the propositional variables \( p_1, \ldots, p_u \), as well as the new variables \( q_1, \ldots, q_u \).

• For each clause \( c_j = l_{j,1} \lor l_{j,2} \lor l_{j,3} \) of \( \varphi \), we add the following 10 clauses to \( \psi \):

\[
(l_{j,1}), (l_{j,2}), (l_{j,3}), (q_j), \\
(\neg l_{j,1} \lor \neg l_{j,2}), (\neg l_{j,1} \lor \neg l_{j,3}), (\neg l_{j,2} \lor \neg l_{j,3}), \\
(l_{j,1} \lor q_j), (l_{j,2} \lor \neg q_j), (l_{j,3} \lor \neg q_j).
\]

That is \( \psi \) consists of the conjunction of the 10m resulting clauses.

• We let \( k = 7m \).

(a) Show that this reduction is correct.

(b) Show that if there is a polynomial-time \( \rho \)-approximation algorithm for MAX2SAT for each \( \rho < 1 \), then \( P = NP \).

– Hint: use the function \( f \) described above.

(c) Give a polynomial-time \( \frac{1}{2} \)-approximation algorithm for MAX2SAT.