Characterizations of Sequential Scoring Rules in Rank Aggregation

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Abstract

In rank aggregation, the goal is to combine multiple input rankings into a single output ranking. In computational social choice, this problem is typically modeled via social preference functions (SPFs), which map profiles of strict and complete input rankings to non-empty sets of output rankings. In this paper, we provide axiomatic characterizations of four sequential SPFs, namely Instant Runoff, Sequential Plurality, and their Veto-based analogues. Specifically, our characterizations combine relaxations of the axioms that feature in Young's (1988) influential characterization of the Kemeny rule with novel proportionality and stability properties. These "Kemeny-style" characterizations of sequential scoring rules show that they are serious alternatives to the Kemeny rule in rank aggregation, as they share the same normative basis while addressing known stability and proportionality failures of the Kemeny rule.

1 Introduction

In *rank aggregation*, the task is to combine multiple input rankings into a single output ranking. For example, this problem arises when a hiring committee aggregates the preferences of the committee members over the applicants into a single ranking (Kuhlman and Rundensteiner 2020), when computing an overall ranking of AI models with respect to their performance on multiple tasks (Lanctot et al. 2025a,b), or when ranking athletes based on the results of multiple races (Kondratev, Ianovski, and Nesterov 2024). The problem of rank aggregation also lies at the heart of *social choice theory*, as Arrow's (1951) impossibility theorem was originally stated for this setting. In this field, rank aggregation is formalized via *social preference functions (SPFs)*, which map profiles of strict and complete input rankings over some candidates to non-empty sets of winning rankings (ideally, there is one winning ranking, but there may be ties).

In social choice theory and many applications of rank aggregation, the predominant SPF is the Kemeny rule, which chooses the ranking that minimizes the total swap distance to the input rankings (Kemeny 1959). The status of this SPF is typically justified by its appealing axiomatic properties and characterizations (e.g., Young and Levenglick 1978; Young 1988, 1995; Lederer 2024). However, the Kemeny rule has also several drawbacks. For example, it is well-known that computing the Kemeny ranking is NP-hard (Bartholdi, III, Tovey, and Trick 1989; Hemaspaandra, Spakowski, and Vogel 2005). Moreover, Lederer, Peters, and Wąs (2024) have recently observed that the Kemeny rule is not suitable if a proportional output ranking is desired, and an example by Eberl and Lederer (2026) shows that a small change in the input profile can almost reverse its output ranking.

Motivated by such drawbacks, Boehmer, Bredereck, and Peters (2026) have studied variants of positional scoring rules for rank aggregation. In particular, these authors found that sequential scoring rules, which either repeatedly top-rank the candidate with maximal total score or bottom-rank the candidate with minimal score, satisfy similar axiomatic properties as the Kemeny rule. However, while these authors as well as Freeman, Brill, and Conitzer (2014) provide characterization of such sequential scoring rules, these rely on a technical and unconvincing axiom called reinforcement at the top (resp. bottom). Thus, a clean characterization of such sequential scoring rules is missing. This is also acknowledged by Boehmer, Bredereck, and Peters (2026, p. 14) who write that it is an interesting open problem "whether Sequential-Loser rules can be characterized based on a reinforcement-like".

Contribution. In this paper, we derive "Kemeny-style" characterizations of four sequential scoring rules, namely Instant Runoff, Sequential Plurality, Veto Runoff, and Sequential Veto. Roughly, *Instant Runoff* (resp. *Sequential Plurality*) repeatedly puts the candidate that is top-ranked by the fewest (resp. most) voters at the lowest (resp. highest) available position of the output ranking, deletes this candidate from the profile, and updates the scores of the remaining candidates. *Veto Runoff* and *Sequential Veto* work analogously but focus on the bottom-ranked candidates. All four of these SPFs belong to the classes of *sequential-loser and sequential-winner scoring rules* by Boehmer, Bredereck, and Peters (2026), which respectively generalize Instant Runoff and Sequential Plurality by allowing for arbitrary positional score functions instead of the plurality score.

For our characterizations, we observe that all our SPFs satisfy three basic properties called anonymity, neutrality, and continuity. Henceforth, we will refer to the conjunction of these axioms by standardness. Moreover, a result by Boehmer, Bredereck, and Peters (2026) implies that Sequential Plurality and Sequential Veto satisfy *independence at the top (top IIA)*, whereas Instant Runoff and Veto Runoff satisfy *independence at the bottom (bottom IIA)*. Roughly, these axioms require that, if we delete the top-ranked (resp. bottom-ranked) candidate of a winning ranking from the profile, the truncated winning ranking remains chosen for the reduced profile. For example, bottom-IIA is desirable for evaluating AI models with respect to different benchmarks, because it ensures that introducing bad models does not affect the order of the top-ranked ones. Lastly, we prove in Proposition 1 that all sequential-winner and sequential-loser scoring rules satisfy a population consistency axiom called quasi-reinforcement but fail the more common notion of reinforcement (e.g., Smith 1973; Young 1975). Roughly, *reinforcement* requires that, if the SPF chooses some common rankings for two disjoint profiles, it should select precisely these common rankings for the joint profile. By contrast, *quasi-reinforcement* applies the same reasoning only if the rankings chosen for one profile are a subset of those chosen for the other profile.

Based on these conditions, we prove the following results (see also Table 1 for a concise overview of our contribution).

(1) Instant Runoff is the only SPF that satisfies standardness, quasi-reinforcement, bottom IIA, and minority protection (Theorem 1). The last axiom is a mild proportionality condition which requires that, if more than $\frac{1}{m}$ of the voters top-rank a candidate, this candidate should not be bottom-ranked in an output ranking. Here, m is the number of candidates in the considered profile.

(2) Sequential Plurality is the only SPF that satisfies standardness, quasi-reinforcement, top IIA, faithfulness, and prefix-stability (Theorem 2). Here, faithfulness refers to a basic condition, requiring that, if there is a single voter, we need to choose his input ranking as the output ranking. Prefix-stability, on the other hand, demands that the selected prefix of the chosen ranking is immune to small changes in the voters' rankings below the considered prefix: if a voter swaps two candidates that he ranks below all the candidates in the considered prefix, we still need to choose a ranking with the same prefix.

(3) Our characterizations of Sequential Veto and Veto Runoff are dual to our results on Instant Runoff and Sequential Plurality, because these SPFs are, in a formal sense, reverse to each other. Using this duality, we derive that Sequential Veto is the only SPF that satisfies standardness, quasi-reinforcement, top IIA, and that candidates that are bottom-ranked by more than $\frac{1}{m}$ of the voters are never top-ranked in an output ranking. Further, Veto Runoff is the only SPF that satisfies standardness, quasi-reinforcement, bottom IIA, faithfulness, and suffix-stability (Corollary 1).

These results can be directly compared to a variant of Young's (1988) characterization of the Kemeny rule due to Lederer (2024), who has shown that the Kemeny rule is the only SPF satisfying standardness, faithfulness, reinforcement, and both top and bottom IIA. In comparison, our results only replace some of these conditions with proportionality and stability axioms. Thus, our paper makes a strong case for sequential scoring rules, as they share the normative basis of the Kemeny rule while addressing its drawback regarding proportionality and stability notions.

Rule	Reinf.	IIA	Extra conditions
Kemeny (Lederer 2024)	Full	Top and bottom	Faithfulness
Instant Runoff (Thm. 1)	Quasi	Bottom	Minority protection
Veto Runoff (Cor. 1)	Quasi	Bottom	Faithfulness Suffix-stability
Seq. Plurality (Thm. 2)	Quasi	Top	Faithfulness Prefix-stability
Seq. Veto (Cor. 1)	Quasi	Top	Minority rejection

Table 1: Summary of our results. Each SPF is characterized by the given axioms and standardness. In the "Reinf." column, "Full" refers to the classical reinforcement notion, while "Quasi" refers to quasi-reinforcement. In the "IIA" column, "Top" and "Bottom" indicate top and bottom IIA, respectively.

Related work. For a general overview of rank aggregation in (computational) social choice, we refer to the books by Arrow, Sen, and Suzumura (2011) and Brandt et al. (2016). In the social choice literature, our paper is related to the characterizations of positional scoring rules by Smith (1973) and Young (1975) as well as characterizations of the Kemeny rule (Young and Levenglick 1978; Young 1988, 1995). In particular, these results are also driven by reinforcement and IIA conditions. We further note that, analogous to our work, these classic papers inspired numerous characterizations in modern voting settings (e.g., Lackner and Skowron 2018; Skowron, Faliszewski, and Slinko 2019; Dong and Lederer 2023, 2024; Lederer, Peters, and Wąs 2024).

The two closest papers to ours are due to Freeman, Brill, and Conitzer (2014) and Boehmer, Bredereck, and Peters (2026). These authors also characterize sequential scoring rules, but their results rely on axioms called reinforcement at the top and bottom, respectively. For example, reinforcement at the bottom requires that, when some common candidates are bottom-ranked by the output rankings for two disjoint profiles, precisely these candidates are bottom-ranked by the output rankings for the joint profile. Since we struggle to see why this condition would be desirable, we believe that a more natural characterization of such sequential scoring rules, as provided by this paper, is important. This sentiment is also echoed by Boehmer, Bredereck, and Peters (2026) who explicitly name characterizations of their rules based on a global reinforcement property as an open problem.

Finally, our work addresses two topics that have recently gained increased attention. Firstly, there is a growing stream of works that, motivated by novel applications such as ranking AI models, study rank aggregation through the lens of computational social choice (e.g., Lederer, Peters, and Wąs 2024; Aziz et al. 2025; Eberl and Lederer 2026). Secondly, our paper contributes to the body of works that reason in favor of Instant Runoff (e.g., Tomlinson, Ugander, and Kleinberg 2023, 2024; Delemazure and Peters 2024; Durand 2025).

2 Preliminaries

In this paper, we will define SPFs for variable electorates and variable agendas. Hence, let $\mathbb{N} = \{1, 2, \dots\}$ denote the infinite set of all possible voters and let $C = \{c_1, \dots, c_m\}$ denote the finite set of possible candidates. An *electorate* is a finite and non-empty subset of $\mathbb{N}$ and an *agenda* is a finite and non-empty subset of C. Less formally, an electorate is the set of voters who actually vote and the agenda is the set of candidates that are available. Let $\mathcal{F}(X) = \{Y \subseteq X : Y \text{ is finite and non-empty}\}$ denote the set of all finite and non-empty subsets of a given set X. Then, the set of all electorates is $\mathcal{F}(\mathbb{N})$ and the set of all agendas is $\mathcal{F}(C)$.

Given an electorate $N \in \mathcal{F}(\mathbb{N})$ and agenda $X \in \mathcal{F}(C)$, we assume that every voter $i \in N$ reports a *ranking* $\succ_i$ over the candidates in X. Formally, a ranking is a strict linear order over the given agenda. We define the set of all rankings over an agenda $X \in \mathcal{F}(C)$ by $\mathcal{R}(X)$. We write rankings as sequences of candidates; for example, $\succ = x_1x_2x_3$ is the ranking where x_1 is preferred to x_2 and x_2 to x_3. For all agendas $X, Y \in \mathcal{F}(C)$ with $Y \subseteq X$ and rankings $\succ \in \mathcal{R}(X)$, we denote by $\succ|_Y = \succ \cap Y^2$ the restriction of $\succ$ to Y.

Next, a *(ranking) profile* $R = (\succ_i)_{i \in N}$ for an electorate $N \in \mathcal{F}(\mathbb{N})$ and agenda $X \in \mathcal{F}(C)$ assigns every voter $i \in N$ to a ranking $\succ_i \in \mathcal{R}(X)$. Consequently, the set of all ranking profiles for an electorate N and agenda X is $\mathcal{R}(X)^N$ and the set of all ranking profiles is $\mathcal{R}^* = \bigcup_{N \in \mathcal{F}(\mathbb{N})} \bigcup_{X \in \mathcal{F}(C)} \mathcal{R}(X)^N$. Given a profile $R \in \mathcal{R}^*$, we let N_R denote the set of voters that report a ranking in R and C_R the set of candidates that the voters rank. Further, $R|_X = (\succ_i|_X)_{i \in N_R}$ is the restriction of R to the candidates in $X \subseteq C_R$. Finally, given two profiles R, R' with $N_R \cap N_{R'} = \emptyset$ and $C_R = C_{R'}$, we define by $R'' = R + R'$ the profile on $N_{R''} = N_R \cup N_{R'}$ such that $\succ''_i = \succ_i$ for all $i \in N_R$ and $\succ''_i = \succ_i'$ for all $i \in N_{R'}$.

For deciding which ranking(s) to choose for a given profile, we use *social preference functions (SPFs)*. These functions map every profile $R \in \mathcal{R}^*$ to a non-empty set of rankings $W \subseteq \mathcal{R}(C_R)$. We emphasize that SPFs may choose multiple tied winning rankings, which is necessary to, e.g., allow for fair outcomes in symmetric profiles. Further, to improve readability, we use the convention that $\succ$ refers to input rankings and $\rhd$ to output rankings.

2.1 Sequential Scoring Rules

In this paper, we will focus on four SPFs called Instant Runoff, Sequential Plurality, Sequential Veto and Veto Runoff. To define these SPFs, we will introduce the classes of *sequential-winner scoring rules* and *sequential-loser scoring rules* due to Boehmer, Bredereck, and Peters (2026). Intuitively, these rules compute the winning ranking sequentially, either top-down or bottom-up, by repeatedly selecting the candidate that maximizes or minimizes a score function.

To formalize this, we define the rank of a candidate $x \in X$ with respect to a ranking $\succ \in \mathcal{R}(X)$ by $r(\succ, x) = 1 + |\{y \in X \setminus \{x\} : y \succ x\}|$. Further, we let $t(\succ)$ and $b(\succ)$ respectively denote the top-ranked and bottom-ranked candidates in a ranking $\succ$. For example, if $\succ = x_1x_2x_3x_4$, then $t(\succ) = x_1$, $b(\succ) = x_4$, and $r(\succ, x_2) = 2$. Next, we call a function $s : \{1, \dots, |C|\} \times \{1, \dots, |C|\} \to \mathbb{R}$ a *score function* if $s(1, k) \geq s(2, k) \geq \dots \geq s(k, k)$ and $s(1, k) > s(k, k)$ for all $k \in \{2, \dots, |C|\}$. Intuitively, $s(i, k)$ is the score a voter assigns to this i-th ranked candidate if there are k candidates in the agenda. The total score of a candidate x in a profile R is given by $s(R, x) = \sum_{i \in N_R} s(r(\succ_i, x), |C_R|)$. Lastly, we define the sets of candidates that respectively maximize and minimize the total score in a profile R by $g_s^+(R) = \arg\max_{x \in C_R} s(R, x)$ and $g_s^-(R) = \arg\min_{x \in C_R} s(R, x)$.

Now, both sequential-winner and sequential-loser scoring rules are defined by a score function and repeatedly top-rank (resp. bottom-rank) the candidate with maximal (resp. minimal) total score and delete this candidate from the profile.

Definition 1. An SPF f is a *sequential-winner scoring rule* if there is a score function s such that, for all profiles $R \in \mathcal{R}^*$, it holds that $\rhd \in f(R)$ if and only if $t(\rhd) \in g_s^+(R)$ and $\rhd|_{C_R \setminus \{t(\rhd)\}} \in f(R|_{C_R \setminus \{t(\rhd)\}})$ when $|C_R| > 1$.

Definition 2. An SPF f is a *sequential-loser scoring rule* if there is a score function s such that, for all profiles $R \in \mathcal{R}^*$, it holds that $\rhd \in f(R)$ if and only if $b(\rhd) \in g_s^-(R)$ and $\rhd|_{C_R \setminus \{b(\rhd)\}} \in f(R|_{C_R \setminus \{b(\rhd)\}})$ when $|C_R| > 1$.

Finally, *Instant Runoff* and *Sequential Plurality* are the sequential-loser and sequential-winner scoring rules defined by the Plurality score function $s_{\texttt{PL}}$, given by $s_{\texttt{PL}}(1, k) = 1$ and $s_{\texttt{PL}}(i, k) = 0$ for all $k \in \{1, \dots, |C|\}$, $i \in \{2, \dots, k\}$. On the other hand, *Veto Runoff* (which is also known as Coombs rule (Coombs 1964)) and *Sequential Veto* are the sequential-loser and sequential-winner scoring rules defined by the Veto score function $s_{\texttt{V}}$, given by $s_{\texttt{V}}(i, k) = 1$ and $s_{\texttt{V}}(k, k) = 0$ for all $k \in \{1, \dots, |C|\}$, $i \in \{1, \dots, k-1\}$.

Example 1. To illustrate our SPFs, consider the following profile with 3 candidates and 12 voters.

5: $x_1x_3x_2$ 4: $x_2x_3x_1$ 3: $x_3x_2x_1$

For this profile, Sequential Plurality selects the ranking $x_1x_3x_2$ because x_1 is top-ranked by the most voters and, after deleting this candidate, x_3 is top-ranked by more voters than x_2. By contrast, Instant Runoff selects the ranking $x_2x_1x_3$ because x_3 is top-ranked by the fewest voters and, after deleting x_3, x_2 is top-ranked by 7 voters but x_1 only by 5 voters. It can be further verified that Sequential Veto and Veto Runoff both select $x_3x_2x_1$ for this profile.

2.2 Axioms

We will next introduce the axioms that form the basis of our characterizations. We note that all of the conditions below are well-known in the literature (e.g., Smith 1973; Young 1975; Lederer 2024; Boehmer, Bredereck, and Peters 2026).

Standardness. First, we will introduce three basic axioms called anonymity, neutrality, and continuity. Since these properties are satisfied by all commonly studied SPFs, we refer to their conjunction as standardness. In more detail, we say an SPF is *standard* if it satisfies the following three conditions:

- An SPF f is *anonymous* if $f(R) = f(\pi(R))$ for all profiles $R \in \mathcal{R}^*$ and permutations $\pi : \mathbb{N} \to \mathbb{N}$. Here, $R' = \pi(R)$ is the profile given by $R'_{\pi(i)} = R_i$ for all $i \in N_R$. Intuitively, this axiom states that all voters should be treated equally.

- An SPF f is *neutral* if $f(\tau(R)) = \{\tau(\rhd) \colon \rhd \in f(R)\}$ for all profiles $R \in \mathcal{R}^*$ and permutations $\tau : C_R \to C_R$. Here, $\succ' = \tau(\succ)$ is the ranking given by $\tau(x) \succ' \tau(y)$ if and only if $x \succ y$ for all $x, y \in C_R$, and $\tau(R)$ is the profile defined by $\tau(R) = (\tau(\succ_i))_{i \in N_R}$. Less formally, this condition demands that all candidates are treated equally.
- An SPF f is *continuous* if for all profiles $R, R' \in \mathcal{R}^*$ with $C_R = C_{R'}$, there is $\lambda \in \mathbb{N}$ such that $f(\lambda R + R') \subseteq f(R)$. Here, λR is a profile that consists of λ copies of R. Intuitively, this axiom means that we can always marginalize the influence of a group of voters by cloning another electorate sufficiently often.

Reinforcement and Quasi-reinforcement. Reinforcement is a well-known population consistency condition that features in many prominent results in social choice theory (e.g., Smith 1973; Young 1975; Fishburn 1978; Lackner and Skowron 2018; Dong and Lederer 2024). This axiom requires that when combining two profiles with common winning rankings, precisely these common winning rankings are chosen for the joint profile. More formally, an SPF f is *reinforcing* if $f(R + R') = f(R) \cap f(R')$ for all profiles $R, R' \in \mathcal{R}^*$ such that $N_R \cap N_{R'} = \emptyset$, $C_R = C'_R$, and $f(R) \cap f(R') \neq \emptyset$. Intuitively, this condition states that, if some ranking is considered optimal for two disjoint profiles, then it should also be considered optimal in the joint profile.

As we will show in Proposition 1, all sequential-winner and sequential-loser scoring rules fail reinforcement. However, they satisfy a variant of this condition that we call quasi-reinforcement. This axiom weakens reinforcement by requiring that the set of winning rankings for one profile is contained in the set of winning rankings for the other profile. Hence, an SPF f satisfies *quasi-reinforcement* if $f(R + R') = f(R)$ for all profiles $R, R' \in \mathcal{R}^*$ such that $N_R \cap N_{R'} = \emptyset$, $C_R = C_{R'}$, and $f(R) \subseteq f(R')$. In practice, the difference between quasi-reinforcement and reinforcement seems negligible, because these axioms coincide when f chooses a single output ranking for R or R'.

Top and Bottom IIA. Top IIA and Bottom IIA (aka independence at the top and bottom) are variants of Arrow's independence of irrelevant alternatives (Arrow 1951) and have been considered before by, e.g., Freeman, Brill, and Conitzer (2014) and Boehmer, Bredereck, and Peters (2026). In particular, these conditions require that, when deleting the top-ranked (resp. bottom-ranked) candidate of a winning ranking from the input profile, the truncated winning ranking should remain chosen. Formally, we say an SPF f satisfies

- *top IIA* if $\rhd \in f(R)$ implies that $\rhd|_{C_R \setminus \{t(\rhd)\}} \in f(R|_{C_R \setminus \{t(\rhd)\}})$ for all profiles $R \in \mathcal{R}^*$ with $|C_R| > 1$.
- *bottom IIA* if $\rhd \in f(R)$ implies that $\rhd|_{C_R \setminus \{b(\rhd)\}} \in f(R|_{C_R \setminus \{b(\rhd)\}})$ for all profiles $R \in \mathcal{R}^*$ with $|C_R| > 1$.

We note that the conjunction of top and bottom IIA is sometimes called *local IIA* (Young 1988; Lederer 2024). Further, both conditions seem important in practice: bottom-IIA ensures that introducing bad candidates does not affect the order of the top-ranked candidates, whereas top-IIA is useful for dynamic settings (e.g., hiring committees) where the top-ranked candidate may not be available in the end.

3 Results

We are now ready to present our results. Specifically, in Section 3.1, we first recall some known properties of sequential-winner and sequential-loser scoring rules and show that they also satisfy quasi-reinforcement. After this, we prove our characterizations of Instant Runoff, Sequential Plurality, and the Veto-based rules in Sections 3.2 to 3.4, respectively. Due space restrictions, we defer the proofs of all results to the supplementary material and discuss proof sketches instead.

3.1 Properties of Sequential Scoring Rules

For proving our characterizations, we first recall some of the properties of sequential-winner and sequential-loser scoring rules. In particular, in Claim (1) of Proposition 1, we merely restate insights of Boehmer, Bredereck, and Peters (2026). By contrast, in Claim (2), we show that all sequential-winner and sequential-loser scoring rules satisfy quasi-reinforcement but fail reinforcement.[1] This result clarifies an ambiguity by Boehmer et al. who incorrectly claim in their conference paper that their rules satisfy reinforcement, but later report in the journal version without explanation that the rules fail this axiom (Boehmer, Bredereck, and Peters 2023, 2026). The full proof of the following result can be found in Appendix A.2.

