Exercise 1 (4pt). Consider the following problem EXACTLY-2-IN-4-SAT:

**Instance:** A propositional formula $\varphi$ in 4CNF—that is, a formula of the form $\varphi = c_1 \land \cdots \land c_m$, where each $c_i$ is of the form $c_i = l_{i,1} \lor l_{i,2} \lor l_{i,3} \lor l_{i,4}$, where $l_{i,1}, l_{i,2}, l_{i,3}, l_{i,4}$ are propositional literals.

**Question:** Is there a truth assignment $\alpha$ to the variables occurring in $\varphi$ that sets exactly 2 literals in each clause $c_i$ to true?

Prove that EXACTLY-2-IN-4-SAT is NP-complete—that is, prove that is in NP and that it is NP-hard.

- **Hint:** reduce from a suitable variant of 3SAT.\(^1\)

Exercise 2 (2pt). Show that NP ⊆ EXP.

Exercise 3 (2pt). Let $A$ be an NP-complete language. Let $p$ be a polynomial and let $M_A$ be a polynomial-time Turing machine such that, for all $x \in \{0,1\}^*$:

$$x \in A \text{ if and only if there exists some } u \in \{0,1\}^{p(|x|)} \text{ such that } M_A(x,u) = 1.$$  

(a) Define the set $B = \{ \langle x, z \rangle \mid \text{there exists } z' \in \{0,1\}^* \text{ such that } |zz'| \leq p(|x|) \text{ and } M_A(x,zz') = 1 \}$. Prove that $B$ is in NP.

(b) Suppose that we have access to $A$ as an oracle. Basically this means that we have a subroutine that, given a string $y$, tells in a single step whether $y \in A$. (See Definition 3.4 of Arora & Barak, 2009.) Construct a polynomial-time Turing machine $M_{\text{search}}$ that, given $x \in \{0,1\}^*$, if $x \in A$ outputs a string $u$ such that $M_A(x,u) = 1$ and if $x \notin A$ outputs 0. Use (a).

Exercise 4 (2pt). Let $A$ be a language. When a Turing machine $M$ has access to the oracle $A$, we write $M^A$. We say that $A$ is auto-reducible if there is a polynomial-time Turing machine $M$ such that for all $x \in \{0,1\}^*$:

$$x \in A \text{ if and only if } M^A(x) = 1,$$

with the special requirement that on input $x$ the Turing machine $M^A$ is not allowed to query the oracle $A$ for $x$.

Suppose that $A$ is NP-complete. Prove that $A$ is auto-reducible. Use Exercise 3.

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