Computational Complexity

Homework Sheet 4

Hand in before March 7, 23:59

Preferably by email to `J.M.Czajkowski@cwi.nl`

Exercise 1 (2pt)**.** Define the complexity class

$$\mathrm{DP} = \{\ A \cap B \mid A \in \mathrm{NP}, B \in \mathrm{coNP}\ \}.$$

Let $G = (V, E)$ be an undirected graph. A subset $D \subseteq V$ of vertices is called a *dominating set* if for every $v \in V$ it holds that (i) $v \in D$ or (ii) there is some $u \in D$ and there is an edge between u and v in G. Consider the problem EXACT-DS:

$$\textsc{Exact}\text{-DS} = \{\ \langle G, k\rangle \mid G \text{ is an undirected graph that has a dominating set of size } k \text{ but has no dominating set of size } k-1\ \}.$$

(a) Prove that $\textsc{Exact}\text{-DS} \in \mathrm{DP}$.

(b) Prove that if $\mathrm{DP} \subseteq \mathrm{NP}$, then the Polynomial Hierarchy collapses.

Exercise 2 (3pt)**.** Prove that there is no polynomial-time algorithm A with access to a SAT oracle such that, on each input $x \in \{0,1\}^*$ representing a propositional formula φ:

- A makes at most $O(\log |x|)$ queries to the oracle, and
- if φ is satisfiable, A outputs a truth assignment α that satisfies φ,

unless $\mathrm{P} = \mathrm{NP}$.

Exercise 3 (3pt)**.** Define:

$$\mathrm{P}_{/\log} = \bigcup_{c,d \in \mathbb{N}} \mathrm{DTIME}(n^c)/(d \log n).$$

That is, $\mathrm{P}_{/\log}$ is the class of all languages that can be decided in polynomial time with $O(\log n)$ bits of advice. Prove that $\mathrm{SAT} \notin \mathrm{P}_{/\log}$, unless $\mathrm{P} = \mathrm{NP}$.

- *Hint:* iterate over all possible advice strings of length $O(\log n)$.
- *Hint:* you may assume that for any string x that represents a propositional formula φ and any truth assignment α to (some of) the variables of φ, one can in polynomial-time encode the formula $\varphi[\alpha]$ as a string x' that is of the same length as x—where $\varphi[\alpha]$ is obtained from φ by instantiating each variable z in the domain of α by $\alpha(z)$.

Exercise 4 (2pt)**.** Let $\mathrm{P}_{/1}$ be the class of languages that can be decided in polynomial time with a single bit of advice (for each input size). That is $\mathrm{P}_{/1} = \bigcup_{c \in \mathbb{N}} \mathrm{DTIME}(n^c)/1$. Prove that $\mathrm{P} \subsetneq \mathrm{P}_{/1}$.

- *Hint:* use the following undecidable language UHALT:

$$\mathrm{UHALT} = \{\ 1^n \mid n\text{'s binary expansion encodes a pair } \langle M, x\rangle \text{ such that } M \text{ is a Turing machine that halts on input } x\ \}.$$