Proposition 1. *The following claims are true.*

(1) All sequential-winner scoring rules are standard and satisfy top-IIA. Further, all sequential-loser scoring rules are standard and satisfy bottom-IIA.

(2) All sequential-winner and sequential-loser scoring rules satisfy quasi-reinforcement but fail reinforcement.

Proof Sketch. Since Claim (1) was shown by Boehmer, Bredereck, and Peters (2026), we focus on Claim (2). Further, we restrict attention to sequential-winner scoring rules as the arguments for sequential-loser scoring rules are symmetric. Hence, let s denote a score function and let f be the induced sequential-winner scoring rule. To show that f fails reinforcement, we construct two profiles R and R' such that *(i)* $f(R)$ and $f(R')$ intersect in a single ranking $\rhd^*$ and *(ii)* there are rankings $\rhd \in f(R)$ and $\rhd' \in f(R')$ with $t(\rhd) = t(\rhd')$ and $t(\rhd) \neq t(\rhd^*)$. For these profiles, reinforcement requires that $f(R + R') = \{\rhd^*\}$. However, the candidate $t(\rhd)$ maximizes the total score $s(R + R', t(\rhd)) = s(R, t(\rhd)) + s(R', t(\rhd))$, so a ranking that top-ranks this candidate must be in $f(R + R')$.

Next, to show that f satisfies quasi-reinforcement, let R and R' be profiles such that $N_R \cap N_{R'} = \emptyset$, $C_R = C_{R'}$, and $f(R) \subseteq f(R')$. First, let $\rhd = x_1 \dots x_k$ be a ranking in $f(R) \subseteq f(R')$ and set $C_i = \{x_i, \dots, x_k\}$ for all $i \in \{1, \dots, k\}$. By definition, x_i maximizes the scores $s(R|_{C_i}, x_i)$ and $s(R'|_{C_i}, x_i)$ for all $i \in \{1, \dots, k\}$. Hence, it also maximizes $s(R + R'|_{C_i}, x_i)$ for all $i \in \{1, \dots, k\}$, which implies that $\rhd \in f(R + R')$. Next, let $\rhd' = y_1 \dots y_k$ be a ranking that is not in $f(R)$. Further, let $\rhd = x_1 \dots x_k$ denote the ranking in $f(R)$ that has the longest matching prefix to $\rhd'$, i.e., $\rhd$ maximizes the integer j with $x_i = y_i$

[1] In an unpublished manuscript, Merlin (1996) also observed that sequential-loser scoring rules are quasi-reinforcing.

for all $i \in \{1, \dots, j-1\}$. Defining C_i as before, this means that $s(R|_{C_j}, x_j) > s(R|_{C_j}, y_j)$. Indeed, x_j maximizes this score by definition and, if y_j was also a maximizer, there would be a ranking in $f(R)$ with longer matching prefix. Because $f(R) \subseteq f(R')$, it holds also that $\rhd \in f(R')$ and thus $s(R'|_{C_j}, x_j) \geq s(R'|_{C_j}, y_j)$. Combining our insights shows that $\rhd' \notin f(R+R')$ since $s(R+R'|_{C_j}, x_j) > s(R+R'|_{C_j}, y_j)$. This proves that $f(R+R') = f(R)$. $\square$

Remark 1. One may think that the axioms in Proposition 1 are close to characterizing sequential-winner and sequential-loser scoring rules. Unfortunately, this is not the case since there is a large class of SPFs that satisfy these conditions and even full reinforcement. To introduce these, we define the top-score $s_t(R, \rhd)$ of a ranking $\rhd = x_1 \dots x_k$ in a profile R by $s_t(R, \rhd) = \sum_{i=1}^{k} s(R|_{\{x_i, \dots, x_k\}}, x_i)$, where s is an arbitrary score function. Every rule that picks all rankings that maximize the top score for a given score function satisfy our axioms. Since the Kemeny score is the top score of the Borda scoring function, this class contains the Kemeny rule and many variants of it. We hence believe that it is of independent interest. Moreover, one can define an analogous bottom score and the rules that choose the rankings minimizing this score satisfy standardness, reinforcement, and bottom IIA.

3.2 Characterization of Instant Runoff

We now turn to our first main result, the characterization of Instant Runoff. Additionally to the axioms discussed so far, we need one more condition for this result, which we call minority protection. The idea of this axiom is that a candidate that is top-ranked by more than $\frac{1}{|C_R|}$ voters in a profile R should not be bottom-ranked by the output ranking.

Minority Protection An SPF f satisfies *minority protection* if $b(\rhd) \neq x$ for all profiles $R \in \mathcal{R}^*$, candidates $x \in C_R$, and rankings $\rhd \in f(R)$ such that $|C_R| > 1$ and $|\{i \in N : t(\succ_i) = x\}| > \frac{|N_R|}{|C_R|}$.

Minority protection is a weakening of a recently suggested axiom called rank-PSC Aziz et al. (2025), which adapts proportionality for solid coalitions (PSC), a well-known proportionality condition for committee voting (Dummett 1984; Aziz and Lee 2021), to rank aggregation. Hence, our characterization establishes that, axiomatically, Instant Runoff can be seen as a more proportional variant of the Kemeny rule. Moreover, it fits in with a recent trend of analyzing proportional representation in rank aggregation (e.g., Lederer, Peters, and Wąs 2024; Aziz et al. 2025; Lederer 2025). The full proof of Theorem 1 can be found in Appendix A.5.

Theorem 1. *Instant Runoff is the only SPF that satisfies standardness, quasi-reinforcement, bottom IIA, and minority protection.*

Proof Sketch. For the direction from left to right, we note that Instant Runoff satisfies standardness, quasi-reinforcement, and bottom IIA since it is a sequential-loser scoring rule. Further, this SPF satisfies minority protection, because, by the pigeon hole principle, there is in every profile R a candidate that is top-ranked by at most $\frac{|N_R|}{|C_R|}$ voters. Hence, if a candidate x is top-ranked by strictly more than $\frac{|N_R|}{|C_R|}$ voters in a profile R, it cannot have the minimal plurality score, so it is not bottom-ranked by Instant Runoff.

For the other direction, let f denote an SPF that satisfies all given axioms. As a first step, we show that quasi-reinforcement suffices to apply a hyperplane argument, as first demonstrated by Young (1975). Specifically, we prove that, for every agenda $X \in \mathcal{F}(C)$, f can be extended to a function $\hat{g} : \mathbb{Q}^{|X|!} \to \mathcal{F}(\mathcal{R}(X))$, which intuitively computes f (for profiles over X) only based on how often each ranking in $\mathcal{R}(X)$ is reported. Based on this function, we define by $D_\rhd = \{v \in \mathbb{Q}^{|X|!} : \rhd \in \hat{g}(v)\}$ the preimage of every $\rhd \in \mathcal{R}(X)$ and by $\bar{D}_\rhd$ the closure of this set with respect to $\mathbb{R}^{|X|!}$. By quasi-reinforcement, it follows that these sets are convex cones, and we show that they have disjoint interiors. Hence, we can use the hyperplane argument for convex sets to find non-zero vectors $u^{\rhd,\rhd'}$ for all distinct $\rhd, \rhd' \in \mathcal{R}(X)$ such that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_{\rhd'}$ (where $vu^{\rhd,\rhd'}$ denotes the standard scalar product between two vectors). Further, these vectors fully describe the sets $\bar{D}_\rhd$ and thus f because $\bar{D}_\rhd = \{v \in \mathbb{R}^{|X|!} : \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} : vu^{\rhd,\rhd'} \geq 0\}$.

We next analyze the vectors $u^{\rhd,\rhd'}$ in more detail. To this end, we note that minority protection and continuity almost immediately imply that f coincides with Instant Runoff when $|X| = 2$. Hence, we let X be an agenda with $|X| \geq 3$ and we inductively assume that f is Instant Runoff for every agenda $X' \subsetneq X$. In this case, we say two rankings $\rhd, \rhd' \in \mathcal{R}(X)$ are prefix-consistent if, whenever $\rhd$ and $\rhd'$ agree on the best $k \in \{2, \dots, m-1\}$ candidates, they also agree on the order of these candidates. We use minority protection and our induction hypothesis to construct for every pair of prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$ a profile R such that *(i)* $\{\rhd, \rhd'\} \subseteq f(R)$, *(ii)* no ranking $\rhd'' \in f(R) \setminus \{\rhd, \rhd'\}$ bottom-ranks $b(\rhd)$ or $b(\rhd')$, and *(iii)* every candidate is bottom-ranked by some ranking in $f(R)$. This profile is of interest because its corresponding vector v belongs to both $\bar{D}_\rhd$ and $\bar{D}_{\rhd'}$, so $vu^{\rhd,\rhd'} = 0$. By analyzing this profile and variants thereof, we derive that $u_i^{\rhd,\rhd'}$ is -1 if the corresponding input ranking $\succ_i$ top-ranks $b(\rhd)$, $+1$ if $\succ_i$ top-ranks $b(\rhd')$, and 0 otherwise. Less formally, this means that $\rhd$ can only be chosen for a profile if there is no prefix-consistent ranking $\rhd'$ whose bottom-ranked candidate $b(\rhd')$ is top-ranked by less voters than $b(\rhd)$. Thus, for every ranking $\rhd \in f(R)$, the bottom-ranked candidate $b(\rhd)$ must have the minimal Plurality score. By combining this insight with bottom IIA and our induction hypothesis, it follows that f is Instant Runoff. $\square$

Remark 2. Most of the axioms of Theorem 1 are necessary for the characterization. When only omitting minority protection, the Kemeny rule satisfy all remaining axioms. The reverse sequential rules by Aziz et al. (2025), such as reverse sequential STV, satisfy all axioms of Theorem 1 except quasi-reinforcement. The plurality score (which ranks all candidates in decreasing order of their plurality scores in the original profile) only fails bottom IIA. Without standardness—or, more specifically, continuity—we can refine Instant Runoff by, e.g., only choosing the rankings among those selected by

Instant Runoff that maximize the Kemeny score. By contrast, we show in Section A.6 that anonymity is implied by the other axioms of Theorem 1, so it is not necessary for our result. Lastly, we leave it open whether neutrality is required for our characterization.

Remark 3. Minority protection is a weak proportionality axiom, which makes the "only if" part of Theorem 1 stronger. On the other hand, Instant Runoff satisfies a more demanding proportionality condition: for all profiles R, sets of voters $S \subseteq N_R$, integers k, and sets of candidates $Y \subseteq C_R$ such that $|S| > \frac{|N_R|}{k}$ and all voters in S prefer all candidates in Y to all candidates in $C_R \setminus Y$, every ranking $\rhd$ chosen by Instant Runoff places a candidate $y \in Y$ among the top $k-1$ candidates of the output ranking.

Remark 4. Instant Runoff has recently attracted significant attention. For example, Tomlinson, Ugander, and Kleinberg (2024) have shown that Instant Runoff has a moderating effect and Durand (2023, 2025) that it is very resilient to coalitional manipulations. Moreover, Tomlinson, Ugander, and Kleinberg (2023) have analyzed the effect of ballot length for this SPF, while Delemazure and Peters (2024) study the effect of weak input rankings. Theorem 1 thus relates to a significant amount of recent works and provides another strong argument in favor of Instant Runoff.

3.3 Characterization of Sequential Plurality

We next turn to our second main result, our characterization of Sequential Plurality. For this result, we will use two more axioms, namely faithfulness and prefix-stability.

Faithfulness. Faithfulness is a mild condition and requires that, if there is a single voter, the SPF should simply return the input ranking of the voter. Formally, an SPF f is faithful if $f(R) = \{\succ_i\}$ for all profiles $R \in \mathcal{R}^*$ that consist only of a single voter i with ranking $\succ_i$.

Prefix-stability. The idea of prefix-stability is that the prefix of the chosen ranking should be independent of voters' ordering of the candidates that are ranked below all candidates in the prefix. To formalize this, we define by $\texttt{Pref}(\rhd, \ell)$ the length-ℓ prefix of a ranking $\rhd$. Further, given a ranking $\succ$ and a length-ℓ prefix $\texttt{Pref}(\rhd, \ell)$ of some ranking $\rhd$, we define the covering prefix $\texttt{CP}(\succ, \texttt{Pref}(\rhd, \ell))$ of $\succ$ regarding $\texttt{Pref}(\rhd, \ell)$ as the shortest prefix of $\succ$ that contains all candidates in $\texttt{Pref}(\rhd, \ell)$. For example, if $\rhd = x_1x_2x_3x_4$ and $\succ = x_2x_3x_1x_4$, then $\texttt{Pref}(\succ, 2) = x_1x_2$ and $\texttt{CP}(\succ, \texttt{Pref}(\rhd, 2)) = x_2x_3x_1$. Now, an SPF is *prefix-stable* if $\rhd \in f(R)$ implies that there is a ranking $\rhd' \in f(R')$ with $\texttt{Pref}(\rhd, \ell) = \texttt{Pref}(\rhd', \ell)$ for all profiles $R, R' \in \mathcal{R}^*$, rankings $\rhd \in \mathcal{R}(C_R)$, and integers $\ell \in \{1, \dots |C_R|\}$ such that $N_R = N_{R'}$, $C_R = C_{R'}$, and $\texttt{CP}(\succ_i, \texttt{Pref}(\rhd, \ell)) = \texttt{CP}(\succ'_i, \texttt{Pref}(\rhd, \ell))$ for all $i \in N_R$.

Prefix-stability is desirable whenever the top-ranked candidates are of particular interest. For example, in the hiring committee example, it seems useful that the top-ranked candidate x does not change if a voter swap two other candidates at the bottom of his ranking. However, a recent example by Eberl and Lederer (2026, proof of Theorem 3) shows that the Kemeny rule severely fails this axiom because a single swap at the bottom of an input ranking can almost reverse the output ranking of the Kemeny rule. By contrast, we will characterize Sequential Plurality based on prefix-stability.

Theorem 2. *Sequential Plurality is the only SPF that satisfies standardness, quasi-reinforcement, top IIA, faithfulness, and prefix-stability.*

Proof Sketch. For the direction from left to right, we observe that Sequential Plurality is a sequential-winner scoring rule, so it satisfies standardness, quasi-reinforcement, and top-IIA by Proposition 1. Further, it is easy to see that Sequential Plurality is faithful. Finally, for prefix-stability, let R and R' be two profiles, $\rhd \in \mathcal{R}(C_R)$ a ranking, and $\ell \in \{1, \dots, |C_R|\}$ an integer such that $\rhd \in f(R)$ and all preconditions of prefix-stability are satisfied. Since R and R' only differ on the order of candidates that the voters rank below all candidates in $\texttt{Pref}(\rhd, \ell)$, it holds for all sets $Y \subseteq C_R$ with $Y \cap X \neq \emptyset$ that $s_{PL}(R|_Y, y) = s_{PL}(R'|_Y, y)$ for all $y \in Y$. Hence, the execution of Sequential Plurality is the identical for R and R' until all candidates in $\texttt{Pref}(\rhd, \ell)$ have been chosen. This proves that there is a ranking $\rhd' \in f(R')$ with $\texttt{Pref}(\rhd, \ell) = \texttt{Pref}(\rhd', \ell)$, so prefix-stability is satisfied.

For the other direction, assume that f is an SPF that satisfies all given axioms. First, we can use the same hyperplane techniques as for Theorem 1. In more detail, for every agenda X with $k = |X| \geq 3$, there is a function $\hat{g}$ that extends f from profiles to $\mathbb{Q}^{k!}$. Further, for every $\rhd \in \mathcal{R}(X)$, we denote by $D_\rhd = \{v \in \mathbb{Q}^{k!} \colon \rhd \in \hat{g}(v)\}$ the preimage of $\hat{g}$ with respect to $\rhd$ and by $\bar{D}_\rhd$ the closure of these sets. Now, the central insight of our proof is that f satisfies the following condition: for all voter-disjoint profiles R and R' with $C_R = C_{R'}$ and candidates $x \in C_R$, if $f(R)$ and $f(R')$ both contain a ranking that top-rank x, then $f(R + R')$ also contains a ranking that top-ranks x. Motivated by this insight, we define by $D_x = \bigcup_{\rhd \in \mathcal{R}(X) \colon t(\rhd) = x} D_\rhd$ the domain where $\hat{g}$ selects a ranking that top-ranks x. Further, we let $\bar{D}_x$ denote the closure of D_x with respect to $\mathbb{R}^{k!}$ and note that $\bar{D}_x$ is a convex set. Since $\bar{D}_x = \bigcup_{\rhd \in \mathcal{R}(X) \colon t(\rhd) = x} \bar{D}_\rhd$, we derive that the interiors of $\bar{D}_x$ and $\bar{D}_y$ are disjoint for any two distinct $x, y \in X$. From this point on, we can mirror the proof of Young (1975) to show that there is a score function s such that a candidate x is top-ranked by a ranking in $f(R)$ if and only if x maximizes the total scores $s(R, x)$. Next, by prefix-stability, we infer that s must be the Plurality scoring function. By combining this insight with top IIA, it follows that f is the Sequential Plurality rule. $\square$

Remark 5. All the axioms are necessary for Theorem 2. Without prefix-stability, the Kemeny rule satisfies all given axioms. The trivial rule, which always returns all rankings over the given agenda, satisfies all axioms but faithfulness. The following variant of Sequential Plurality satisfies all conditions but top IIA: in each step, we choose the candidate that maximizes the total plurality score over all previous rounds. That is, if the candidates $x_1, \dots, x_i$ were chosen in the first i rounds, we select next the candidate y maximizing $\sum_{j=0}^{i} s_{PL}(R|_{C_R \setminus \{x_1, \dots, x_i\}}, y)$. Without quasi-reinforcement, the following sequential SPF f_1 satisfies all conditions: let $g(R)$ denote the set of candidates that are top-ranked by at least $\frac{|N_R|}{|C_R|}$ voters in the profile R. Then, f_1 chooses all rankings that concatenate a candidate $x \in g(R)$ with a ranking in $f_1(R|_{C_R\setminus\{x\}})$. Without anonymity, we can count the "even" voters $i \in \{2, 4, 6, \dots\}$ twice during the execution of Sequential Plurality. Similarly, without neutrality, we can duplicate the Plurality score of a fixed candidate and compute Sequential Plurality. Finally, the following variant f_2 of Sequential Plurality violates only continuity: first, f_2 computes the set of rankings $h(R)$ that concatenates each Plurality winner x with each ranking in $f_2(R|_{C_R\setminus\{x\}})$. Then, f_2 returns all rankings $\rhd \in h(R)$ where the second-rank candidate x_2 maximizes the plurality score in R among all candidates that are second-ranked by some ranking in $h(R)$.

Remark 6. We do not need full prefix-stability for the proof of Theorem 2. Instead, it suffices to use prefix-stability for prefixes of length 1, i.e., for the top-ranked candidate. Further, we note that Instant Runoff is also prefix-stable.

3.4 Characterizations of Veto-based SPFs

Lastly, we will present our characterizations of Sequential Veto and Veto Runoff. To this end, we note that these rules can be seen as the reverse of Instant Runoff and Plurality Veto. More formally, let $rev(\succ)$ denote the reverse ranking of $\succ$, i.e., if $\succ = x_1 \dots x_k$, then $rev(\succ) = x_k \dots x_1$. Further, $rev(R) = (rev(\succ_i))_{i\in N_R}$ is the reverse profile of R. Then, the reverse SPF f^r of an SPF f returns the reverse of every ranking that f chooses for the reverse profile, i.e., $f^r(R) = \{rev(\rhd) \colon \rhd \in f(rev(R))\}$. Now, Boehmer, Brederecck, and Peters (2026, Lemma 3.5) have observed that the reverse SPF of Instant Runoff is Sequential Veto and that the reverse SPF of Sequential Plurality is Veto Runoff. Moreover, these authors show that, if the original rule satisfies standardness, quasi-reinforcement, and faithfulness, its reverse SPF satisfies these conditions, too. Further, if the original rule satisfies top IIA (resp. bottom IIA), the reverse rule satisfies bottom IIA (resp. top IIA). Hence, we only need the "reverse" properties of minority protection and prefix-stability to characterize our veto-based SPFs, which we present next.

Minority rejection. Minority rejection is the reverse property of minority protection. It states that a candidate that is bottom-ranked by by more than $\frac{|N_R|}{|C_R|}$ voters cannot be top-ranked by an output ranking. Hence, an SPF f satisfies *minority rejection* if $t(\rhd) \neq x$ for all profiles R, candidates $x \in C_R$, and rankings $\rhd \in f(R)$ such that $|\{i \in N_R \colon b(\succ_i) = x\}| > \frac{|N_R|}{|C_R|}$.

Suffix-stability. Suffix-stability is the reverse property of prefix-stability. It requires that, if voters only reorder candidates that they rank ahead of all candidates in a suffix of an output ranking, we still need to choose a ranking with the same suffix. To formalize this, we define by $\texttt{Suff}(\rhd, \ell)$ the length-ℓ suffix of a ranking $\rhd$ and by $\texttt{CS}(\succ, \texttt{Suff}(\rhd, \ell))$ the shortest suffix of $\succ$ that contains all candidates in $\texttt{Suff}(\rhd, \ell)$. Then, an SPF is *suffix-stable* if $\rhd \in f(R)$ implies that there $\rhd' \in f(R')$ with $\texttt{Suff}(\rhd, \ell) = \texttt{Suff}(\rhd', \ell)$ for all profiles $R, R' \in \mathcal{R}^*$, rankings $\rhd \in \mathcal{R}(C_R)$, and $\ell \in \{1, \dots |C_R|\}$ such that $N_R = N_{R'}$, $C_R = C_{R'}$, and $\texttt{SC}(\succ_i, \texttt{Suff}(\rhd, \ell)) = \texttt{SC}(\succ'_i, \texttt{Suff}(\rhd, \ell))$ for all $i \in N_R$.

Based on the relation between our Plurality-based and Veto-based SPFs, we derive the following results. We note that the reverse SPFs of those given in Remarks 2 and 5 show that most axioms are necessary for the claims in Corollary 1.

Corollary 1. *The following claims are true.*

(1) Sequential Veto is the only SPF that satisfies standardness, quasi-reinforcement, top IIA, and minority rejection.

(2) Veto Runoff is the only SPF that satisfies standardness, quasi-reinforcement, bottom IIA, faithfulness, and suffix-stability.

