Exercise 1 (2pt). Define the complexity class

\[ \text{DP} = \{ A \cap B \mid A \in \text{NP}, B \in \text{coNP} \}. \]

Let \( G = (V, E) \) be an undirected graph. A subset \( D \subseteq V \) of vertices is called a dominating set if for every \( v \in V \) it holds that (i) \( v \in D \) or (ii) there is some \( u \in D \) and there is an edge between \( u \) and \( v \) in \( G \). Consider the problem \text{EXACT-DS}:

\[ \text{EXACT-DS} = \{ \langle G, k \rangle \mid G \text{ is an undirected graph that has a dominating set of size } k \text{ but has no dominating set of size } k - 1 \}. \]

(a) Prove that \text{EXACT-DS} \in \text{DP}.

(b) Prove that if \text{DP} \subseteq \text{NP}, then the Polynomial Hierarchy collapses.

Exercise 2 (3pt). Prove that there is no polynomial-time algorithm \( A \) with access to a SAT oracle such that, on each input \( x \in \{0, 1\}^* \) representing a propositional formula \( \varphi \):

- \( A \) makes at most \( O(\log |x|) \) queries to the oracle, and
- if \( \varphi \) is satisfiable, \( A \) outputs a truth assignment \( \alpha \) that satisfies \( \varphi \),

unless \( P = \text{NP} \).

Exercise 3 (3pt). Define:

\[ P_{/ \log} = \bigcup_{c,d \in \mathbb{N}} \text{DTIME}(n^c)/(d \log n). \]

That is, \( P_{/ \log} \) is the class of all languages that can be decided in polynomial time with \( O(\log n) \) bits of advice. Prove that \( \text{SAT} \not\in P_{/ \log} \), unless \( P = \text{NP} \).

- **Hint**: iterate over all possible advice strings of length \( O(\log n) \).

- **Hint**: you may assume that for any string \( x \) that represents a propositional formula \( \varphi \) and any truth assignment \( \alpha \) to (some of) the variables of \( \varphi \), one can in polynomial-time encode the formula \( \varphi[\alpha] \) as a string \( x' \) that is of the same length as \( x \)—where \( \varphi[\alpha] \) is obtained from \( \varphi \) by instantiating each variable \( z \) in the domain of \( \alpha \) by \( \alpha(z) \).

Exercise 4 (2pt). Let \( P_{/1} \) be the class of languages that can be decided in polynomial time with a single bit of advice (for each input size). That is \( P_{/1} = \bigcup_{c \in \mathbb{N}} \text{DTIME}(n^c)/1 \). Prove that \( P \not\subseteq P_{/1} \).

- **Hint**: use the following undecidable language \( \text{UHALT} \):

\[ \text{UHALT} = \{ 1^n \mid n \text{'s binary expansion encodes a pair } \langle M, x \rangle \text{ such that } M \text{ is a Turing machine that halts on input } x \}. \]