Proof. The central insight for this corollary is that Sequential Plurality and Veto Runoff (resp. Instant Runoff and Sequential Veto) are the reverse of each other. To see this, we note that, in each step, Veto Runoff bottom-ranks the candidate that is bottom-ranked by the most voters. After reversing the profile, these candidates are the one that are top-ranked by the most voters, which are precisely the candidates that Sequential Plurality top-ranks. Hence, after reversing the output rankings of Sequential Plurality on the reversed profile, we get the output rankings of Veto Runoff. An analogous argument also applies to Instant Runoff and Sequential Veto.

Based on this insight, one merely needs to verify that, after reversing an SPF f that satisfies the axioms in Claim (1) or (2), respectively, we obtain an SPF that meets all conditions of Theorems 1 and 2. Since this follows almost directly by the definitions of the axioms, we leave this task to the reader. $\square$

4 Conclusion

In this paper, we provide "Kemeny-style" characterizations of four sequential SPFs, namely Instant Runoff, Sequential Plurality, Veto Runoff, and Sequential Veto. In more detail, we characterize these SPFs by combining relaxations of the axioms that characterize the Kemeny rule (Young 1988; Lederer 2024) with novel but natural proportionality and stability conditions. For our result on Instant Runoff, we combine the relaxed axioms of the Kemeny characterization with a mild proportionality condition, which requires that a candidate should not be bottom-ranked if it is top-ranked by more than a $\frac{1}{|C_R|}$ fraction of all voters. By contrast, for our characterization of Sequential Plurality, we combine the Kemeny axioms with a stability notion for the prefix of the output ranking. Further, our characterizations of Sequential Veto and Veto Runoff use dual properties since these are the reverse SPFs of Instant Runoff and Sequential Plurality.

An immediate follow-up question to our work is to characterize the classes of sequential-winner and sequential-loser scoring rules. We are, however, skeptical that such a characterization is possible without technical auxiliary properties, since we found a large class of SPFs that satisfy similar properties to sequential-winner and sequential-loser scoring rules. Thus, it may be interesting to study this novel class of SPFs in more depth. Further, our results may also be a helpful for a characterization of related rules such as Single Transferable Vote (Tideman 1995; Delemazure and Peters 2024)

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A Omitted proofs

In this appendix, we provide the proofs of Proposition 1 and Theorems 1 and 2. To this end, we first recall the required notation and introduce new concepts in Section A.1. In Section A.2, we then prove Proposition 1 by formalizing the ideas mentioned in the main body. Next, we discuss in Section A.3 the hyperplane framework that drives the proofs of Theorems 1 and 2. Based on this insight, we prove Theorem 2 in Section A.4 and Theorem 1 in Section A.5. We note that we first prove the characterization for Sequential Plurality because it is significantly easier and thus serves as warm-up for the characterization of Instant Runoff.

A.1 Overview of notation

Before discussing our proofs, we will recall old notation and introduce new one. To this, end fix an agenda $X \in \mathcal{F}(C)$.

- The rank of a candidate $x \in X$ in a ranking $\succ \in \mathcal{R}(X)$ is $r(\succ, x) = 1 + |y \in X : y \succ x|$.
- The top-ranked candidate of a ranking $\succ \in \mathcal{R}(X)$ is denoted by $t(\succ)$. Formally, $t(\succ)$ is the candidate in X such that $t(\succ) \succ x$ for all $x \in X \setminus \{t(\succ)\}$.
- The bottom-ranked candidate of a ranking $\succ \in \mathcal{R}(X)$ is denoted by $b(\succ)$. Formally, $b(\succ)$ is the candidate in X such that $x \succ b(\succ)$ for all $x \in X \setminus \{b(\succ)\}$.
- The set of the top-ℓ candidates of a ranking $\succ \in \mathcal{R}(X)$ is given by $\mathtt{T}(\succ, \ell) = \{x \in X : r(\succ, x) \leq \ell\}$. Note that $\mathtt{T}(\succ, 1) = \{t(\succ)\}$.
- the set of the bottom-ℓ candidates of a ranking $\succ \in \mathcal{R}(X)$ is $\mathtt{B}(\succ, \ell) = \{x \in X : r(\succ, x) \geq |X| + 1 - \ell\}$. Note that $\mathtt{B}(\succ, 1) = \{b(\succ)\}$.
- The length-ℓ prefix of a ranking $\succ \in \mathcal{R}(X)$ is defined by $\mathtt{Pref}(\succ, \ell) = \succ|_{\mathtt{T}(\succ, \ell)}$.
- The length-ℓ suffix of a ranking $\succ \in \mathcal{R}(X)$ is defined by $\mathtt{Suff}(\succ, \ell) = \succ|_{\mathtt{B}(\succ, \ell)}$.

A.2 Proof of Proposition 1

Proposition 1. *The following claims are true.*

(1) All sequential-winner scoring rules are standard and satisfy top-IIA. Further, all sequential-loser scoring rules are standard and satisfy bottom-IIA.

(2) All sequential-winner and sequential-loser scoring rules satisfy quasi-reinforcement but fail reinforcement.

Proof. We will only prove Claim (2) and, moreover, restrict attention to sequential-winner scoring rules as the arguments for sequential-loser scoring rules are symmetric. In particular, to show that sequential-loser scoring rules satisfy quasi-reinforcement, we can simply apply the following argument bottom-up instead of top-down. Further, to show that every such SPF fails reinforcement, it suffices to reverse the profiles we use to show that sequential-winner scoring rules fail this condition. Now, we fix a sequential-winner scoring rule f and let s be its corresponding score function. We will prove that f satisfies quasi-reinforcement but fails reinforcement.

f **is quasi-reinforcing.** For proving this claim, we fix two profiles R and R' such that $N_R \cap N_{R'} = \emptyset$, $C_R = C_{R'}$, and $f(R) \subseteq f(R')$. Further, we set $R'' = R + R'$ and $k = |C_R|$. Our goal is to show that $f(R'') = f(R)$. To this end, we first fix $\rhd = x_1 \dots x_k$ in $f(R)$ and let $C_i = \{x_i, \dots, x_k\}$. Because $f(R) \subseteq f(R')$, we also have $\rhd \in f(R')$. By the definition of f, it follows that, for all $i \in \{1, \dots, k\}$, x_i maximizes the scores $s(R|_{C_i}, x)$ and $s(R'|_{C_i}, x)$. This implies that x_i also maximizes $s(R''|_{C_i}, x_i) = s(R|_{C_i}, x_i) + s(R'|_{C_i}, x_i)$ for all $i \in \{1, \dots, k\}$, so $\rhd \in f(R'')$. Since $\rhd$ was chosen arbitrarily, this means that $f(R) \subseteq f(R'')$.

Next, let $\rhd' = y_1 \dots y_k$ denote a ranking that is not in $f(R)$. Further, let $\rhd = x_1 \dots x_k$ denote a ranking in $f(R)$ that maximizes the integer $\ell \in \{1, \dots, k\}$ such that $x_j = y_j$ for all $j \in \{1, \dots, \ell - 1\}$. Less formally, $\rhd$ is the ranking in $f(R)$ that has the longest matching prefix with $\rhd'$ (where a prefix of length 0 is possible). Since $\rhd \in f(R) \subseteq f(R')$, we again know that, for all $i \in \{1, \dots, k\}$, x_i maximizes both $s(R|_{C_i}, x)$ and $s(R'|_{C_i}, x)$ among all candidates $x \in C_i$, where C_i is again $\{x_i, \dots, x_k\}$. On the other hand, because $\rhd' \notin f(R)$ and $\rhd$ has the longest matching prefix to $\rhd'$ among the rankings in $f(R)$, we derive that $s(R|_{C_\ell}, x_\ell) > s(R|_{C_\ell}, y_\ell)$. Indeed, if $s(R|_{C_\ell}, x_\ell) = s(R|_{C_\ell}, y_\ell)$, there would be a ranking $\rhd'' \in f(R)$ and an integer $\ell' > \ell$ such that $\mathtt{Pref}(\rhd, \ell' - 1) = \mathtt{Pref}(\rhd'', \ell' - 1)$, which contradicts the definition of ℓ. By combining our insights, we now derive that $s(R''|_{C_\ell}, x_\ell) = s(R|_{C_\ell}, x_\ell) + s(R'|_{C_\ell}, x_\ell) > s(R|_{C_\ell}, y_\ell) + s(R'|_{C_\ell}, y_\ell) = s(R''|_{C_\ell}, x_\ell)$. This proves that $\rhd' \notin f(R'')$, so $f(R'') \subseteq f(R)$. In combination with the insights of the previous paragraph, this proves that $f(R'') = f(R)$, as desired.

f **fails reinforcement.** To prove this claim, we fix an agenda $X = \{x_1, \dots, x_k\}$ of size $k \geq 3$. We consider two profiles R^1 and R^2. In R^1, there are k voters who report the rankings $x_i \dots x_k x_1 \dots x_{i-1}$ for $i \in \{1, \dots, k\}$. Further, to define R^2, we let R^x denote the profile on $(k-1)!$ voters where every ranking in $\mathcal{R}(X)$ that top-ranks x is reported once. Then, R^2, consists of $k - 1$ copies of R^{x_1} and $k + 1 - i$ copies of R^{x_i} for each $i \in \{2, \dots, k\}$. We will now analyze the behavior of f on these two profiles.

First, in R^1, every alternative appears at every rank $r \in \{1, \dots, k\}$ exactly once, so $s(R^1, x) = s(R^1, y)$ for all $x, y \in X$. This means that, for every candidate $x \in X$, there is a ranking in $f(R^1)$ that top-ranks x. Let us consider the case that candidate x_1 is picked first. After deleting this candidate from R^1, we obtain the profile where the ranking $x_2 \dots x_k$ is reported twice (namely by the voters who originally reported $x_1 x_2 \dots x_k$ and $x_2 \dots x_k x_1$), whereas each ranking $x_i \dots, x_k x_2 \dots x_{i-1}$ is reported once. Hence, the score of each candidate x_i is $s(R^1|_{X \setminus \{x_1\}}, x_i) = s(i-1, k-1) + \sum_{j=1}^{k-1} s(j, k-1)$. Since s is non-increasing in the first argument and $s(1, x) > s(x, x)$ for all $x \in \{1, \dots, k\}$, $s(R^1|_{X \setminus \{x_1\}}, x_2)$ is maximal and strictly larger than $s(R^1|_{X \setminus \{x_1\}}, x_k)$. This means that there is a ranking $\rhd \in f(R^1)$ with prefix $x_1 x_2$ but no ranking with prefix $x_1 x_k$. Next, we consider the situation where x_1 and x_2 have been removed from R^1. In this case, the ranking $x_3 \dots x_k$ is reported by three voters (namely the two voters reporting $x_2 \dots x_k$ in $R^1|_{X \setminus \{x_1\}}$ and the one reporting $x_3 \dots x_k x_2$), whereas every ranking $x_i \dots x_k x_3 \dots x_{i-1}$ for $i \in \{3, \dots, k\}$ is reported once. Hence, x_3 maximizes the total score, possibly among other candidates. By repeating this reasoning, we infer that the ranking $x_1 \dots x_k$ is in $f(R^1)$. Further, as observed before, no ranking $\rhd$ with $\texttt{Pref}(\rhd, 2) = x_1 x_k$ is in $f(R^1)$. By symmetry, this further means that $\rhd \notin f(R^1)$ for all rankings $\rhd$ with $\texttt{Pref}(\rhd, 2) = x_2 x_1$.

Next, for R^2, we first note that, in every profile R^x, the total score of x is $z_1 = (k-1)! \cdot s(1,k)$ and the score of every other candidate is $z_2 = (k-2)! \sum_{j=2}^{k} s(j,k)$. Further, it holds that $z_1 > z_2$ because $s(1,k) > s(k,k)$. Next, we let $t = (k-1) + \sum_{j=1}^{k-1} j$ denote the total number of copies of profiles R^x for all candidates $x \in X$ in R^2. We have that $s(R^2, x_1) = (k-1)z_1 + (t-(k-1))z_2$ and $s(R^2, x_i) = (k+1-i)z_1 + (t-(k+1-i))z_2$ for all $i \in \{2, \dots, k\}$. Hence, we conclude that $s(R^2, x_1) = s(R^2, x_2) > s(R^2, x_3) > \dots > s(R^2, x_k)$, so every ranking in $f(R^2)$ has to top-rank x_1 or x_2. Next, we note that, for every set $Y \subseteq X$ and candidate $x \notin Y$, it holds that $R^x|_Y$ contains every ranking in $\mathcal{R}(Y)$ equally often and thus $s(R^x|_Y, y) = s(R^x|_Y, z)$ for all $y, z \in Y$. On the other hand, if $x \in Y$, then it is still top-ranked by all voters, whereas all other candidates are fully symmetric to each other. This means that $s(R^x|_Y, x) > s(R^x|_Y, y) = s(R^x|_Y, z)$ for all $x, y \in Y \setminus \{x\}$. These insights imply that f picks the candidates in decreasing order of the number of copies of R^{x_i} in R^2. Thus, $f(R^2) = \{x_1 x_2 x_3 \dots x_k,\ x_2 x_1 x_3 \dots x_k\}$.

Lastly, we note that $f(R^1) \cap f(R^2) = \{x_1 x_2 \dots x_k\}$ because $f(R^1)$ contains no ranking with prefix $x_2 x_1$. Hence, reinforcement requires that $f(R^1 + R^2) = \{x_1 x_2 \dots x_k\}$. However, it holds that $s(R^1 + R^2, x_1) = s(R^1, x_1) + s(R^2, x_1) = s(R^1, x_2) + s(R^2, x_2) = s(R^1 + R^2, x_2)$. This means that, if $f(R^1 + R^2)$ contains a ranking that top-ranks x_1, it also contain one that top-ranks x_2. Thus, f fails reinforcement. □

A.3 Hyperplanes and the Young framework

In this appendix, we will provide the basic setup of the proofs of Theorems 1 and 2. Specifically, we will establish that quasi-reinforcement, paired with mild side conditions, is enough to replicate Young's hyperplane framework for analyzing voting rules. To this end, we will fix throughout this section an agenda $X \in \mathcal{F}(C)$ with $|X| \geq 2$ and consider only profiles defined over the agenda X. Further, we set $k = |X|$. In this section, f will denote an SPF that is standard, quasi-reinforcing, and non-imposing. The last condition requires that for every agenda $Y \in \mathcal{F}(C)$ and $\rhd \in \mathcal{R}(Y)$, there is a profile $R \in \mathcal{R}^*$ such that $f(R) = \{\rhd\}$. Of course, since we only focus on the the agenda X in this section, it is technically enough if f is only non-imposing for X.

We aim to show that f allows for hyperplane arguments similar to the ones originally used by Young (1975). Moreover, we will also mirror some of the technical insights of more recent works (e.g., Lederer 2024; Lederer, Peters, and Wąs 2024), which will be helpful for our proofs. Towards these hyperplane arguments, we first prove that f satisfies another variant of reinforcement: it holds that $f(R) \cap f(R') \subseteq f(R + R')$ for all profiles $R, R' \in \mathcal{R}^*$ with $N_R \cap N_{R'} = \emptyset$. We call this property inclusion-reinforcement.

Lemma 1. *f satisfies inclusion-reinforcement.*

Proof. Fix two voter-disjoint profiles R and R'. If $f(R) \cap f(R') = \emptyset$, there is nothing to show. Hence, suppose that $f(R) \cap f(R') \neq \emptyset$ and let $\rhd$ denote a ranking in this intersection. Assume for contradiction that $\rhd \notin f(R + R')$. Let R^* denote a profile such that $f(R^*) = \{\rhd\}$; such a profile exists by non-imposition. By continuity, there exists $\lambda \in \mathbb{N}$ such that $f(\lambda(R + R') + R^*) \subseteq f(R + R')$, so $\rhd \notin f(\lambda(R + R') + R^*)$. On the other hand, quasi-reinforcement implies that $f(\lambda R) = f(R)$ and $f(\lambda R') = f(R')$, so $\rhd \in f(\lambda R)$ and $\rhd \in f(\lambda R')$. This means that $f(R^*) = \{\rhd\} \subseteq f(\lambda R)$, so $f(\lambda R + R^*) = \{\rhd\}$ by quasi-reinforcement. Moreover, this proves that $f(\lambda R + R^*) \subseteq f(\lambda R')$, so $f(\lambda R + \lambda R' + R^*) = \{\rhd\}$. This contradicts our previous observation that $\rhd \notin f(\lambda R + \lambda R' + R^*)$, so our assumption that $\rhd \notin f(R + R')$ must have been wrong. □

We next aim to move from profiles to a numerical space. To this end, we let $\succ_1, \dots, \succ_{k!}$ denote an arbitrary enumeration of all rankings in $\mathcal{R}(X)$. Since f is standard and thus anonymous, we can compute f only based on the number of voters that report a given ranking. Hence, let $v = v(R)$ denote the vector such that v_ℓ states the number of voters reporting $\succ_\ell$ in a profile R. By anonymity, there is a function $g : \mathbb{N}_0^{k!} \setminus \{0\} \to \mathcal{F}(\mathcal{R}(X))$ such that $f(R) = g(v(R))$ for all $R \in \mathcal{R}^*$. Moreover, since g is merely a more compact representation of f, it satisfies the following properties:

- Neutrality: $g(\tau(v)) = \{\tau(\rhd) \colon \rhd \in g(v)\}$ for all vectors $v \in \mathbb{N}_0^{k!} \setminus \{0\}$ and permutations $\tau : X \to X$. Here, we denote by $v' = \tau(v)$ the vector such that, for all $i \in \{1, \dots, k!\}$, $v'_i = v_j$, where j is chosen such that $\succ_i = \tau(\succ_j)$. Put differently, this means that $\tau(v(R)) = v(\tau(R))$ for every profile R.
- Quasi-reinforcement: $g(v + v') = g(v)$ for all vectors $v, v' \in \mathbb{N}_0^{k!} \setminus \{0\}$ such that $g(v) \subseteq g(v')$.
- Inclusion-reinforcement: $g(v) \cap g(v') \subseteq g(v + v')$ for all vectors $v, v' \in \mathbb{N}_0^{k!} \setminus \{0\}$.

We will next show that we can extend g to the domain $\mathbb{Q}^{k!}$ while preserving these properties.

Lemma 2. *There is a function $\hat{g} : \mathbb{Q}^{k!} \to \mathcal{F}(\mathcal{R}(X))$ such that (i) $\hat{g}(v(R)) = f(R)$ for all $R \in \mathcal{R}^*$ and (ii) $\hat{g}$ is neutral, quasi-reinforcing, and inclusion-reinforcing.*

Proof. Let g denote the function defined before the lemma such that $f(R) = g(v(R))$ for all $R \in \mathcal{R}^*$. We will extend g first to $\mathbb{Z}^{k!}$ and then to $\mathbb{Q}^{k!}$.

Step 1: Extension to $\mathbb{Z}^{k!}$. Let R^* denote the profile where every ranking $\succ \in \mathcal{R}(X)$ is reported by one voter and let $v^* = v(R)$ be the corresponding vector. By anonymity and neutrality, it follows that $g(v^*) = f(R^*) = \mathcal{R}(X)$ because every permutation maps v^* back to itself. Based on v^*, we define $\bar{g}$, the extension of g to $\mathbb{Z}^{k!}$, as follows: given a vector $v \in \mathbb{Z}^{k!}$, we let $\bar{g}(v) = g(v + \lambda v^*)$, where $\lambda \in \mathbb{N}_0$ is an arbitrary scalar such that $v + \lambda v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$.

We first show that $\bar{g}$ is well-defined despite not fully specifying λ. To this end, fix a vector $v \in \mathbb{Z}^{k!}$ and two distinct scalars λ and λ' such that $v + \lambda v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$ and

$v + \lambda' v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$. Without loss of generality, we suppose that $\lambda < \lambda'$. This means that $g(v+\lambda' v^*) = g(v+\lambda v^* + (\lambda' - \lambda)v^*) = g(v+\lambda v^*)$. Here, the last equality follows from quasi-reinforcement since $g((\lambda'-\lambda)v^*) = g(v^*) = \mathcal{R}(X)$. This proves that $\hat{g}$ is invariant under the exact choice of λ and thus well-defined. Moreover, we infer from this that, for all profiles R, $\bar{g}(v(R)) = g(v(R) + 0v^*) = f(R)$.

Next, we will show that $\bar{g}$ satisfies neutrality, quasi-reinforcement, and inclusion-reinforcement. To this end, fix a vector $v \in \mathbb{Z}^{k!}$ and let $\lambda \in \mathbb{N}_0$ denote a scalar such that $v + \lambda v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$.

- To see that $\hat{g}$ is neutral, let $\tau : X \to X$ be a permutation. Since $\tau(v^*) = v^*$, it holds that $\bar{g}(\tau(v)) = g(\tau(v) + \lambda v^*) = g(\tau(v + \lambda v^*)) = \{\tau(\rhd) \colon \rhd \in g(v + \lambda v^*)\} = \{\tau(\rhd) \colon \rhd \in \bar{g}(v)\}$.
- For quasi-reinforcement, we fix a second vector $v' \in \mathbb{Z}^{k!}$ with corresponding scalar λ' such that $\bar{g}(v) \subseteq \bar{g}(v')$. This means that $g(v + \lambda v^*) \subseteq g(v' + \lambda' v^*)$. Since $v + v' + (\lambda + \lambda')v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$, we infer that $\bar{g}(v + v') = g(v + v' + (\lambda + \lambda')v^*) = g(v + \lambda v^*) = \bar{g}(v)$, where the second equality uses the quasi-reinforcement of g.
- For inclusion-reinforcement, we again fix a second vector $v' \in \mathbb{Z}^{k!}$ with corresponding scalar λ'. Analogously to quasi-reinforcement, it holds that $\bar{g}(v) \cap \bar{g}(v') = g(v + \lambda v^*) \cap g(v + \lambda' v^*) \subseteq g(v + v' + (\lambda + \lambda')v^*) = \bar{g}(v + v')$.

Step 2: Extension to $\mathbb{Q}^{k!}$. Next, we extend $\bar{g}$ to a function $\hat{g}$ on $\mathbb{Q}^{k!}$. Specifically, we let $\hat{g}(v) = \bar{g}(\eta v)$ for all $v \in \mathbb{Q}^{k!}$, where $\eta \in \mathbb{N}$ is an arbitrary scalar such that $\eta v \in \mathbb{Z}^{k!}$.

Just as for $\bar{g}$, we show that $\hat{g}$ is well-defined despite not fully specifying η. To this end, fix a vector $v \in \mathbb{Q}^{k!}$ and two distinct scalars $\eta, \eta' \in \mathbb{N}$ such that $\eta v \in \mathbb{Z}^{k!}$ and $\eta' v \in \mathbb{Z}^{k!}$. This implies that $\eta \cdot \eta' \cdot v \in \mathbb{Z}^{k!}$. By the quasi-reinforcement of $\bar{g}$, we conclude that $\bar{g}(\eta v) = \bar{g}(\eta \cdot \eta' \cdot v) = \bar{g}(\eta' v)$. This shows that $\hat{g}$ is independent of the exact choice of η and thus well-defined. Further, this insight implies that $\hat{g}(v(R)) = \bar{g}(1 \cdot v(R)) = f(R)$ for all profiles $R \in \mathcal{R}^*$, thus verifying Condition (i) of the lemma.

Next, we show that $\hat{g}$ satisfies neutrality, quasi-reinforcement, and inclusion-reinforcement, which completes the proof of this lemma. To this end, fix a vector $v \in \mathbb{Q}^{k!}$ and let $\eta \in \mathbb{N}$ such that $\eta v \in \mathbb{Z}^{k!}$.

- For neutrality, we fix a permutation $\tau : X \to X$. Since $\eta\tau(v) = \tau(\eta v)$, it follows that $\hat{g}(\tau(v)) = \bar{g}(\eta\tau(v)) = \bar{g}(\tau(\eta v)) = \{\tau(\rhd) \colon \rhd \in \bar{g}(\eta v)\} = \{\tau(\rhd) \colon \tau \in \hat{g}(v)\}$. Here, the third equality holds as $\bar{g}$ is neutral.
- To show that $\hat{g}$ is quasi-reinforcing, we fix a second vector $v' \in \mathbb{Q}^{k!}$ with corresponding scalar $\eta' \in \mathbb{N}$ such that $\hat{g}(v) \subseteq \hat{g}(v')$. We observe that $\eta \cdot \eta' \cdot v \in \mathbb{Z}^{k!}$ and $\eta \cdot \eta' \cdot v' \in \mathbb{Z}^{k!}$, so $\hat{g}(v) = \bar{g}(\eta v) = \bar{g}(\eta \cdot \eta' \cdot v)$ and $\hat{g}(v') = \bar{g}(\eta' v') = \bar{g}(\eta \cdot \eta' \cdot v')$ by the quasi-reinforcement of $\bar{g}$. Since this means that $\bar{g}(\eta \cdot \eta' \cdot v) \subseteq \bar{g}(\eta \cdot \eta' \cdot v')$, we derive that $\hat{g}(v + v') = \bar{g}(\eta \cdot \eta' \cdot (v + v')) = \bar{g}(\eta \cdot \eta' \cdot v) = \hat{g}(v)$.
- Lastly, for inclusion-reinforcement, we fix again a second vector $v' \in \mathbb{Q}^{k!}$ with corresponding scalar $\eta' \in \mathbb{N}$. Analogous to the argument for quasi-reinforcement, we have that $\hat{g}(v) \cap \hat{g}(v') = \bar{g}(\eta \cdot \eta' \cdot v) \cap \bar{g}(\eta \cdot \eta' \cdot v') \subseteq \bar{g}(\eta \cdot \eta' \cdot (v + v')) = \hat{g}(v + v')$. □

Following Young (1975), we next define by $D_\rhd = \{v \in \mathbb{Q}^{k!} \colon v \in \hat{g}(v)\}$ the subdomain of $\mathbb{Q}^{k!}$ where $\hat{g}$ picks the ranking $\rhd \in \mathcal{R}(X)$. Since $\hat{g}$ is neutral, these sets are symmetric to each other. That is, for every permutation $\tau : X \to X$ and ranking $\rhd$, it holds that $v \in D_\rhd$ implies that $\tau(v) \in D_{\tau(\rhd)}$. Further, since $\hat{g}$ satisfies quasi-reinforcement and inclusion-reinforcement, the sets $D_\rhd$ are $\mathbb{Q}$-convex (i.e., for every $v_1, v_2 \in D_\rhd$ and $\lambda \in (0,1) \cap \mathbb{Q}$, it holds that $\lambda v_1 + (1-\lambda)v_2 \in \mathbb{D}_\rhd$). To see this, we note that, by definition, $\hat{g}$ is invariant under scaling the input vector, so $\hat{g}(\lambda v_1) = \hat{g}(v_1)$ and $\hat{g}((1-\lambda)v_2) = \hat{g}(v_2)$. In turn, inclusion-reinforcement implies that $\rhd \in \hat{g}(\lambda v_1 + (1-\lambda)v_2)$ and thus $\lambda v_1 + (1-\lambda)v_2 \in D_\rhd$ if $\rhd \in \hat{g}(v_1)$ and $\rhd \in \hat{g}(v_2)$. Further, we observe that $\bigcup_{\rhd \in \mathcal{R}(X)} D_\rhd = \mathbb{Q}^{k!}$ since the domain of $\hat{g}$ is $\mathbb{Q}^{k!}$. Next, for every $\rhd \in \mathcal{R}(X)$, let $\bar{D}_\rhd$ denote the closure of $D_\rhd$ with respect to $\mathbb{R}^{k!}$. By Lemma 2 of Young (1975), the sets $\bar{D}_\rhd$ are convex. Further, it holds that $\bigcup_{\rhd \in \mathcal{R}(X)} \bar{D}_\rhd = \mathbb{R}^{k!}$. Lastly, since all sets $\bar{D}_\rhd$ are symmetric to each other and there are only finitely many such sets, they must have non-empty interior.

Based on these insights, we next show that there is a hyperplane separating $\bar{D}_\rhd$ from $\bar{D}_{\rhd'}$ for all distinct rankings $\rhd, \rhd' \in \mathcal{R}(X)$. Given two vectors $u, v \in \mathbb{R}^{k!}$, we subsequently write $uv = \sum_{i=1}^{k!} u_i v_i$ for the standard scalar product between two vectors.

Lemma 3. *For all distinct $\rhd, \rhd' \in \mathcal{R}(X)$, there is a non-zero vector $u^{\rhd,\rhd'} \in \mathbb{R}^{k!}$ such that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_{\rhd'}$.*

Proof. Fix two distinct rankings $\rhd, \rhd' \in \mathcal{R}(X)$. To prove this claim, we aim to show that the interiors of $\bar{D}_\rhd$ and $\bar{D}_{\rhd'}$ are disjoint, i.e., $\text{int } \bar{D}_\rhd \cap \text{int } \bar{D}_{\rhd'} = \emptyset$. Since the interior of convex sets is also convex, this allows us to apply the separating hyperplane theorem (see, e.g., Rockafeller 1970), from which the theorem then follows. For proving our auxiliary claim, we first recall that the interiors $\text{int } \bar{D}_\rhd$ and $\text{int } \bar{D}_{\rhd'}$ are non-empty as otherwise $\bigcup_{\rhd'' \in \mathcal{R}(X)} \bar{D}_{\rhd''} \neq \mathbb{R}^{k!}$.

Now, assume for contradiction that $\text{int } \bar{D}_\rhd \cap \text{int } \bar{D}_{\rhd'} \neq \emptyset$. Since both $\text{int } \bar{D}_\rhd$ and $\text{int } \bar{D}_{\rhd'}$ are open sets, their intersection is also open. Hence, we can find a vector $v \in \text{int } \bar{D}_\rhd \cap \text{int } \bar{D}_{\rhd'} \cap \mathbb{Q}^{k!}$. Next, by Lemma 1 of Young (1975), it holds that $D_\rhd = \text{cvx } D_\rhd \cap Q^{k!}$, where $\text{cvx } D_\rhd$ denotes the convex hull of $D_\rhd$. This implies that $\bar{D}_\rhd = \overline{\text{cvx } D_\rhd \cap Q^{k!}} = \overline{\text{cvx } D_\rhd}$. Since the interior of the closure of any convex sets Z is a subset of Z, we get that $\text{int } \bar{D}_\rhd = \text{int } \overline{\text{cvx } D_\rhd} \subseteq \text{cvx } D_\rhd$. Finally, we have by assumption that $v \in \mathbb{Q}^{k!} \cap \text{int } \bar{D}_\rhd$, so $v \in Q^{k!} \cap \text{cvx } D_\rhd = D_\rhd$. An analogous argument also shows that $v \in D_{\rhd'}$, which proves that $\{\rhd, \rhd'\} \subseteq \hat{g}(v)$.

Now, let v' denote a vector such that $\hat{g}(v') = \rhd$. Such a vector exists as f is non-imposing. By quasi-reinforcement, it follows that $\hat{g}(\epsilon v') = \rhd$ for every $\epsilon \in (0, \infty) \cap \mathbb{Q}$. Thus, $\hat{g}(\epsilon v') \subset \hat{g}(v)$ and, using again quasi-reinforcement, we derive that $\hat{g}(v + \epsilon v') = \{\rhd\}$ for all $\epsilon \in (0, \infty) \cap \mathbb{Q}$. However, since $v \in \text{int } \bar{D}_{\rhd'}$, there is $\lambda \in (0, \infty) \cap \mathbb{Q}$ such that $v + \lambda v' \in \text{int } \bar{D}_{\rhd'}$. Because $v + \lambda v' \in \mathbb{Q}^{k!}$, we can use the same arguments as before to show that this means that $v + \lambda v' \in D_{\rhd'}$ and thus $\rhd' \in \hat{g}(v + \lambda v')$. This contradicts our previous insight, so $\text{int } \bar{D}_\rhd \cap \text{int } \bar{D}_{\rhd'} = \emptyset$.

We can now use the separating hyperplane theorem for convex sets to find a non-zero vector $u^{\rhd,\rhd'} \in \mathbb{R}^{k!}$ and a constant c such that $vu^{\rhd,\rhd'} > c$ if $v \in \text{int } \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} < c$ if $v \in \text{int } \bar{D}_{\rhd'}$. By taking the closure of our sets, this implies that $vu^{\rhd,\rhd'} \geq c$ if $v \in \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} \leq c$ if $v \in \bar{D}_{\rhd'}$. Finally, we note that both $\bar{D}_\rhd$ and $\bar{D}_{\rhd'}$ are cones, i.e. these sets are closed under multiplication with a positive scalar. Hence, if $c > 0$, we can choose any vector $v \in \bar{D}_\rhd$. Clearly, there is a sufficiently small $\epsilon > 0$ such that $\epsilon vu^{\rhd,\rhd'} < c$, but $\epsilon v \in \bar{D}_\rhd$. This contradicts the choice of c, so $c \leq 0$. By repeating this argument for $D_{\rhd'}$ when $c < 0$, it follows that $c = 0$, thus completing the proof. □

Motivated by Lemma 3, we subsequently define by $u^{\rhd,\rhd'}$ the non-zero vectors that separates $\bar{D}_\rhd$ from $\bar{D}_{\rhd'}$ for all pairs of rankings $\rhd, \rhd \in \mathcal{R}(X)$, i.e., it holds that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_{\rhd'}$. We next show that these vectors fully specify the sets $\bar{D}_\rhd$.

Lemma 4. *It holds for every ranking $\rhd \in \mathcal{R}(X)$ that $\bar{D}_\rhd = \{v \in \mathbb{R}^{k!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd,\rhd'} \geq 0\}$.*

Proof. Fix a ranking $\rhd$ and the corresponding set $\bar{D}_\rhd$. For easier notation, we define $S = \{v \in \mathbb{R}^{k!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd,\rhd'} \geq 0\}$. Our goal is to show that $\bar{D}_\rhd = S$. To this end, we first observe that, by definition, if $v \in \bar{D}_\rhd$, then $vu^{\rhd,\rhd'} \geq 0$ for all $\rhd' \in \mathcal{R}(X) \setminus \{\rhd\}$. This proves $\bar{D}_\rhd \subseteq S$. For the other direction, we observe that int $S \neq \emptyset$ since int $\bar{D}_\rhd \neq \emptyset$ and int $\bar{D}_\rhd \subseteq$ int S. Hence, let $v \in$ int S, which means that $vu^{\rhd,\rhd'} > 0$ for all $\rhd' \in \mathcal{R}(X) \setminus \{\rhd\}$. By the definition of $u^{\rhd,\rhd'}$, it follows that $v \notin D_{\rhd'}$ for every ranking $\rhd' \in \mathcal{R}(X) \setminus \{\rhd\}$. Since $\bigcup_{\rhd' \in \mathcal{R}(X)} \bar{D}_{\rhd'} = \mathbb{R}^{k!}$, this proves that $v \in \bar{D}_\rhd$. Thus, int $S \subseteq \bar{D}_\rhd$. Finally, by taking the closure of both sets, we derive that $S \subseteq \bar{D}_\rhd$, because $\bar{D}_\rhd$ is already a closed set. □

We note that f fully specifies $\hat{g}$, which, in turn, specifies the sets $D_\rhd$. Hence, if $\rhd \in \hat{g}(v)$ for some vector $v \in \mathbb{Q}^{k!}$, then $v \in \bar{D}_\rhd$. We will next show that this is an equivalence, which means that the sets $\bar{D}_\rhd$ fully describes $\hat{g}$ and thus f.

Lemma 5. *The following claims hold for every vector $v \in \mathbb{Q}^{k!}$ and ranking $\rhd \in \mathcal{R}(X)$.*

(1) If $\hat{g}(v) = \{\rhd\}$, then $vu^{\rhd,\rhd'} > 0$ for all $\rhd' \in \mathcal{R}(X) \setminus \{\rhd\}$.
(2) $\rhd \in \hat{g}(v)$ if and only if $v \in \bar{D}_\rhd$.

Proof. Fix a vector $v \in \mathbb{Q}^m!$ and a ranking $\rhd \in \mathcal{R}(X)$. We will show the claims of the lemma separately.

Claim (1): First, assume that $\hat{g}(v) = \{\rhd\}$. By definition, this means that $v \in D_\rhd$ and thus also in $\bar{D}_\rhd$. In turn, by the definition of the vectors $u^{\rhd,\rhd'}$, we have that $vu^{\rhd,\rhd'} \geq 0$ for all $\rhd' \in \mathcal{R}(X) \setminus \{\rhd\}$. Now, assume for contradiction that $vu^{\rhd,\rhd'} = 0$ for some ranking $\rhd'$. Since $u^{\rhd,\rhd'}$ is a non-zero vector, there is $v' \in \mathbb{Q}^{k!}$ such that $vu^{\rhd,\rhd'} < 0$. Now, let v^* denote the vector corresponding to the profile where every ranking $\succ \in \mathcal{R}(C)$ is reported by one voter, and note that $\hat{g}(v^*) = \mathcal{R}(X)$. Further, let $\lambda, \lambda' \in \mathbb{N}_0$ and $\eta \in \mathbb{N}$ such that $\eta v + \lambda v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$ and $\eta v' + \lambda' v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$. By the quasi-reinforcement of $\hat{g}$, it holds that $\hat{g}(v) = \hat{g}(\eta v + \lambda v^*)$ and $\hat{g}(v') = \hat{g}(\eta v' + \lambda' v^*)$. Next, there are profiles R and R' such that $v(R) = \eta v + \lambda v^*$ and $v(R') = \eta v' + \lambda v^*$. Further, by condition (i) in Lemma 2, it holds that $f(R) = \hat{g}(\eta v + \lambda v^*) = \hat{g}(v) = \{\rhd\}$ and $f(R') = \hat{g}(\eta v' + \lambda v^*) = \hat{g}(v')$. Finally, we recall that f is continuous, so there is $\lambda^* \in \mathbb{N}$ such that $f(\lambda^* R + R') \subseteq f(R)$. Using quasi-reinforcement, this means that $\hat{g}(\lambda^* v + v') = \hat{g}(\lambda^* \cdot \eta v + \eta' v + (\lambda^* \lambda + \lambda')v^*) = \hat{g}(\lambda^* v(R) + v(R')) = \{\rhd\}$, too. By definition, this requires that $\lambda^* v + v' \in \bar{D}_\rhd$. However, we have that $(\lambda^* v + v')u^{\rhd,\rhd'} < 0$ because $vu^{\rhd,\rhd'} = 0$ and $v'u^{\rhd,\rhd'} < 0$. This contradicts that $v \in \bar{D}_\rhd$, so assumption that $vu^{\rhd,\rhd'} = 0$ must have been wrong.

Claim (2): For the next claim, we assume that $\rhd \in \hat{g}(v)$. By definition, this implies that $v \in D_\rhd \subseteq \bar{D}_\rhd$, thus proving the direction from left to right. For the converse, assume for contradiction that $v \in \bar{D}_\rhd$ but $\rhd \notin \hat{g}(v)$. It holds for all $\rhd' \in \hat{g}(v)$ that $v \in \bar{D}_{\rhd'}$, so we get that $vu^{\rhd,\rhd'} = 0$ for all these rankings. Now, let R' be a profile such that $f(R') = \{\rhd\}$ and let $v' = v(R)$ be the corresponding vector. By the insights of the last claim, it holds that $v'u^{\rhd,\rhd'} > 0$ for all $\rhd' \in \mathcal{R}(X) \setminus \{\rhd\}$. Further, just as in the last case, let λ and η such that $\eta v + \lambda v^* \in \mathbb{N}_0^{k!} \setminus \{0\}$. There is again a profile R such that $v(R) = \eta v + \lambda v^*$. By continuity, there must be $\lambda^* \in \mathbb{N}$ such that $f(\lambda^* R + R') \subseteq f(R)$. This also means that $\hat{g}(\lambda^* \eta v + v') \subseteq \hat{g}(\eta v)$. However, we note that, for every $\rhd' \in \hat{g}(\eta v) = \hat{g}(v)$, it holds that $(\lambda^* \eta v + v')u^{\rhd,\rhd'} > 0$ because $vu^{\rhd,\rhd'} = 0$ and $v'u^{\rhd,\rhd'} > 0$. This contradicts that $\lambda^* \eta v + v' \in \bar{D}_{\rhd'}$ for any $\rhd' \in \hat{g}(v)$. Hence, our assumption must be wrong and $v \in \bar{D}_\rhd$ implies that $\rhd \in \hat{g}(v)$. □

A.4 Proof of Theorem 2

We now turn to the proof of Theorem 2. Since we have shown in the main body that Sequential Plurality satisfies all given properties, we focus here on the converse. Thus, let f denote an SPF that satisfies standardness, quasi-reinforcement, top IIA, faithfulness, and prefix-stability. We aim to show that f is Sequential Plurality, which we denote by $f_{\texttt{SeqPL}}$. To this end, we fix an agenda $X \in \mathcal{F}(C)$ of size $k = |X| \geq 3$ and analyze the behavior of f for profiles over X.

First, we note that faithfulness implies non-imposition, so we can use all results of Section A.3. In particular, this means that f is inclusion-reinforcing. Further, given a set of rankings $Z \subseteq \mathcal{R}(X)$, we let $W(Z) = \{x \in X \colon \exists \rhd \in Z \colon t(\rhd) = x\}$ denote the set of candidates that are first-ranked by some ranking in Z. We will next show that f satisfies inclusion-reinforcement for the set of top-ranked candidates.

Lemma 6. *It holds for all candidates $x \in X$ and profiles $R, R' \in \mathcal{R}^*$ with $N_R \cap N_{R'} = \emptyset$ and $C_R = C_{R'} = X$ that $W(f(R)) \cap W(f(R', X)) \subseteq W(f(R + R'))$.*

Proof. Fix a candidate $x \in X$ and two voter-disjoint profiles $R, R' \in \mathcal{R}^*$ such that $x \in W(f(R)) \cap W(f(R'))$. Further, let $\rhd \in f(R)$ and $\rhd' \in f(R)$ denote the corresponding rankings that top-rank x. We need to show that there is a ranking $\rhd'' \in f(R + R')$ that also top-ranks x. First, if $\rhd = \rhd'$, this follows directly since Lemma 1 shows that f satisfies inclusion-reinforcement. Hence, assume that $\rhd \neq \rhd'$. In this case, we let R^* denote the profile where every ranking $\succ \in \mathcal{R}(X)$ is reported once. By anonymity and neutrality, we have that $f(R^*) = \mathcal{R}(X)$. This means that $f(R + \lambda R^*) = f(R)$ for every $\lambda \in \mathbb{N}$ by quasi-reinforcement.

Next, since f is faithful and quasi-reinforcing, there is a profile $\hat{R}$ over $X \setminus \{x\}$ such that $f(\hat{R}) = \{\texttt{Suff}(\rhd', |X|-1)\}$. Further, let $\hat{\lambda} \in \mathbb{N}$ such that the number of voters in $\hat{\lambda}\hat{R}$ is a multiple of $(k-1)!$. That is, there is $\lambda \in \mathbb{N}$ such that $\lambda(k-1)! = \hat{\lambda}|N_{\hat{R}}|$. By quasi-reinforcement, it holds that $f(\hat{\lambda}\hat{R}) = \{\texttt{Suff}(\rhd', |X|-1)\}$. Lastly, let $\hat{R}^*$ denote the following profile on X: first, for every candidate $y \in X \setminus \{x\}$, each ranking that top-ranks y is reported by λ voters. Secondly, we add $\lambda(k-1)!$ voters that top-rank x and order the remaining candidates as in $\hat{\lambda}\hat{R}$. We claim that $f(\hat{R}^*|_{X\setminus\{x\}}) = \{\texttt{Suff}(\rhd', |X|-1)\}$. To see this, let R^y denote the profile where every ranking $\succ \in \mathcal{R}(X)$ that top-ranks y is reported once. It holds that $\hat{R}^*|_{X\setminus\{x\}} = \hat{\lambda}\hat{R} + \lambda \sum_{y\in X\setminus\{x\}} R^y|_{X\setminus\{x\}}$. Further, in $\lambda \sum_{y\in X\setminus\{x\}} R^y|_{X\setminus\{x\}}$, each ranking $\succ \in \mathcal{R}(X \setminus \{x\})$ is reported by the same number of voters. By anonymity and neutrality, we hence conclude that $f(\lambda \sum_{y\in X\setminus\{x\}} R^y|_{X\setminus\{x\}}) = \mathcal{R}(X \setminus \{x\})$. In turn, quasi-reinforcement implies that $f(\hat{R}^*|_{X\setminus\{x\}}) = f(\hat{\lambda}\hat{R}|_{X\setminus\{x\}}) = \{\texttt{Suff}(\rhd', |X|-1)\}$.

Finally, by continuity, there is $\lambda' \in \mathbb{N}$ such that $f(R|_{X\setminus\{x\}} + \lambda'\hat{R}^*|_{X\setminus\{x\}}) = \{\texttt{Suff}(\rhd', |X|-1)\}$. Further, we note that $R + \lambda'\hat{R}^*$ only differs from $R + \lambda' \cdot \lambda R^*$ in the rankings of the voters in $\lambda'\hat{R}^*$ that top-rank x. Since these voters top-rank x in both $R + \lambda'\hat{R}^*$ and $R + \lambda' \cdot \lambda R^*$ and $f(R + \lambda' \cdot \lambda R^*) = f(R)$ contains a ranking that top-ranks x, prefix-stability shows that there must also be a ranking $\hat{\rhd}$ that top-ranks x in $f(R + \lambda'\hat{R}^*)$. In turn, top IIA requires that $\texttt{Suff}(\hat{\rhd}, |X|-1) \in f((R+\lambda'\hat{R}^*)|_{X\setminus\{x\}})$. This means $\texttt{Suff}(\hat{\rhd}, |X|-1) = \texttt{Suff}(\rhd', |X|-1)$, so we derive that that $\hat{\rhd} = \rhd'$ and thus $\rhd' \in f(R + \lambda'\hat{R}^*, X)$.

Next, by inclusion-reinforcement, it follows that $\rhd' \in f(R + R' + \lambda'\hat{R}^*)$. By prefix-stability, it further follows that there is a ranking $\rhd'' \in f(R + R' + \lambda' \cdot \lambda R^*)$ that top-ranks x because such a ranking is in $f(R + R' + \lambda'\hat{R}^*)$ and we derive $R + R' + \lambda' \cdot \lambda R^*$ from $R + R' + \lambda'\hat{R}^*$ by only reordering the preferences of voters below x. Lastly, by quasi-reinforcement, it holds that $f(R + R' + \lambda' \cdot \lambda R^*) = f(R + R')$ because $f(\lambda' \cdot \lambda R^*) = \mathcal{R}(X)$. Thus, there is a ranking $\rhd'' \in f(R + R')$ that top-ranks x. □

Next, since f satisfies all premises of Section A.3, we can apply the hyperplane framework to f. In particular, we recall that we can represent profiles as vectors $v \in \mathbb{N}_0^{k!} \setminus \{0\}$ such v_i states how often the ranking $\succ_i$ is reported in R. The vector corresponding to a profile R is typically denoted by $v(R)$. Then, we have shown the following insights:

(1) There is a function $\hat{g} : \mathbb{Q}^{k!} \to \mathcal{F}(\mathcal{R}(X))$ that is neutral, quasi-reinforcing, inclusion-reinforcing and satisfies that $f(R) = \hat{g}(v(R))$ for all profiles R over X.

(2) This function induces the sets $D_\rhd = \{v \in \mathbb{Q}^{k!} \colon \rhd \in \hat{g}(v)\}$ for all $\rhd \in \mathcal{R}(X)$. Further, the closure of these sets with respect to $\mathbb{R}^{k!}$ is denoted by $\bar{D}_\rhd$. These sets are convex, symmetric to each other (i.e., $\tau(v) \in \bar{D}_{\tau(\rhd)}$ for every permutation $\tau : X \to X$ and vector $v \in \bar{D}_\rhd$) and satisfy that $\bigcup_{\rhd\in\mathcal{R}(X)} \bar{D}_\rhd = \mathbb{R}^{k!}$.

(3) For any two distinct rankings $\rhd, \rhd' \in \mathcal{R}(X)$, there is a non-zero vector $u^{\rhd,\rhd'} \in \mathbb{R}^{k!}$ such that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_{\rhd'}$. Further, it holds for every ranking $\rhd \in \mathcal{R}(X)$ that $\bar{D}_\rhd = \{v \in \mathcal{R}^{k!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd,\rhd'} \geq 0\}$.

We will next aim to replicate these results for the top-ranked candidates in the rankings chosen by $\hat{g}$. Specifically, for every $x \in X$, we let $D_x = \{v \in \mathbb{Q}^{k!} \colon x \in W(\hat{g}(v))\}$. Since Lemma 6 extends to $\hat{g}$, the sets D_x are $\mathbb{Q}$-convex. This means that the closures of these sets, denoted by $\bar{D}_x$, are convex. Further, it holds by definition that $D_x = \bigcup_{\rhd\in\mathcal{R}(X) \colon t(\rhd)=x} D_\rhd$. Since this is a finite union, we have also that $\bar{D}_x = \bigcup_{\rhd\in\mathcal{R}(X) \colon t(\rhd)=x} \bar{D}_\rhd$. This means that $\bigcup_{x\in X} \bar{D}_x = \mathbb{R}^{k!}$. We will next show that, for any two candidates $x, y \in X$, there is a non-zero vector $u^{x,y} \in \mathbb{R}^{k!}$ such that $vu^{x,y} \geq 0$ for all $v \in \bar{D}_x$ and $vu^{x,y} \leq 0$ for all $v \in \bar{D}_y$.

Lemma 7. *For all distinct candidates $x, y \in X$, there is a non-zero vector $u^{x,y} \in \mathbb{R}^{k!}$ such that $vu^{x,y} \geq 0$ for all $v \in \bar{D}_x$ and $vu^{x,y} \leq 0$ for all $v \in \bar{D}_y$.*

Proof. Fix two candidates $x, y \in X$. By our previous discussion, the sets $\bar{D}_x$ and $\bar{D}_y$ are convex. Hence, we aim to show that the intersection of their interior, $I = \text{int } \bar{D}_x \cap \text{int } \bar{D}_y$, is empty, because this allows us to use the hyperplane argument for convex sets. To this end, we first note that $\text{int } \bar{D}_x \neq \emptyset$ and $\text{int } \bar{D}_y \neq \emptyset$. The reason for this is that $\bar{D}_x = \bigcup_{\rhd\in\mathcal{R}(X) \colon t(\rhd)=x} \bar{D}_\rhd$ and $\bar{D}_y = \bigcup_{\rhd\in\mathcal{R}(X) \colon t(\rhd)=y} \bar{D}_\rhd$, and all sets $\bar{D}_\rhd$ have non-empty interior.

Now, assume for contradiction that $I \neq \emptyset$. Since $\text{int } \bar{D}_x$ and $\text{int } \bar{D}_y$ are convex and open sets, the same holds for I. Because $I \subseteq \bar{D}_x = \bigcup_{\rhd\in\mathcal{R}(X) \colon t(\rhd)=x} \bar{D}_\rhd$, it holds that $I \cap \bar{D}_\rhd$ for some ranking $\rhd \in \mathcal{R}(X)$ that top-ranks x. Moreover, since I is open and thus fully dimensional, there must be a ranking $\rhd \in \mathcal{R}(X)$ that top-ranks x such that $\text{int } (I \cap \bar{D}_\rhd) \neq \emptyset$. Otherwise, I would be the union of a finite number of boundary sets and could thus not be open (as it would not even be fully dimensional). Similarly, we have that $\text{int } (I \cap \bar{D}_\rhd) \subseteq \bar{D}_y = \bigcup_{\rhd'\in\mathcal{R}(X) \colon t(\rhd')=y} \bar{D}_{\rhd'}$ because I is a subset of $\bar{D}_y$. Since $\text{int } (I \cap \bar{D}_\rhd)$ is an open set, we can again find a ranking $\rhd' \in \mathcal{R}(X)$ such that $t(\rhd') = y$ and $\text{int } (I \cap \bar{D}_\rhd \cap \bar{D}_{\rhd'}) \neq \emptyset$. However, this means that $\text{int } \bar{D}_\rhd \cap \text{int } \bar{D}_{\rhd'} \neq \emptyset$ as we have found a common subset of the two. This contradicts the central insight of the proof of Lemma 3, where we showed that $\text{int } \bar{D}_\rhd \cap \text{int } \bar{D}_{\rhd'} = \emptyset$ for all distinct $\rhd, \rhd' \in \mathcal{R}(X)$. Thus, our assumption that $I \neq \emptyset$ must have been wrong.

Now, by the separating hyperplane theorem for convex sets, there is a non-zero vector $u^{x,y}$ and a constant $c \in \mathbb{R}$ such that $vu^{x,y} > c$ if $v \in \text{int } \bar{D}_x$ and $vu^{x,y} < c$ if $v \in \text{int } \bar{D}_y$. By using the same arguments as for Lemma 3, this implies that $vu^{x,y} \geq 0$ if $v \in \bar{D}_x$ and and $vu^{x,y} \leq 0$ if $v \in \bar{D}_y$. □

We will infer from the hyperplanes $u^{x,y}$ a score representation that defines when a candidate can be top-ranked in a ranking chosen by f. We note that the proof of the subsequent lemma closely follows the work of Young (1975). Since we subsequently only consider our fixed agenda X, we omit the second parameter of the score function s. Further, we note that the score function that we derive in the next paragraph is not yet guaranteed to be monotone decreasing.

Lemma 8. *There is a score function $s : \{1, \dots, k\} \to \mathbb{R}$ such that $x \in W(f(R))$ if and only if $x \in \arg\max_{y \in X} s(R, y)$ for all profiles R with $C_R = X$ and candidates $x \in X$.*

Proof. Let $u^{x,y}$ denote the vectors derived in Lemma 7 for all distinct $x, y \in X$. We will prove this lemma in multiple steps. First, we will show that there is a bivariate score function $\hat{s} : \{1, \dots, k\} \times \{1, \dots, k\} \to \mathbb{R}$ such that *(i)* $\hat{s}(i, j) \neq 0$ for at least one pair of distinct indices $i, j \in \{1, \dots, k\}$, *(ii)* $\hat{s}(i, j) = -\hat{s}(j, i)$ for all distinct indices $i, j \in \{1, \dots, k\}$, and *(iii)* for all distinct $x, y \in X$ and vectors $v \in \mathbb{R}^{k!}$, it holds that $\sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, x), r(\succ_i, y)) \geq 0$ if $v \in \bar{D}_x$ and $\sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, x), r(\succ_i, y)) \leq 0$ if $v \in \bar{D}_y$. The existence of $\hat{s}$ effectively follows from the symmetry of the sets $\bar{D}_x$. Next, we infer from $\hat{s}$ a non-constant score function $s : \{1, \dots, k\} \to \mathbb{R}$ and show for all $x, y \in X$ and $v \in \mathbb{R}^{k!}$ that $\sum_{i=1}^{k!} v_i s(r(\succ_i, x)) \geq \sum_{i=1}^{k!} v_i s(r(\succ_i, y))$ if $v \in \bar{D}_x$ and $\sum_{i=1}^{k!} v_i s(r(\succ_i, x)) \leq \sum_{i=1}^{k!} v_i s(r(\succ_i, y))$ if $v \in \bar{D}_y$. Based on these insights, we prove the lemma in the last step.

Step 1: First, we will show that there is a function $\hat{s} : \{1, \dots, k\} \times \{1, \dots, k\} \to \mathbb{R}$ such that *(i)* $\hat{s}(i, j) \neq 0$ for at least one pair of distinct indices $i, j \in \{1, \dots, k\}$, *(ii)* $\hat{s}(i, j) = -\hat{s}(j, i)$ for all distinct indices $i, j \in \{1, \dots, k\}$, and *(iii)* for all distinct $x, y \in X$ and vectors $v \in \mathbb{R}^{k!}$, it holds that $\sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, x), r(\succ_i, y)) \geq 0$ if $v \in \bar{D}_x$ and $\sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, x), r(\succ_i, y)) \leq 0$ if $v \in \bar{D}_y$. The main observation for this is that the sets $\bar{D}_x$ are symmetric to each other. In more detail, for every permutation $\tau : X \to X$, it holds that $\tau(v) \in \bar{D}_{\tau(x)}$ if $v \in \bar{D}_x$. To see this, we recall that $v \in \bar{D}_x$ implies that $v \in \bar{D}_\rhd$ for some ranking $\rhd$ that top-ranks x. Now, the sets $\bar{D}_\rhd$ are symmetric to each other as $\hat{g}$ is neutral. This means that $\tau(v) \in \bar{D}_{\tau(\rhd)}$. Since $\tau(\rhd)$ top-ranks $\tau(x)$, this shows that $\tau(v) \in \bar{D}_{\tau(x)}$.

Now, we fix two candidates $x, y \in X$ and consider the non-zero vector $u^{x,y}$ given by Lemma 7. In particular, this vector satisfies that $vu^{x,y} \geq 0$ for all $v \in \bar{D}_x$, $vu^{x,y} > 0$ for all $v \in \text{int } \bar{D}_x$, $vu^{x,y} \leq 0$ for all $v \in \bar{D}_y$, and $vu^{x,y} < 0$ for all $v \in \text{int } \bar{D}_y$. We claim that, for every permutation $\tau : X \to X$, the vector $\tau(u^{x,y})$ satisfies that $v\tau(u^{x,y}) \geq 0$ for all $v \in \bar{D}_{\tau(x)}$ and $v\tau(u^{x,y}) \leq 0$ for all $v \in \bar{D}_{\tau(y)}$. To see this, fix such a permutation τ and let $v \in \bar{D}_{\tau(x)}$. By the symmetry of our sets, this means that $\tau^{-1}(v) \in \bar{D}_x$, where τ^{-1} is the inverse permutation of τ. By definition of $u^{x,y}$, this means that $\tau^{-1}(v)u^{x,y} \geq 0$. Since the scalar product is invariant under permuting both vectors, it holds that $v\tau(u^{x,y}) = \tau^{-1}(v)u^{x,y}$. By combining these insights, we derive that $v\tau(u^{x,y}) \geq 0$ for all $v \in \bar{D}_{\tau(x)}$ and symmetric argument shows that $v\tau(u^{x,y}) \leq 0$ for all $v \in \bar{D}_{\tau(y)}$. Further, an analogous argument also shows that $v\tau(u^{x,y}) > 0$ for all $v \in \text{int } \bar{D}_{\tau(x)}$ and $v\tau(u^{x,y}) < 0$ for all $v \in \text{int } \bar{D}_{\tau(y)}$.

Next, we let $\mathcal{T}$ denote the set of permutations on X with $\tau(x) = x$ and $\tau(y) = y$. Further, we define $\hat{u}^{x,y} = \sum_{\tau \in \mathcal{T}} \tau(u^{x,y})$. By the definition of $\mathcal{T}$ and our previous insights, it holds for all $\tau \in \mathcal{T}$ that $v\tau(u^{x,y}) \geq 0$ for all $v \in \bar{D}_x$, $v\tau(u^{x,y}) \leq 0$ for all $v \in \bar{D}_y$, and $v\tau(u^{x,y}) > 0$ for all $v \in \text{int } \bar{D}_x$. This implies that $v\hat{u}^{x,y} \geq 0$ for all $v \in \bar{D}_x$, $v\hat{u}^{x,y} \leq 0$ for all $v \in \bar{D}_y$, and $v\hat{u}^{x,y} > 0$ for all $v \in \text{int } \bar{D}_x$. Put differently, $\hat{u}^{x,y}$ is a non-zero vector that separates $\bar{D}_x$ from $\bar{D}_y$. Moreover, it holds by definition that $\hat{u}^{x,y}_\succ = \hat{u}^{x,y}_{\tau(\succ)}$ for all $\tau \in \mathcal{T}$.

Next, let $\tilde{\tau}$ be the permutation with $\tilde{\tau}(x) = y$, $\tilde{\tau}(y) = x$, and $\tilde{\tau}(z) = z$ for all $z \in X \setminus \{x, y\}$. The vector $\tilde{\tau}(\hat{u}^{x,y})$ satisfies that $v\tilde{\tau}(\hat{u}^{x,y}) \geq 0$ for all $v \in \bar{D}_y$, $v\tilde{\tau}(\hat{u}^{x,y}) \leq 0$ for all $v \in \bar{D}_x$, and $v\tilde{\tau}(\hat{u}^{x,y}) < 0$ for all $v \in \text{int } \bar{D}_x$. This implies for the vector $\tilde{u}^{x,y} = \hat{u}^{x,y} - \tilde{\tau}(\hat{u}^{x,y})$ that $v\tilde{u}^{x,y} \geq 0$ for all $v \in \bar{D}_x$, $v\tilde{u}^{x,y} \leq 0$ for all $v \in \bar{D}_y$, and $v\tilde{u}^{x,y} > 0$ for all $v \in \text{int } \bar{D}_x$. Hence, $\tilde{u}^{x,y}$ is again a non-zero vector separating $\bar{D}_x$ from $\bar{D}_y$. Moreover, this vector preserves that $\tilde{u}^{x,y}_\succ = \tilde{u}^{x,y}_{\tau(\succ)}$ for all $\tau \in \mathcal{T}$ and additionally satisfies that $\tilde{\tau}(\tilde{u}^{x,y}) = \tilde{\tau}(\hat{u}^{x,y} - \tilde{\tau}(\hat{u}^{x,y})) = \tilde{\tau}(\hat{u}^{x,y}) - \hat{u}^{x,y} = -\hat{u}^{x,y}$. In particular, this means that $\tilde{u}^{x,y}_i = \tilde{\tau}(\tilde{u}^{x,y})_j = -\tilde{u}^{x,y}_j$ for all $i, j \in \{1, \dots, k!\}$ such that $\succ_i = \tau(\succ_j)$.

We next claim that the vector $\tilde{u}^{x,y}$ induces a bivariate scoring function $\hat{s}\colon \{1, \dots, k!\} \times \{1, \dots, k\} \to \mathbb{R}$ that satisfies our requirements for x, y. In more detail, for all distinct i, j, we define $\hat{s}$ by $\hat{s}(i, j) = \tilde{u}^{x,y}_\ell$, where ℓ is an arbitrary index such that $r(\succ_\ell, x) = i$ and $r(\succ_\ell, y) = j$. Moreover, we set $\hat{s}(i, i) = 0$ for all $i \in \{1, \dots, k!\}$ and note that these entries will not matter. Now, we will first prove that the choice of the ranking $\succ_\ell$ does not matter. To this end, we observe that, for any two rankings $\succ_\ell, \succ_{\ell'} \in \mathcal{R}(X)$ with $r(\succ_\ell, x) = r(\succ_{\ell'}, x) = i$ and $r(\succ_\ell, y) = r(\succ_{\ell'}, y) = j$, there is a permutation τ with $\tau(x) = x$ and $\tau(y) = y$ such that $\tau(\succ_\ell) = \succ_{\ell'}$. This means that $\tilde{u}^{x,y}_\ell = \tilde{u}^{x,y}_{\ell'}$ by our previous analysis. Put differently, this shows that $\tilde{u}^{x,y}_\ell = \hat{s}^{x,y}(r(\succ_\ell, x), r(\succ_\ell, y))$ for all $\ell \in \{1, \dots, k!\}$. Thus, $\hat{s}^{x,y}$ is not always 0 since $\tilde{u}^{x,y}$ is a non-zero vector. Lastly, we note for all $\ell \in \{1, \dots, k!\}$ that $\hat{s}(r(\succ_\ell, x), r(\succ_\ell, y)) = \tilde{u}^{x,y}_\ell = -\tilde{u}^{x,y}_{\tilde{\tau}(\ell)} = -\hat{s}(r(\tilde{\tau}(\succ_\ell), x), r(\tilde{\tau}(\succ_\ell), y)) = -\hat{s}(r(\succ_\ell, y), r(\succ_\ell, x))$. This proves that $\hat{s}(i, j) = -\hat{s}(j, i)$ for all distinct $i, j \in \{1, \dots, |S|\}$. Hence, $\hat{s}$ satisfies Conditions *(i)* and *(ii)*.

Finally, we need to show for all candidates $w, z \in X$ that $\sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, w), r(\succ_i, z)) \geq 0$ for all $v \in \bar{D}_w$ and $\sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, w), r(\succ_i, z)) \leq 0$ for all $v \in \bar{D}_z$. To ease notation, let $\hat{s}(v, w, z) = \sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, w), r(\succ_i, z))$ for all $v \in \mathbb{R}^{m!}$, and $w, z \in X$. Now, let $\tau : X \to X$ denote a permutation such that $\tau(x) = w$ and $\tau(y) = z$, and let $v \in \bar{D}_w$. This means that $\tau^{-1}(v) \in \bar{D}_x$, so $\hat{s}(\tau^{-1}(v), x, y) \geq 0$. Further, for every ranking $\succ \in \mathcal{R}(C)$, it holds that $r(\tau^{-1}(\succ), x) = r(\succ, w)$ and $r(\tau^{-1}(\succ), y) = r(\succ, z)$. This means that $\hat{s}(r(\tau^{-1}(\succ), x), r(\tau^{-1}(\succ), y)) = \hat{s}(r(\succ, w), r(\succ, z))$ for all $\succ \in \mathcal{R}(C)$. Thus, we conclude that $\hat{s}(\tau^{-1}(v), x, y) = \hat{s}(v, w, z)$. Lastly, if $v \in \bar{D}_w$, then $\tau^{-1}(v) \in \bar{D}_x$ and so $\hat{s}(v, w, z) = \hat{s}(\tau^{-1}(v), x, y) \geq 0$.

Further a symmetric argument between z and y shows that $\hat{s}(v, w, z) \leq 0$ if $v \in \bar{D}_z$. Consequently, $\hat{s}$ satisfies all properties required by this step.

Step 2: We will next show that there is a non-constant score function $s : \{1, \dots, |X|\} \to \mathbb{R}$ such that, for all $x, y \in X$ and $v \in \bar{D}_x$, it holds $\sum_{i=1}^{k!} v_i s(r(\succ_i, x)) \geq \sum_{i=1}^{k!} v_i s(r(\succ_i, y))$ if $v \in \bar{D}_x$ and $\sum_{i=1}^{k!} v_i s(r(\succ_i, x)) \leq \sum_{i=1}^{k!} v_i s(r(\succ_i, y))$ if $v \in \bar{D}_y$. We recall that $\hat{s}(v, x, y) = \sum_{i=1}^{k!} v_i \hat{s}(r(\succ_i, x), r(\succ_i, y))$ for all candidates $x, y \in X$ and $v \in \mathbb{R}^{k!}$. For defining s, we show that $\hat{s}(r_1, r_2) + \hat{s}(r_2, r_3) = \hat{s}(r_1, r_3)$ for all ranks $r_1, r_2, r_3 \in \{1, \dots, k\}$. To this end, we need to construct several auxiliary profiles.

First, consider the profile R^1 that consists of a single voter reporting a ranking $\succ \in \mathcal{R}(X)$ where x is top-ranked. By faithfulness, we have that $f(R^1) = \{\succ\}$, so $v(R^1) \in \bar{D}_x$. Moreover, we claim that $\hat{s}(r(\succ, x), r(\succ, y)) > 0$ for all $y \in X \setminus \{x\}$. Assume for contradiction that this is not true. Since $v(R^1) \in \bar{D}_x$, it holds that $\hat{s}(r(\succ, x), r(\succ, y)) \geq 0$ for all $y \in X \setminus \{x\}$ by the analysis of Step 1. Hence, there is a candidate $y \in X \setminus \{x\}$ with $\hat{s}(r(\succ, x), r(\succ, y)) = 0$. Now, let $v \in \mathbb{N}_{\geq 0} \setminus \{0\}$ denote a vector such that $\hat{s}(v, x, y) < 0$. Such a vector must exist since $\bar{D}_y$ is a fully dimensional cone that is closed under addition with the all-1 vector v^* given by $v^*_i = 1$ for all $i \in \{1, \dots, k!\}$. Let R^2 be a profile such that $v(R^2) = v$. This means that $f(R^2)$ does not contain a ranking that top-ranks x because this would entail that $\hat{s}(v, x, y) \geq 0$. Next, by continuity, there is $\lambda \in \mathbb{N}$ such that $f(\lambda R^1 + R^2, X) \subseteq f(R^1, X)$ and thus $\lambda v(R^1) + R^2 \in \bar{D}_x$. However, for every λ, it holds that $\hat{s}(v(\lambda R^1 + R^2), x, y) = \hat{s}(R^2, x, y) < 0$ because $\hat{s}(v(R^1), x, y) = 0$. This proves that $s(r(\succ, X, x), r(\succ, X, y)) > 0$ for all $y \in X \setminus \{x\}$.

Now, let R^x denote the profile where every ranking $\succ \in \mathcal{R}(C)$ that top-ranks x is reported once. By the previous discussion, we have that $\hat{s}(r(\succ, X, x), r(\succ, X, y)) > 0$ for each ranking $\succ$ in R^x and candidate $y \in X \setminus \{x\}$, so $\hat{s}(v(R^x), x, y) > 0$ for all $y \in X \setminus \{x\}$. On the other hand, it holds that $s(v(R^x), y, z) = 0$ for all $y, z \in X \setminus \{x\}$ since the profile is symmetric regarding y and z. That is, for each ranking $\succ$ in R^x, there is also the ranking that swapped the position of y and z in R^x. Since $\hat{s}(i, j) = -\hat{s}(j, i)$ for all $i, j \in \{1, \dots, k\}$, this implies our claim.

Finally, fix three alternatives x, y, z and let R^* denote the profile consisting of R^x, R^y, and R^z. By our previous observation, it holds for all $w \in X \setminus \{x, y, z\}$ that $\hat{s}(v(R^*), x, w) = \hat{s}(v(R^x), x, w) + \hat{s}(v(R^y), x, w) + \hat{s}(v(R^z), x, w) > 0$, so f can only top-rank x, y, or z. Further, we note that x, y, and z are symmetric in R^*, so anonymity and neutrality require that if $f(R^*)$ contains a ranking that top-ranks a candidate in $\{x, y, z\}$, it must also contain rankings that top-rank the other two candidates. This means that $v(R^*) \in \bar{D}_x \cap \bar{D}_y \cap \bar{D}_z$, so $\hat{s}(v(R^*), x, y) = \hat{s}(v(R^*), y, z) = \hat{s}(v(R^*), x, z) = 0$.

Now, assume for contradiction that there are three indices r_1, r_2, r_3 such that $\hat{s}(r_1, r_2) + \hat{s}(r_2, r_3) \neq \hat{s}(r_1, r_3)$. Let R be a profile on 3 voters such that $r(\succ_1, x) = r(\succ_2, y) = r(\succ_3, z) = r_1$, $r(\succ_1, y) = r(\succ_2, z) = r(\succ_3, x) = r_2$, and $r(\succ_1, z) = r(\succ_2, x) = r(\succ_3, y) = r_3$. It holds that $s(v(R), x, y) = s(v(R), y, z) = s(v(R), z, x) = s(r_1, r_2) + s(r_2, r_3) + s(r_3, r_1) = s(r_1, r_2) + s(r_2, r_3) - s(r_1, r_3) \neq 0$. This proves that $v(R) \notin D_w$ for $w \in \{x, y, z\}$. Finally, let $\lambda \in \mathbb{N}$ such that $s(\lambda v(R^*) + v(R), x, w) > 0$ for all $w \notin \{x, y, z\}$. Hence, $\lambda v(R^*) + v(R) \notin \bar{D}_w$ for all $w \in X \setminus \{x, y, z\}$. On the other hand, $s(\lambda v(R^*) + v(R), x, y) = s(v(R), x, y)$, $s(\lambda v(R^*) + v(R), y, z) = s(v(R), y, z)$, and $s(\lambda v(R^*) + v(R), z, x) = s(v(R), z, x)$, so we also have that $\lambda v(R^*) + v(R) \notin \bar{D}_w$ for $w \in \{x, y, z\}$. However, this means that $\lambda v(R^*) + v(R) \notin \bar{D}_w$ for all $w \in X$, which implies that $f(\lambda R^* + R) = \emptyset$. This contradicts the definition of an SPF, so our assumption that $s(r_1, r_2) + s(r_2, r_3) \neq s(r_1, r_3)$ must have been wrong.

Based on the insight of Step 2, we now define the score function s by $s(i) = \hat{s}(i, k)$ for all $i \in \{1, \dots, k-1\}$ and $s(k) = 0$. By our previous insight, it holds for all $i, j \in \{1, \dots, k-1\}$ with $i \neq j$, that $\hat{s}(i, j) = \hat{s}(i, k) + \hat{s}(k, j) = \hat{s}(i, k) - \hat{s}(j, k) = s(i) - s(j)$. Further, if $j = k$, then $\hat{s}(i, j) = s(i) = s(i) - s(j)$ by definition, and an analogous argument holds if $i = k$. Therefore, we conclude that $\hat{s}(v, x, y) \geq 0$ if and only if $\sum_{i=1}^{k!} v_i s(r(\succ_i, x)) - s(r(\succ_i, y)) \geq 0$ for all vectors $v \in \mathbb{R}^{k!}$ and candidates $x, y \in X$. Moreover, since $\hat{s}(i, j) = 0$ is not true for all $i, j \in \{1, \dots, k\}$, s is not constant. This proves our claim.

Step 3: We are now ready to prove this lemma. To this end, let s be the score function defined in Step 2 and let $s(R, x) = \sum_{i \in N_R} s(r(\succ_i, x))$ for every profile R with $C_R = X$ and candidate $x \in X$. Further, we denote by f_s the social choice function given by $f_s(R) = \arg\max_{x \in X} s(R, x)$. We will show that $W(f(R)) = f_s(R)$ for all profiles R with $C_R = X$, which proves the lemma. To this end, we first note that for every profile R with $x \in W(f(R))$, it holds that $v(R) \in \bar{D}_x$. By Step 2, this means that $s(R, x) \geq s(R, y)$ for all $y \in X \setminus \{x\}$. Hence, $x \in f_s(R)$. This proves that $W(f(R)) \subseteq f_s(R)$ for all profiles R over X.

For the converse set inclusion, assume for contradiction that there is a candidate $x \in f_s(R)$ that is not in $W(f(R, X))$. Let R^x be the profile where every ranking $\succ \in \mathcal{R}(X)$ that top-ranks x is reported once. In Step 2, we have shown that $\hat{s}(R^x, x, y) > 0$ for all $y \in X \setminus \{x\}$, which means that $\arg\max_{y \in X} s(R^x, y) = \{x\}$. By continuity, there must be λ such that $f(\lambda R + R^x) \subseteq f(R)$, which means also that $W(f(\lambda R + R^x)) \subseteq W(f(R))$. On the other hand, it holds that $W(f(\lambda R + R^x)) \subseteq f_s(\lambda R + R^x) = \{x\}$. The final equation here follows since x uniquely maximizes $s(R^x, \cdot)$ and it maximizes $s(R, \cdot)$ (possibly among other candidates). However, this contradicts that $W(f(\lambda R + R')) \subseteq W(f(R))$ because $x \notin W(f(R))$. This contradiction shows that $f_S(R) \subseteq W(f(R))$ for all profiles $R \in \mathcal{R}^*$ with $C_R = X$, which completes the proof of this lemma. ☐

Based on Lemma 8, we now prove our characterization.

Theorem 2. *Sequential Plurality is the only SPF that satisfies standardness, quasi-reinforcement, top IIA, faithfulness, and prefix-stability.*

Proof. Since we have shown in the main body that Sequential Plurality satisfies all our conditions, we focus here on the converse. Hence, suppose that f satisfies all axioms listed in the theorem. To show that f is Sequential Plurality, we proceed inductively over the size of the considered agenda. First, if $|X| = 2$, it follows from anonymity, neutrality, quasi-reinforcement, and faithfulness that f is Sequential Plurality. In more detail, let $X = \{x, y\}$ and define $n_{xy}(R)$ (resp. $n_{yx}(R)$) as the number of voters that prefer x to y (resp. y to x) in R. By anonymity and neutrality, it holds that $f(R) = \{xy, yx\}$ for all profiles R with $C_R = \{x, y\}$ and $n_{xy}(R) = n_{yx}(R)$. Further, if $n_{xy}(R) > n_{yx}(R)$, we can decompose R into a profile R^1 where all voters prefer x to y, and a profile R^2 with $n_{xy}(R^2) = n_{yx}(R^2)$. By our previous insight, we have that $f(R^2) = \{xy, yx\}$. Further, faithfulness and quasi-reinforcement require that $f(R^1) = \{xy\}$. Hence, $f(R) = f(R^1+R^2) = \{xy\}$ by quasi-reinforcement, so f also agrees with sequential plurality in this case. This concludes the induction basis.

Next, fix an agenda X with $|X| \geq 3$ and suppose that $f(R) = f_{\texttt{SeqPL}}(R)$ for all profiles R with $C_R \subsetneq X$. By Lemma 8, there is a score function s such that $W(f(R)) = \arg\max_{x\in X} \sum_{i\in N_R} s(r(\succ_i, x))$ for all profiles R with $C_R = X$. We will show that s is equivalent to the plurality scoring function $s_{\texttt{PL}}$ given by $s_{\texttt{PL}}(1) = 1$ and $s_{\texttt{PL}}(i) = 0$ for all $i \in \{2, \dots, k\}$. To this end, we first note that $s(1) > s(x)$ for all $x \in \{2, \dots, |X|\}$. In more detail, if $s(1) \leq s(x)$ for some $x \in \{2, \dots, |X|\}$, then faithfulness would be violated because $W(f(R)) \neq \{x\}$ for the profile that consists of one input ranking $\succ \in \mathcal{R}(X)$ that top-ranks x. Now, let R^x be the profile where every ranking on X that top-ranks x is reported once. It holds that $\sum_{i\in N_{R^x}} s(r(\succ_i, x)) > \sum_{i\in N_{R^x}} s(r(\succ_i, y))$ for all $y \in X \setminus \{x\}$ because x is top-ranked by every voter. Further, by symmetry of the profile, $\sum_{i\in N_{R^x}} s(r(\succ_i, y)) = \sum_{i\in N_{R^x}} s(r(\succ_i, z))$ for all $y, z \in X \setminus \{x\}$. Next let $\hat{R} = R^x + R^y$ be the profile that concatenates two of these profiles for distinct candidates $x, y \in X$. It is easy to see that $W(f(\hat{R})) = \arg\max_{z\in X} \sum_{i\in N_{\hat{R}}} s(r(\succ_i, z)) = \{x, y\}$.

Now, assume for contradiction that there are two ranks $r_1, r_2 \in \{2, \dots, |X|\}$ such that $s(r_1) \neq s(r_2)$. Without loss of generality, we assume that $s(r_1) < s(r_2)$. Since $\hat{R}$ contains every ranking that top-ranks x, there is a voter who prefers x the most and puts y at rank r_1. Let $\hat{R}'$ be the profile where this voter moves y to position r_2. This means that $\sum_{i\in N_{\hat{R}'}} s(r(\succ_i, y)) > \sum_{i\in N_{\hat{R}}} s(r(\succ_i, y))$, but $\sum_{i\in N_{\hat{R}'}} s(r(\succ_i, x)) = \sum_{i\in N_{\hat{R}}} s(r(\succ_i, x))$. Thus, $x \notin W(f(\hat{R}'))$ anymore. However, prefix-stability requires that such a ranking is chosen, because x was a prefix in a chosen ranking and the voter only changes his preferences below x. This is the desired contradiction, proving that $s(r_1) = s(r_2)$ for all $r_1, r_2 \in \{2, \dots, |X|\}$. Now, since f is invariant under additive shifts to s, this means we can assume that $s(2) = \dots = s(|X|) = 0$. Further, since $s(1) > s(2)$ and f is invariant under multiplying s with a positive scaler, we can further assume that $s(1) = 1$. This means that s is the plurality scoring function.

So far, we have that $W(f(R)) = W(f_{\texttt{SeqPL}}(R))$ for all profiles R with $C_R = X$. By top IIA, it holds for every $\rhd \in f(R)$ that $\texttt{Suff}(\rhd, |X| - 1) \in f(R|_{X\setminus\{t(\rhd)\}})$. Further, by our induction hypothesis, $f(R|_{X\setminus\{t(\rhd)\}}) = f_{\texttt{SeqPL}}(R|_{X\setminus\{t(\rhd)\}})$ for all profiles R with $C_R = X$ and rankings $\rhd \in f(R)$. This means that $f(R) \subseteq f_{\texttt{SeqPL}}(R)$ for all profiles R on X. Finally, assume that this is a strict inclusion for some profile R with $C_R = X$, so there is a ranking $\rhd \in f_{\texttt{SeqPL}}(R) \setminus f(R)$. Let R' be a profile where a single voter reports $\rhd$. By faithfulness, $f(R') = \{\rhd\}$. Since Sequential Plurality satisfies quasi-reinforcement, we have that $f_{\texttt{SeqPL}}(\lambda R + R') = \{\rhd\}$ for all $\lambda \in \mathbb{N}$. Further, because f always picks a non-empty subset of sequential plurality, this means that $f(\lambda R + R') = \{\rhd\}$ for all $\lambda \in \mathbb{N}$. However, this contradicts the continuity of f, which postulates that there is $\lambda \in \mathbb{N}$ such that $f(\lambda R + R') \subseteq f(R)$ and thus $\rhd \notin f(\lambda R + R')$. This is the desired contradiction, so $f(R) = f_{\texttt{SeqPL}}(R)$ for all R with $C_R = X$. This completes the induction step and thus proves the theorem. ☐

A.5 Proof of Theorem 1

In this section, we turn to our characterization of Instant Runoff, denoted by $f_{\texttt{IR}}$. Since we have shown that Instant Runoff satisfies all axioms listed in Theorem 1, we focus on the converse and let f denote an SPF that satisfies standardness, quasi-reinforcement, bottom IIA, and minority protection. For showing that f is Instant Runoff, we first prove that f is non-imposing, which allows us to use the insights of Section A.3. We recall here that $s_{PL}(R, x) = |\{i \in N_R : r(\succ_i, x) = 1\}|$ denotes the Plurality score of a candidate $x \in C_R$.

Lemma 9. *f is non-imposing.*

Proof. Fix an agenda $X = \{x_1, \dots, x_k\} \in \mathcal{F}(C)$ and a ranking $\rhd = x_k \dots x_1 \in \mathcal{R}(X)$. We will construct a profile R with $C_R = X$ such that $f(R) = \{\rhd\}$. To this end, let $\succ^* = x_1, \dots, x_k$ denote the ranking that sorts the candidates in X exactly inverse to $\rhd$. Further, let R denote the profile on $k!-1$ voters where every ranking in $\mathcal{R}(X)\setminus\{\succ^*\}$ is reported once. Lastly, let $X_i = \{x_i, \dots, x_k\}$ for all $i \in \{1, \dots, k\}$. We will inductively show that $f(R|_{X_i}) = \{x_k \dots x_i\}$ for all $i \in \{1, \dots, k\}$. For $i = 1$, this proves our lemma as $\rhd = x_k \dots x_1$.

First, the base case $i = k$ is trivial since there is only a single ranking in this case. Next, we inductively assume that our statement holds for some $i \in \{2, \dots, k\}$ and aim to show that $f(R|_{X_{i-1}}) = \{x_k \dots x_{i-1}\}$. To this end, let R^* denote the profile where every ranking in $\mathcal{R}(X)$ is reported by one voter. By the symmetry of R^*, it follows that each candidate $x \in X_{i-1}$ is top-ranked by the same number of voters. Further, this holds even when restricting R^* to X_{i-1}, i.e., $s_{PL}(R^*|_{X_{i-1}}, x) = \frac{k!}{|X_{i-1}|}$ for all $x \in X_{i-1}$. Further, R arises from R^* by deleting the voter reporting $\succ^*$. Since $t(\succ^*|_{X_{i-1}}) = x_{i-1}$, this means that $s_{PL}(R|_{X_{i-1}}, x) = \frac{k!}{|X_{i-1}|}$ for all $x \in X_{i-1} \setminus \{x_{i-1}\}$ and $s_{PL}(R|_{X_{i-1}}, x_{i-1}) = \frac{k!}{|X_{i-1}|} - 1$. Further, because $\frac{k!}{|X_{i-1}|} > \frac{k!-1}{|X_{i-1}|} = \frac{|N_R|}{|X_{i-1}|}$, minority protection requires that no candidate $x \in X_{i-1} \setminus \{x_{i-1}\}$ is bottom-ranked by a ranking in $f(R|_{X_{i-1}})$. Put differently, $\rhd \in f(R|_{X_{i-1}})$ implies that $\rhd$ bottom-ranks x_{i-1}. Further, bottom IIA requires that, if $\rhd \in$

$f(R|_{X_{i-1}})$, then $\texttt{Pref}(\rhd, |X_{i-1}|-1) \in f(R|_{X_{i-1}\setminus\{b(\rhd)\}})$. By the induction hypothesis and the fact that $b(\rhd) = x_{i-1}$, we know that $f(R|_{X_{i-1}\setminus\{b(\rhd)\}}) = f(R|_{X_i}) = \{x_k \dots x_i\}$. Hence, we now derive that $f(R|_{X_{i-1}}) = \{x_k \dots x_{i-1}\}$, which proves the induction step. □

Based on Lemma 9, we can now invoke the results from Section A.3 as f satisfies all necessary properties. To recall these results, we fix an agenda $X \in \mathcal{F}(C)$ with $|X| \geq 2$, denote by $\succ_1, \dots, \succ_{|X|!}$ the rankings in $\mathcal{R}(X)$, and let $v(R)$ be the vector such that v_i states how often the ranking $\succ_i \in \mathcal{R}(X)$ is reported in R. Then, the following claims are true

(1) For every ranking $\rhd \in \mathcal{R}(X)$, there is a convex set $\bar{D}_\rhd \subseteq R^{|X|!}$ such that $\{R \in \mathcal{R}^* \colon \rhd \in f(R)\} = \{R \in \mathcal{R}^* \colon v(R) \in \bar{D}_\rhd\}$.
(2) For every two rankings $\rhd, \rhd' \in \mathcal{R}(X)$, there are non-zero vectors $u^{\rhd,\rhd'} \in \mathbb{R}^{|X|!}$ such that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_\rhd$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_{\rhd'}$.
(3) For every ranking $\rhd \in \mathcal{R}(X)$, it holds that $\bar{D}_\rhd = \{v \in \mathbb{R}^{|X|!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd,\rhd'} \geq 0\}$.

We emphasize that the sets $\bar{D}_\rhd$ and vectors $u^{\rhd,\rhd'}$ for different agendas have different dimensions and are defined with respect to different input rankings. Thus, it is not immediately possible to compare two such sets. To address this issue, we define a projection for vectors from higher-dimensional spaces to lower-dimensional ones. In more detail, fix two agendas $X, Y \in \mathcal{F}(C)$ with $Y \subseteq X$, and let $\succ_1, \dots, \succ_{|X|!}$ and $\succ'_1, \dots, \succ'_{|Y|!}$ denote the corresponding enumerations of all rankings in $\mathcal{R}(X)$ and $\mathcal{R}(Y)$, respectively. Given a vector $v \in \mathbb{R}^{|X|}$, we define by $v^{X\to Y}$ the vector in $\mathbb{R}^{|Y|!}$ such that $v_i^{X\to Y} = \sum_{j \in \{1,\dots,|X|!\} \colon \succ_j|_X = \succ_i} v_j$. To make this definition more intuitive, we consider a profile R over the candidates in X with corresponding vector $v = v(R)$. Then, it holds that $v^{X\to Y} = v(R|_Y)$, i.e., our operation simply maps a profile to the restricted set of input rankings.

In the following, we will analyze the vectors $u^{\rhd,\rhd'}$ for rankings on a given agenda X in depth. To this end, we will first discuss an alternative representation of the sets $\bar{D}_\rhd$ for $\rhd \in \mathcal{R}(X)$. In particular, while Lemma 4 shows that $\bar{D}_\rhd = \{v \in \mathbb{R}^{|X|!} \colon \forall \rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon vu^{\rhd,\rhd'} \geq 0\}$, not all vectors $u^{\rhd,\rhd'}$ will be necessary to define $\bar{D}_\rhd$. To see this, we recall that, by bottom IIA, $\rhd \in f(R)$ implies that $\rhd|_{C_R\setminus\{b(\rhd)\}} \in f(R|_{C_R\setminus\{b(\rhd)\}})$. As we will show, this means that if $v \in \bar{D}_\rhd$, then $v^{X\to \texttt{T}(\rhd,|X|-1)} \in \bar{D}_{\texttt{Pref}(\rhd,|X|-1)}$. This insight can be used to remove some vectors $u^{\rhd,\rhd'}$ from the hyperplane representation of $\bar{D}_\rhd$.

To make this more formal, we say two ranking $\rhd, \rhd' \in \mathcal{R}(X)$ are *prefix-consistent* if $\texttt{T}(\rhd,\ell) = \texttt{T}(\rhd',\ell)$ implies $\texttt{Pref}(\rhd,\ell) = \texttt{Pref}(\rhd',\ell)$ for all $\ell \in \{2,\dots,|X|-1\}$. Put differently, for two rankings to be prefix-consistent, it is not possible that both rankings put all candidates in a set Y (with $2 \leq |Y| \leq |X|-1$) ahead of all candidates in $X \setminus Y$ but disagree on the order of the candidates in Y. For notational convenience, given a ranking $\rhd$ in $\mathcal{R}(X)$, we denote by $PC(\rhd) = \{\rhd' \in \mathcal{R}(X) \setminus \{\rhd\} \colon \rhd \text{ and } \rhd' \text{ are prefix-consistent}\}$ the set of rankings that are prefix-consistent with $\rhd$. Based on this notation, we will now derive an alternative description of $\bar{D}_\rhd$.

Lemma 10. *For every agenda $X \in \mathcal{F}(C)$ with $k = |X| \geq 3$ and ranking $\rhd \in \mathcal{R}(X)$, it holds that*

$$\bar{D}_\rhd = \{v \in \mathbb{R}^{k!} \colon \forall \rhd' \in PC(\rhd) \colon vu^{\rhd,\rhd'} \geq 0 \text{ and } \forall \ell \in \{2,\dots,k-1\} \colon v^{X\to\texttt{T}(\rhd,\ell)} \in \bar{D}_{\texttt{Pref}(\rhd,\ell)}\}.$$

Proof. Fix an agenda $X \in \mathcal{F}(C)$ with $|X| \geq 3$, let $k = |X|$, and consider a ranking $\rhd \in \mathcal{R}(X)$. Further, for easier notation, we let $S = \{v \in \mathbb{R}^{k!} \colon \forall \rhd' \in PC(\rhd) \colon vu^{\rhd,\rhd'} \geq 0 \text{ and } \forall \ell \in \{2,\dots,k-1\} \colon v^{X\to\texttt{T}(\rhd,\ell)} \in \bar{D}_{\texttt{Pref}(\rhd,\ell)}\}$. Our goal is to show that $\bar{D}_\rhd = S$.

The key idea of this lemma is that $v \in \bar{D}_\rhd$ implies that $v^{\texttt{T}(\rhd,\ell)} \in \bar{D}_{\texttt{Pref}(\rhd,\ell)}$ for every integer $\ell \in \{2,\dots,k-1\}$, and vector $v \in \mathbb{R}^{k!}$. To see this, let $\hat{g}_X$ and $\hat{g}_{\texttt{T}(\rhd,\ell)}$ denote the extension of f defined in Lemma 2 for the agendas X and $\texttt{T}(\rhd,\ell)$. Further, let $\hat{v} \in \mathbb{R}^{k!}$ denote the vector with $\hat{v}_i = 1$ for all $i \in \{1,\dots,k!\}$. For every vector $v \in \mathbb{Q}^{k!}$, there are scalars $\lambda \in \mathbb{N}$ and $\eta \in \mathbb{N}_0$ such that $\lambda v + \eta\hat{v} \in \mathbb{N}_0^{k!} \setminus \{0\}$. Moreover, it holds that $\hat{g}_X(v) = \hat{g}_X(\lambda v + \eta\hat{v})$ by construction. Next, since $\lambda v + \eta\hat{v} \in \mathbb{N}_0^{k!} \setminus \{0\}$, there is a profile R such that $C_R = X$ and $v(R) = \lambda v + \eta\hat{v}$. By Lemma 2, this means that $\hat{g}(v) = \hat{g}(\lambda v + \eta\hat{v}) = f(R)$. Hence, if $\rhd \in \hat{g}(v)$, then $\rhd \in f(R)$. In turn, by applying bottom IIA $k-\ell$ times, it follows that $\texttt{Pref}(\rhd,\ell) \in f(R|_{\texttt{T}(\rhd,\ell)})$. Because $v(R|_{\texttt{T}(\rhd,\ell)}) = (\lambda v + \eta\hat{v})^{X\to\texttt{T}(\rhd,\ell)}$, this implies that $\texttt{Pref}(\rhd,\ell) \in \hat{g}_{\texttt{T}(\rhd,\ell)}((\lambda v + \eta\hat{v})^{X\to\texttt{T}(\rhd,\ell)})$. Finally, we note that $(\lambda v + \eta\hat{v})^{X\to\texttt{T}(\rhd,\ell)} = \lambda v^{X\to\texttt{T}(\rhd,\ell)} + \eta\hat{v}^{X\to\texttt{T}(\rhd,\ell)}$. Since $\hat{v}^{X\to\texttt{T}(\rhd,\ell)}$ takes the same value at every entry, it holds that $\hat{g}_{\texttt{T}(\rhd,\ell)}(v^{X\to\texttt{T}(\rhd,\ell)}) = \hat{g}_{\texttt{T}(\rhd,\ell)}(\lambda v^{X\to\texttt{T}(\rhd,\ell)} + \eta\hat{v}^{X\to\texttt{T}(\rhd,\ell)})$. Thus, $\rhd \in \hat{g}_X(v)$ implies that $\texttt{Pref}(\rhd,\ell) \in \hat{g}_{\texttt{T}(\rhd,\ell)}(v^{X\to\texttt{T}(\rhd,\ell)})$. This means that, if $v \in D_\rhd$, then $v^{X\to\texttt{T}(\rhd,\ell)} \in D_{\texttt{Pref}(\rhd,\ell)}$. Finally, our claim follows since this relation is preserved when taking the closure of $D_\rhd$ and $D_{\texttt{Pref}(\rhd,\ell)}$.

Now, let $v \in \bar{D}_\rhd$. By our discussion so far, this means $v^{X\to\texttt{T}(\rhd,\ell)} \in \bar{D}_{\texttt{Pref}(\rhd,\ell)}$ for all $\ell \in \{2,\dots,k-1\}$. Further, Lemma 4 shows that $vu^{\rhd,\rhd'} \geq 0$ for all $\rhd' \in \mathcal{R}(X)$. This means that $v \in S$ and thus $\bar{D}_\rhd \subseteq S$.

For the other set inclusion, we will show that the interior of S is a subset of $\bar{D}_\rhd$. We hence observe that $\text{int } S \neq \emptyset$ because $\bar{D}_\rhd \subseteq S$ and $\text{int } \bar{D}_\rhd \neq \emptyset$ (see Section A.3). Now, let $v \in \text{int } S$ and assume for contradiction that $v \notin \bar{D}_\rhd$. Because $\bigcup_{\rhd' \in \mathcal{R}(X)} \bar{D}_\rhd = \mathbb{R}^{k!}$, there is a ranking $\rhd^* \in \mathcal{R}(X)$ such that $v \in \bar{D}_{\rhd^*}$. Further, $\rhd^* \notin PC(\rhd)$ because $v \in \text{int } S$ implies that $vu^{\rhd,\rhd'} > 0$ for all rankings $\rhd' \in PC(\rhd)$. Hence, $\rhd^*$ is not prefix-consistent with $\rhd$, so there is an integer $\ell \in \{2,\dots,k-1\}$ such that $\texttt{T}(\rhd,\ell) = \texttt{T}(\rhd^*,\ell)$ but $\texttt{Pref}(\rhd,\ell) \neq \texttt{Pref}(\rhd^*,\ell)$. Just as for $\bar{D}_\rhd$, $v \in \bar{D}_{\rhd^*}$ implies that $v^{X\to\texttt{T}(\rhd^*,\ell)} \in \bar{D}_{\texttt{Pref}(\rhd^*,\ell)}$. However, since $v \in \text{int S}$, it must also hold that $v^{X\to\texttt{T}(\rhd,\ell)} \in \text{int } \bar{D}_{\texttt{Pref}(\rhd,\ell)}$. Using this insight in combination with Lemma 4, we derive that $vu^{\texttt{Pref}(\rhd,\ell),\texttt{Pref}(\rhd^*,\ell)} > 0$. However, this contradicts that $v \in \bar{D}_{\texttt{Pref}(\rhd^*,\ell)}$. Hence, we conclude that $\text{int } S \subseteq \bar{D}_\rhd$ and therefore also $S \subseteq \bar{D}_\rhd$. □

Motivated by Lemma 10, we next aim to understand the vectors $u^{\rhd,\rhd'}$ for prefix-consistent rankings $\rhd, \rhd'$. For this, it will be helpful to have profiles where two arbitrary prefix-consistent rankings are chosen. Indeed, such profiles immediately give constraints on the corresponding vector $u^{\rhd,\rhd'}$. As an intermediate step for this, we will analyze the outcome of f for profiles where each candidate in the considered agenda is top-ranked by the same number of voters. In particular, we will show that, in such profiles, every candidate in the agenda must be bottom-ranked by some output ranking.

Lemma 11. *Let $R \in \mathcal{R}^*$ be a profile such that $s_{PL}(R, x) = s_{PL}(R, y)$ for all $x, y \in C_R$. For every candidate $x \in C_R$, there is a ranking $\rhd \in f(R)$ such that $b(\succ) = x$.*

Proof. Let R denote an arbitrary profile that satisfies the conditions of our lemma. We assume for contradiction that there is an candidate $x \in C_R$ such that no ranking in $f(R)$ bottom-ranks x. In this case, let R' denote a profile with $|X|-1$ voters and $C_{R'} = C_R$ such that, for each candidate $y \in C_R \setminus \{x\}$, there is one voter i who top-ranks y. By continuity, there is $\lambda \in \mathbb{N}$ such that $f(\lambda R + R') \subseteq f(R)$. However, for every $\lambda \in \mathbb{N}$ and $y \in C_R \setminus \{x\}$, it holds that $s_{PL}(\lambda R + R', y) = \lambda \cdot s_{PL}(R, y) + 1 > \frac{|C_R| \cdot \lambda \cdot s_{PL}(R,y) + |C_R| - 1}{|C_R|} = \frac{|N_{\lambda R + R'}|}{|C_R|}$. By minority protection, this means that no candidate $y \in C_R \setminus \{x\}$ can be bottom-ranked in $f(\lambda R + R')$. Or, conversely, each ranking in $f(\lambda R + R')$ has to bottom-rank x. However, this contradicts that $f(\lambda R + R') \subseteq f(R)$ because no ranking in $f(R)$ bottom-ranks x. This is the desired contradiction, which proves the lemma. □

We note that Lemma 11 is quite helpful because it makes no assumption on the order of the lower ranked candidates in the profile. Moreover, this lemma, together with minority protection, implies that f is Instant Runoff for agendas of size 2. In more detail, minority protection immediately postulates that $f(R) = \{xy\}$ for every profile R with $C_R = \{x, y\}$ where more than half of the voters prefer x to y. Further, in the case of a majority tie between x and y, Lemma 11 implies that each candidate needs to be bottom-ranked in one output ranking, so $f(R) = \{xy, yx\}$.

Motivated by this insight, we fix an agenda $X \in \mathcal{F}(C)$ with $|X| \geq 3$ and inductively assume that f is Instant Runoff for every profile R with $C_R \subsetneq X$. Moreover, let $k = |X|$. For showing that f also is Instant Runoff for the agenda X, we will next investigate prefix-consistent rankings in more depth. Specifically, we will show that, for any two prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$, there is a profile R such that both of these rankings are chosen and no other chosen ranking bottom-ranks the same candidate as $\rhd$ or $\rhd'$.

Lemma 12. *Suppose that f is Instant Runoff for every profile R with $C_R \subsetneq X$. For all prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$, there is a profile $R \in \mathcal{R}^*$ with $C_R = X$ such that*

(1) $\rhd \in f(R)$ *and* $b(\rhd) \neq b(\rhd'')$ *for all* $\rhd'' \in f(R) \setminus \{\rhd\}$,
(2) $\rhd' \in f(R)$ *and* $b(\rhd') \neq b(\rhd'')$ *for all* $\rhd'' \in f(R) \setminus \{\rhd'\}$,
(3) $s_{PL}(R, x) = s_{PL}(R, y)$ *for all* $x, y \in X$.

Proof. Fix two rankings $\rhd, \rhd' \in \mathcal{R}(X)$ that are prefix-consistent. For easier notation, we let $\rhd = x_k \dots x_1$ and $\rhd' = y_k \dots y_1$. Now, let $\hat{R}$ denote the profile where every ranking in $\mathcal{R}(X)$ is reported by three voters. We split $\hat{R}$ into three voter-disjoint profiles R^1, R^2, R^3 such that each ranking in $\mathcal{R}(X)$ is reported once in each of these profiles. We will derive our final profile R by modifying $\hat{R}$ as follows:

- In R^1, we replace the ranking $x_1 x_2 \dots x_k$ with $x_1 x_k \dots x_2$ and the ranking $y_1 y_2 \dots y_k$ with $y_1 y_k \dots y_2$. Note that prefix-consistency requires that $x_1 \neq y_1$, so the two modified rankings are distinct from each other.
- For each $\ell \in \{2, \dots, k-2\}$, let $B_\ell^{\rhd}$ denote the set of rankings $\succ$ such that *(i)* $\mathtt{T}(\succ, \ell) = \{x_1, \dots, x_\ell\}$, *(ii)* $\mathtt{T}(\succ, \ell') \neq \{x_1, \dots, x_{\ell'}\}$ for all $\ell' \in \{2, \dots \ell - 1\}$, and *(iii)* $\succ|_{\{x_{\ell+1}, \dots, x_k\}} = x_{\ell+1} \dots x_k$. For all $\ell \in \{2, \dots, k-2\}$, we replace in R^2 every ranking $\succ \in B_\ell^{\rhd}$ with the ranking $\succ' \in \mathcal{R}(X)$ such that $\mathtt{Pref}(\succ', \ell) = \mathtt{Pref}(\succ, \ell)$ and $\succ'|_{\{x_{\ell+1}, \dots, x_k\}} = x_k \dots x_{\ell+1}$. Less formally, $\succ'$ is derived from $\succ$ by reversing the suffix $x_{\ell+1} \dots x_k$ to $x_k \dots x_{\ell+1}$.
- Just as for R^2, we let $B_\ell^{\rhd'}$ denote set of the rankings $\succ$ such that *(i)* $\mathtt{T}(\succ, \ell) = \{y_1, \dots, y_\ell\}$, *(ii)* $\mathtt{T}(\succ, \ell') \neq \{y_1, \dots, y_{\ell'}\}$ for all $\ell' \in \{2, \dots \ell - 1\}$, and *(iii)* $\succ|_{\{y_{\ell+1}, \dots, y_k\}} = y_{\ell+1} \dots y_k$. For each $\ell \in \{2, \dots, k-2\}$, we replace in R^3 every ranking $\succ \in B_\ell^{\rhd'}$ with the ranking $\succ' \in \mathcal{R}(X)$ such that $\mathtt{Pref}(\succ', \ell) = \mathtt{Pref}(\succ, \ell)$ and $\succ'|_{\{y_{\ell+1}, \dots, y_k\}} = y_k \dots y_{\ell+1}$. Less formally, $\succ'$ is derived from $\succ$ by reversing the suffix $y_{\ell+1} \dots y_k$ to $y_k \dots y_{\ell+1}$.

To illustrate these changes, let $\rhd = abcde$. First, in R^1, we replace the ranking $edcba$ with the ranking $eabcd$. Further, in R^2, we replace $edcba$ with $edabc$ and $decba$ with $deabc$ (due to $B_2^{\rhd}$), and $cdeba$ with $cdeab$, $cedab$ with $cedba$, $dceab$ with $dceba$, and $ecdab$ with $ecdba$ (due to $B_3^{\rhd}$). Analogous modifications are performed for the second ranking $\rhd'$.

By the construction of R, we do not change the top-ranked candidate of any input ranking in $\hat{R}$. Hence, every candidate is top-ranked by $3(k-1)!$ voters in R, which means that our profile satisfies Condition (3). By Lemma 11, this also implies that, for each candidate $x \in X$, there is a ranking $\rhd \in f(R)$ that bottom-ranks x. Let $\blacktriangleright \in f(R)$ be a ranking that bottom-ranks x_1. We aim to show that $\blacktriangleright = \rhd$, which proves that $\rhd \in f(R)$ and that $\rhd$ is the only ranking in $f(R)$ that bottom-ranks x_1. To this end, we note that, by bottom IIA, we have that $\blacktriangleright|_{X \setminus \{x_1\}} \in f(R|_{X \setminus \{x_1\}})$. Further, by our induction hypothesis, f is Instant Runoff for every strict subset of X, so $f(R|_{X \setminus \{x_1\}}) = f_{\mathtt{IR}}(R|_{X \setminus \{x_1\}})$. Hence, it suffices to show that $f_{\mathtt{IR}}(R_{X \setminus \{x_1\}}) = \{x_k \dots x_2\}$.

For this, let $X_i = \{x_i, \dots x_k\}$ for all $i \in \{2, \dots, k-1\}$. We will show that, for each $i \in \{2, \dots, k-1\}$, candidate x_i is top-ranked by the fewest voters in $R|_{X_i}$. This implies our claim because instant runoff always eliminates the candidate that is top-ranked by the fewest voters. Now, recall that we derived R from $\hat{R}$. Further, every candidate is top-ranked by the same number of voters in $\hat{R}|_{X_i}$, because, after deleting a candidate from a profile where each ranking is reported once, we simply get several copies of the profile where each ranking over the remaining candidates is reported once. Hence, it suffices to analyze how our changes to $\hat{R}$ modify the plurality scores of the candidates to understand which candidate will be deleted in $R|_{X_i}$. Thus, we fix a round X_i and let y^+ and y^- denote the top-ranked and bottom-ranked candidates in $\rhd'|_{X_i}$, respectively. Then, we make the following observations when comparing $\hat{R}|_{X_i}$ and $R|_{X_i}$.

(1) In R^1, we replace $\succ_1 = x_1 \dots x_k$ with $\bar{\succ}_1 = x_1 x_k \dots x_2$ and $\succ_2 = y_1 \dots y_k$ with $\bar{\succ}_2 = y_1 y_k \dots y_2$. Hence, the best candidate in $\succ_1|_{X_i}$ is x_i and best one in $\bar{\succ}_1|_{X_i}$ is x_k. Further, if $y_1 \in X_i$, the best candidate in both $\succ_2|_{X_i}$ and $\bar{\succ}_2|_{X_i}$ is y_1. Otherwise, the highest ranked candidates in $\succ_2|_{X_i}$ is y^- and the highest ranked one in $\bar{\succ}_2|_{X_i}$ is y^+.

(2) In R^2, for each $\ell \in \{2, \dots, k-2\}$, we replace each ranking $\succ \in B_\ell^{\rhd}$ with the ranking $\succ'$ that reverses the order of the suffix $x_{\ell+1} \dots x_k$. For all $\ell \geq i$ and $\succ \in B_\ell^{\rhd}$, it holds that $X \setminus X_i \subsetneq \mathtt{T}(\succ, \ell)$ because $X \setminus X_i = \{x_1, \dots, x_{i-1}\}$. Since $\succ$ and its replacement $\succ'$ agree on the order of the candidates in $\mathtt{T}(\succ, \ell)$, this means that $\succ|_{X_i}$ and $\succ'|_{X_i}$ top-rank the same candidate. On the other hand, for all $\ell < i$ and $\succ \in B_\ell^{\rhd}$, it holds that $\mathtt{T}(\succ, \ell) \subseteq X \setminus X_i$. Thus, in $\succ'|_{X_i}$, x_k is top-ranked as all higher-ranked candidates were deleted. By contrast, the top-ranked candidate in $\succ|_{X_i}$ is x_i since $\succ|_{\{x_{\ell+1},\dots,x_k\}} = x_{\ell+1} \dots x_k$.

(3) Finally, in R^3, for each $\ell \in \{2, \dots, k-2\}$, we replace each ranking $\succ \in B_\ell^{\rhd'}$ with the ranking $\succ'$ that reverses the order of the suffix $y_{\ell+1} \dots y_k$. In particular, this means that $\succ|_{\mathtt{T}(\succ,\ell)} = \succ'|_{\mathtt{T}(\succ',\ell)}$. Thus, if $\{y_1, \dots, y_\ell\} \not\subseteq X \setminus X_i$, the rankings $\succ|_{X_i}$ and $\succ'|_{X_i}$ agree on the top-ranked candidate. On the other hand, if $\{y_1, \dots, y_\ell\} \subseteq X \setminus X_i$, we delete the common prefix of $\succ$ and $\succ'$. Since $\succ|_{\{y_{\ell+1},\dots,y_k\}} = y_{\ell+1} \dots y_k$ and $\succ'|_{\{y_{\ell+1},\dots,y_k\}} = y_k \dots y_{\ell+1}$, this means that y^- is top-ranked by $\succ|_{X_i}$ and y^+ by $\succ'|_{X_i}$.

It remains to count how much we increase and decrease the Plurality scores of particular candidates. First, if $i = 2$ and thus $X_i = \{x_2, \dots, x_m\}$, it holds that $y_1 \in X_i$ since $y_1 \neq x_1$. Thus, in R^1, we only increase the score of x_k by 1 and decrease the score of x_2 by 1. Further, since we only modify rankings in the sets $B_\ell^{\rhd}$ and $B_\ell^{\rhd'}$ with $\ell \geq 2$ in R^2 and R^3, respectively, we cannot delete the full prefix. Hence, all modifications made in these profiles do not affect the plurality scores of the candidates. This proves that x_2 indeed minimizes the Plurality score in $R|_{X_2}$.

Next, for $i \geq 3$, we have again that, in R^1, we increase the Plurality score of x_k by 1 and decrease the Plurality score of x_i by 1. Moreover, possibly, we increase the score of y^+ by 1 and decrease the score of y^- by 1. Next, in R^2, for each $\ell \in \{2 \dots, i-1\}$, the rankings $\succ \in B_\ell^{\rhd}$ decrease the score of x_i by 1 and increase the score of x_k by 1. To count how many rankings we modified, we observe that, if $\mathtt{T}(\succ, \ell) = \{x_1, \dots, x_\ell\}$ and $\succ|_{\{x_{\ell+1},\dots,x_k\}} = x_{\ell+1} \dots x_k$ for some ranking $\succ$ and integer $\ell \geq 2$, $\succ$ must be in $B_j^{\rhd}$ for $j \leq \ell$. Moreover, because each ranking in $B_\ell^{\rhd}$ orders the candidates $x_{\ell+1}, \dots, x_k$ by their index, it holds that $\succ \notin B_j^{\rhd}$ for any $j \leq \ell$ if $\mathtt{T}(\succ, \ell) \neq \{x_1, \dots, x_\ell\}$ or $\succ|_{\{x_{\ell+1},\dots,x_k\}} \neq x_{\ell+1} \dots x_k$. These two insights imply that $\bigcup_{j \in \{2,\dots,i-1\}} B_j^{\rhd}$ is the set of rankings $\succ$ with $\mathtt{T}(\succ, i-1) = \{x_1, \dots, x_{i-1}\}$ and $\succ|_{\{x_i,\dots x_k\}} = x_i \dots x_k$. Since the sets $B_j^{\rhd}$ are disjoint by definition, we increase the score of x_k by $(i-1)!$ in R^2 and decrease the score of x_i by $(i-1)!$.

Finally, we turn to R^3. First, if $\mathtt{T}(\rhd, k-i+1) = X_i = \mathtt{T}(\rhd', k-i+1)$, it must hold that $x_\ell = y_\ell$ for all $\ell \in \{i, \dots, k\}$. In particular, this implies that $y^- = y_i = x_i$ and $y^+ = y_k = x_k$. Hence, any modifications of the score in R^3 only further decrease the score of x_i and further increase the score of x_k. Since x_i is the only candidate whose score was reduced, this shows that it has the minimal plurality score in this case.

Next, assume that $X_i \neq \mathtt{T}(\rhd', k-i+1)$. Equivalently, this means that $\{x_1, \dots, x_{i-1}\} \neq \{y_1, \dots, y_{i-1}\}$. By our analysis in (3), it holds for all $\succ \in B_\ell^{\rhd'}$ with $\ell \geq i$ that our modifications in R^3 do not affect the best candidate in X_i. Further, for each ranking in $\succ \in B_{i-1}^{\rhd'}$, we have that $\mathtt{T}(\succ, i-1) = \{y_1, \dots, y_{i-1}\}$. Since at least one of these candidates is in X_i, these rankings contribute to the plurality score of the same candidates in $\hat{R}|_{X_i}$ and $R|_{X_i}$. Thus, only the rankings in $B_\ell^{\rhd'}$ with $\ell \in \{2 \dots, i-2\}$ can affect the plurality score. If $i = 3$, this means that the modifications in R^3 do not affect any plurality scores, so x_i is top-ranked by the fewest candidates. On the other hand, if $i \geq 4$, we increase the plurality score of y^+ by at most $(i-2)!$ and decrease the plurality score of y^- by at most $(i-2)!$. In the worst case (if $x_i = y^+$), this means that we decrease the plurality score of x_i by at least $1 + (i-1)! - (1 + (i-2)!) = (i-2)(i-2)!$. By contrast, we decrease the score of y^- by at most $1 + (i-2)! < (i-2)(i-2)!$ if $i \geq 4$. Hence, x_i again has the lowest plurality score in $R|_{X_i}$.

By our analysis, we conclude now that, for all $i \in \{2, \dots, x_k\}$, x_i uniquely minimizes the Plurality score in $R|_{X_i}$, so $f_{\mathtt{IR}}(R|_{X \setminus \{x_1\}}) = \{x_k \dots x_2\}$. This means that the only ranking in $f(R)$ that bottom-ranks x_1 is $\rhd$, thus proving Condition (1). Further, we note that the construction of R is completely symmetric for $\rhd$ and $\rhd'$, so an analogous argument as for $\rhd$ shows Condition (2). □

Based on Lemma 12, we will now analyze the vector $u^{\rhd,\rhd'}$ that separates two prefix-consistent rankings $\rhd, \rhd \in \mathcal{R}(X)$. To ease notation, we will write $u_{\succ}^{\rhd,\rhd'}$ to mean the index of $u^{\rhd,\rhd'}$ corresponding to the ranking $\succ$. Specifically, we will next show that $u^{\rhd,\rhd'}$ only depends on the top-ranked candidate of the input rankings.

Lemma 13. *Fix two prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$ and let $u^{\rhd,\rhd'}$ denote a non-zero vector such that $vu^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_{\rhd}$ and $vu^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_{\rhd'}$. It holds for all $\succ, \succ' \in \mathcal{R}(X)$ with $t(\succ) = t(\succ')$ that $u_{\succ}^{\rhd,\rhd'} = u_{\succ'}^{\rhd,\rhd'}$.*

Proof. Fix two prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$ and let x and y be the bottom-ranked candidates of these rankings. Further, let $u^{\rhd,\rhd'}$ be a non-zero vector that separates $\bar{D}_{\rhd}$ from $\bar{D}_{\rhd'}$, and let R the profile given by Lemma 12. In particular, this means that each candidate $z \in X$ has the same plurality score $s_{PL}(R, z)$, $\rhd$ is the only ranking in $f(R)$

that bottom-ranks x, and $\rhd'$ is the only ranking in $f(R)$ that bottom-ranks y. Next, by quasi-reinforcement and continuity, there is a value λ such that $f(\lambda R + R') \subseteq f(R)$ for each profile R' with $k = |X|$ voters.

Now, consider two rankings $\succ_1$ and $\succ_2$ that top-rank the same candidate z. Let R^1 and R^2 denote two profiles on k voters such that each candidate $x \in X$ is top-ranked by one voter in both R^1 and R^2. Moreover, we suppose that R^1 and R^2 only differ in the ranking of the voter who top-ranks z: in R^1, this voter reports $\succ_1$ and in R^2, he reports $\succ_2$. By the choice of λ, it holds that $f(\lambda R + R^1) \subseteq f(R)$ and $f(\lambda R + R^2) \subseteq f(R)$. Further, for all candidates $x_1, x_2 \in X$, it holds that $s_{PL}(R, x_1) = s_{PL}(R, x_2)$, $s_{PL}(R^1, x_1) = s_{PL}(R^1, x_2)$, and $s_{PL}(R^2, x_1) = s_{PL}(R^2, x_2)$. Thus, the same also holds for $\lambda R + R^1$ and $\lambda R + R^2$. By Lemma 11, this means that, for every candidate w, there are rankings $\rhd_1 \in f(\lambda R + R^1)$ and $\rhd_2 \in f(\lambda R + R^2)$ such that $\rhd_1$ and $\rhd_2$ bottom-rank w. On the other hand, because $f(\lambda R + R^1) \subseteq f(R)$ (resp., $f(\lambda R + R^2) \subseteq f(R)$), the only possible ranking that bottom-ranks x in $f(\lambda R + R^1)$ (resp. $f(\lambda R + R^2)$) is $\rhd$. Similarly, the only ranking in $f(\lambda R + R^1)$ (resp. $f(\lambda R + R^2)$) that can bottom-rank y is $\rhd'$. This implies that $\{\rhd, \rhd'\} \subseteq f(\lambda R + R^1)$ and $\{\rhd, \rhd'\} \subseteq f(\lambda R + R^2)$.

Next, let $v^1 = v(\lambda R + R^1)$ and $v^2 = v(\lambda R + R^2)$. Using the connection between f, $\hat{g}$, $D_\rhd$, and $D_{\rhd'}$, it follows that $v^1, v^2 \in D_\rhd \subseteq \bar{D}_\rhd$ and $v^1, v^2 \in D_{\rhd'} \subseteq \bar{D}_{\rhd'}$. This implies that $v^1 u^{\rhd,\rhd'} = 0$ and $v^2 u^{\rhd,\rhd'} = 0$ and hence also $v^1 u^{\rhd,\rhd'} - v^2 u^{\rhd,\rhd'} = 0$. Finally, we note that $\lambda R + R^1$ and $\lambda R + R^2$ only differ in the ranking of a single voter, who reports $\succ$ in R^1 and $\succ'$ in R^2. Hence, $v^1 u^{\rhd,\rhd'} - v^2 u^{\rhd,\rhd'} = u^{\rhd,\rhd'}_{\succ} - u^{\rhd,\rhd'}_{\succ'}$. Chaining our equalities shows that $u^{\rhd,\rhd'}_{\succ} - u^{\rhd,\rhd'}_{\succ'} = 0$, thus completing the proof of this lemma. $\square$

Based on Lemma 13, we now aim to fully determine the vector $u^{\rhd,\rhd'}$ that separates $\bar{D}_\rhd$ from $\bar{D}_{\rhd'}$.

Lemma 14. *Assume that f agrees with Instant Runoff for every profile R with $C_R \subsetneq Y$, and fix two prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$. Further, let $u \in \mathbb{R}^{k!}$ be the vector such that $u^{\rhd,\rhd'}_{\succ} = -1$ if $t(\succ) = b(\rhd)$, $u^{\rhd,\rhd'}_{\succ} = 1$ if $t(\succ) = b(\rhd')$, and $u^{\rhd,\rhd'}_{\succ} = 0$ otherwise. It holds that $vu \geq 0$ for all $v \in \bar{D}_\rhd$ and $vu \leq 0$ for all $v \in \bar{D}_{\rhd'}$.*

Proof. Fix two prefix-consistent rankings $\rhd, \rhd' \in \mathcal{R}(X)$, and let x and y denote the bottom-ranked candidates of $\rhd$ and $\rhd'$ respectively. Further, let $u^{\rhd,\rhd'} \in \mathbb{R}^{k!}$ be a non-zero vector such that $u^{\rhd,\rhd'} \geq 0$ for all $v \in \bar{D}_\rhd$ and $u^{\rhd,\rhd'} \leq 0$ for all $v \in \bar{D}_\rhd$. By Lemma 13, it holds that $u^{\rhd,\rhd'}_{\succ} = u^{\rhd,\rhd'}_{\succ'}$ for all $\succ, \succ' \in \mathcal{R}(X)$ with $t(\succ) = t(\succ')$. We aim to show that *(i)* $u^{\rhd,\rhd'}_{\succ} = -u^{\rhd,\rhd'}_{\succ'} < 0$ for all $\succ, \succ' \in \mathcal{R}(X)$ with $t(\succ) = b(\rhd)$ and $t(\succ') = b(\rhd')$ and *(ii)* $u^{\rhd,\rhd'}_{\succ} = 0$ for all remaining $\succ \in \mathcal{R}(X)$. This implies our lemma, because we can scale $u^{\rhd,\rhd'}$ by some positive scalar to obtain u.

To ease notation, we will write $u^{\rhd,\rhd'}_z$ for the value that $u^{\rhd,\rhd'}$ assigns to every ranking $\succ$ with $t(\succ) = \{z\}$. Then, we proceed in three steps. Specifically, we will first show that $u^{\rhd,\rhd'}_x = -u^{\rhd,\rhd'}_y$ and $\sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'}_z = 0$. Secondly, we will prove that $u^{\rhd,\rhd'}_x < 0$. In the last step, we will prove that $\sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'}_z = 0$ implies that $u^{\rhd,\rhd'}_z = 0$ for all $z \in X \setminus \{x, y\}$. For each of our claims, we will consider a specific profile R and reason about $f(R)$.

Step 1: As our first step, we will aim to show that $u^{\rhd,\rhd'}_x = -u^{\rhd,\rhd'}_y$ and $\sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'} = 0$. To this end, we define a profile R, which consists of two subprofiles R^1 and R^2. Now, R^1 is the profile such that $R^1|_{X \setminus \{x\}}$ contains every ranking in $\mathcal{R}(X \setminus \{x\})$ once and every ranking in R^1 puts x directly below y. Similarly, R^2 is the ranking such that $R^2|_{X \setminus \{y\}}$ contains every ranking in $\mathcal{R}(X \setminus \{y\})$ once and every ranking in R^2 puts y directly below x. Finally, R is the concatenation of R^1 and R^2. We claim that $\rhd, \rhd' \in f(R)$.

Before proving this claim, we will explain why $\rhd, \rhd' \in f(R)$ implies that $u^{\rhd,\rhd'}_x = -u^{\rhd,\rhd'}_y$ and $\sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'} = 0$. To this end, let R^* denote the profile constructed in Lemma 12 for $\rhd$ and $\rhd'$. In particular, this means that $\rhd, \rhd' \in f(R^*)$. Further, let $v = v(R)$ and $v^* = v(R^*)$. Because $\rhd, \rhd' \in f(R)$ and $\rhd, \rhd' \in f(R^*)$, it follows that $v, v^* \in \bar{D}_\rhd$ and $v, v^* \in \bar{D}_{\rhd'}$. This, in turn, means that $vu^{\rhd,\rhd'} = 0$ and $v^* u^{\rhd,\rhd'} = 0$. Next, we note that, in R^*, every alternative is top-ranked by the same number of voters. This means that there is λ^* such that $v^* u^{\rhd,\rhd'} = \lambda^* \sum_{z \in X} u^{\rhd,\rhd'}_z$. On the other hand, in R, x and y are top-ranked by the same number of voters, but all other candidates are top-ranked by twice as many voters. This holds because no voter in R^1 top-ranks y and no voter in R^2 top-ranks x. Hence, there is λ such that $vu^{\rhd,\rhd'} = \lambda u^{\rhd,\rhd'}_x + \lambda u^{\rhd,\rhd'}_y + 2\lambda \sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'}_z$. Finally, we note that

$$\begin{aligned} 0 &= 2\lambda \cdot v^* u^{\rhd,\rhd'} - \lambda^* \cdot vu^{\rhd,\rhd'} \\ &= 2\lambda \cdot \lambda^* \sum_{z \in X} u^{\rhd,\rhd'}_z \\ &\quad - \lambda \cdot \lambda^* \left(u^{\rhd,\rhd'}_x + u^{\rhd,\rhd'}_y + 2 \sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'}_z \right) \\ &= \lambda \cdot \lambda^* \cdot u^{\rhd,\rhd'}_x + \lambda \cdot \lambda^* \cdot u^{\rhd,\rhd'}_y \end{aligned}$$

This implies that $u^{\rhd,\rhd'}_x = -u^{\rhd,\rhd'}_y$. Moreover, by substituting this into $v^* u^{\rhd,\rhd'} = 0$, it then also follows that $\sum_{z \in X \setminus \{x,y\}} u^{\rhd,\rhd'}_z = 0$.

It remains to show that $\rhd, \rhd' \in f(R)$. Assume for contradiction that this is not the case and assume without loss of generality that $\rhd \notin f(R)$. Further, let $\rhd^*$ denote the profile derived from $\rhd$ by swapping the positions of x and y. Since R is symmetric with respect to x and y, anonymity and neutrality show that $\rhd^* \in f(R)$ if and only if $\rhd \in f(R)$, so we get that $\rhd^* \notin f(R)$. By Claim (2) of Lemma 5, this means that $v(R) \notin \bar{D}_\rhd$ and $v(R) \notin \bar{D}_{\rhd^*}$. Using Lemma 4, this means, in turn, that there are two rankings $\blacktriangleright$ and $\blacktriangleright^*$ such that $vu^{\rhd,\blacktriangleright} < 0$ and $vu^{\rhd^*,\blacktriangleright^*} < 0$. Further, by minority protection, it holds for all $\rhd \in f(R)$ that $\rhd$ bottom-ranks x or

y. Indeed, every alternative $z \in X \setminus \{x,y\}$ is top-ranked by more than $|N_R|/k$ voters and can thus not be bottom-ranked. Thus, as for $\rhd$ and $\rhd^*$, we get for each ranking $\rhd^+$ that does not bottom-rank x or y that there is another ranking $\blacktriangleright^+$ such that $vu^{\rhd^+,\blacktriangleright^+} < 0$.

Lastly, we note that, by assumption, f behaves on $R|_{X\setminus\{x\}}$ and $R|_{X\setminus\{y\}}$ like instant runoff. Hence, we aim to compute instant runoff for these profiles. To this end, we observe that, in R^1 and R^2, each voter ranks x and y consecutively. Hence, by construction of these profiles, both $R^1|_{X\setminus\{x\}}$ and $R^2|_{X\setminus\{x\}}$ correspond to the profile where each ranking in $\mathcal{R}(X \setminus \{x\})$ is reported once. This means that instant runoff chooses every ranking for $R|_{X\setminus\{x\}}$, i.e., $f_{\texttt{IR}}(R|_{X\setminus\{x\}}) = \mathcal{R}(X \setminus \{x\})$. Analogously, it follows that $f_{\texttt{IR}}(R_{X\setminus\{y\}}) = \mathcal{R}(X \setminus \{y\})$. This implies that $v(R) \in \bar{D}_{\rhd_x}$ for all $\rhd_x \in \mathcal{R}(X \setminus \{x\})$ and $v(R) \in \bar{D}_{\rhd_y}$ for all $\rhd_y \in \mathcal{R}(X \setminus \{y\})$. Hence, $v(R|_{X\setminus\{x\}})u^{\rhd_x,\rhd_x'} = 0$ and $v(R_{X\setminus\{y\}})u^{\rhd_y,\rhd_y'} = 0$ for all corresponding rankings.

Finally, let R' denote the profile constructed in Lemma 12 for the rankings $\rhd$ and $\rhd^*$. We note that R' exists since $\rhd$ and $\rhd^*$ are prefix-consistent. To see this, let ℓ be the position such that $r(\rhd, y) = \ell$ and $r(\rhd^*, x) = \ell$. For all $j < \ell$, it holds that $\texttt{Pref}(\rhd, j) = \texttt{Pref}(\rhd^*, j)$ because these two rankings only differ by permuting x and y. On the other hand, for all $j \in \{\ell, \dots, |X|-1\}$, we have that $\texttt{T}(\rhd, j) \neq \texttt{T}(\rhd^*, j)$ because $y \in \texttt{T}(\rhd, j)$ but $y \notin \texttt{T}(\rhd^*, j)$. Further, we have reasoned in the proof of Lemma 12 that $f(R'|_{X\setminus\{x\}}) = \{\texttt{Pref}(\rhd, |X|-1)\}$ and $f(R'|_{X\setminus\{y\}}) = \{\texttt{Pref}(\rhd^*, |X|-1)\}$. By Claim (1) of Lemma 5, this means that $v(R'|_{X\setminus\{x\}})u^{\texttt{Pref}(\rhd,|X|-1),\rhd_x} > 0$ for all $\rhd_x \in \mathcal{R}(X \setminus \{x\}) \setminus \{\texttt{Pref}(\rhd, |X|-1)\}$ and $v(R'|_{X\setminus\{y\}})u^{\texttt{Pref}(\rhd^*,|X|-1),\rhd_y} > 0$ for all $\rhd_y \in \mathcal{R}(X \setminus \{y\}) \setminus \{\texttt{Pref}(\rhd^*, |X|-1)\}$.

Finally, let $v = v(R)$, $v' = v(R')$, and $\epsilon > 0$ such that $vu^{\rhd_1,\rhd_2} < 0$ implies $(v + \epsilon v')u^{\rhd_1,\rhd_2} < 0$ for all $\rhd_1, \rhd_2' \in \mathcal{R}(X)$. By our previous observation, this means that $v + \epsilon v' \notin \bar{D}_{\rhd^+}$ for any $\rhd^+$ with $b(\rhd^*) \notin \{x,y\}$, $v + \epsilon v' \notin \bar{D}_{\rhd}$, and $v + \epsilon v' \notin \bar{D}_{\rhd^*}$. Further, we note that $(v + \epsilon v')^{X \to X\setminus\{x\}} \notin \bar{D}_{\rhd_x}$ for any $\rhd_x \in \mathcal{R}(X \setminus \{x\}) \setminus \{\texttt{Pref}(\rhd, |X|-1)\}$. To see this, we recall that $v^{X\to X\setminus\{x\}}u^{\rhd_x,\rhd_x'} = v(R|_{X\setminus\{x\}})u^{\rhd_x,\rhd_x'} = 0$ for all $\rhd_x, \rhd_x' \in \mathcal{R}(X \setminus \{x\})$ and $(v')^{X\to X\setminus\{x\}}u^{\texttt{Pref}(\rhd,|X|-1),\rhd_x} = v(R'|_{X\setminus\{x\}})u^{\texttt{Pref}(\rhd,|X|-1),\rhd_x} > 0$ for all $\rhd_x \in \mathcal{R}(X \setminus \{x\}) \setminus \{\texttt{Pref}(\rhd, |X|-1)\}$. By Lemma 10, this means that $v+\epsilon v' \notin \bar{D}_{\rhd_1}$ for any ranking $\rhd_1 \in \mathcal{R}(X)\setminus\{\rhd\}$ that bottom-ranks x. Moreover, an analogous argument shows that, among the rankings that bottom rank y, $v + \epsilon v^*$ can only be element of $\bar{D}_{\rhd^*}$. However, we have that $v + \epsilon v^* \notin \bar{D}_{\rhd}$ and $v + \epsilon v^* \notin \bar{D}_{\rhd^*}$, so it follows that $v + \epsilon v^* \notin \bar{D}_{\rhd''}$ for any ranking $\rhd'' \in \mathcal{R}(X)$. This contradicts the basic observation that $\bigcup_{\blacktriangleright \in \mathcal{R}(X)} \bar{D}_{\blacktriangleright} = \mathbb{R}^{|X|!}$. This is the desired contradiction, which proves that $\rhd \in f(R)$.

Step 2: As the next step, we aim to show that $u_x^{\rhd,\rhd'} < 0$. To this end, consider the profile R^1 where all rankings in $\mathcal{R}(X)$ except the inverse of $\rhd$ are reported once. As shown in Lemma 9, it holds that $f(R^1) = \{\rhd\}$. Using Claim (1) of Lemma 5, this means that $v(R^1)u^{\rhd,\rhd'} > 0$. Further, for the profile $\hat{R}$ where every input ranking is present once, it holds that $f(\hat{R}) = \mathcal{R}(X)$ due to anonymity and neutrality. This implies that $v(\hat{R}) \in \bar{D}_{\rhd}$ and $v(\hat{R}) \in \bar{D}_{\rhd'}$, so $v(\hat{R})u^{\rhd,\rhd'} = 0$. Finally, we note that $\hat{R}$ arises from R^1 by adding the voter who reports the inverse of $\rhd$. Since this voter top-ranks x, we get that $0 < v(R^1)u^{\rhd,\rhd'} - v(\hat{R})u^{\rhd,\rhd'} = -u_x^{\rhd,\rhd'}$. By multiplying both sides with -1, our claim follows.

Step 3: Finally, we will show that $u_z^{\rhd,\rhd'} = 0$ for all $z \in X \setminus \{x,y\}$. First, if $|X| = 3$, this follows from the previous step because $\sum_{z\in X\setminus\{y,z\}} u_z^{\rhd,\rhd'} = u_z^{\rhd,\rhd'}$ for the single candidate $z \in X \setminus \{y,z\}$. Hence, assume that $|X| \geq 4$ and recall that $k = |X|$. We will consider again a profile to prove this claim. To define this profile, we denote by R^w, R^{wz}, etc. the profiles where each ranking with prefix w, wz, etc. is reported once. Then, we define the profile $\hat{R}$ as follows:

- For a fixed candidate $c \in X \setminus \{x,y\}$, $\hat{R}$ contains $2(k+1)$ copies of R^c. This subprofile consists of $2(k+1)(k-1)!$ voters.
- For every candidate $z \in X \setminus \{x,y,c\}$, $\hat{R}$ contains $2k$ copies of R^z. In total, this subprofile consists of $(k-3) \cdot 2k(k-1)!$ voters.
- $\hat{R}$ contains $2(k-1)$ copies R^{xz} for $z \in X \setminus \{x,y,z\}$, $3(k-1)$ copies of R^{xy}, and $(k-1)(k-2)$ copies of R^{xyc}. In total, this subprofile consists of $(k-3) \cdot 2(k-1)(k-2)! + 3(k-1)(k-2)! + (k-1)(k-2)(k-3)! = 2(k-1)(k-1)!$ voters
- $\hat{R}$ contains $2(k-1)$ copies R^{yz} for $z \in X \setminus \{x,y,c\}$, $3(k-1)$ copies of R^{yx}, and $(k-1)(k-2)$ copies of R^{yxc}. This subprofile again consists of $2(k-1)(k-1)!$ voters.

By definition, there is a total of $2k \cdot k! - 2(k-1)!$ voters. Since every candidate $z \in X \setminus \{x,y\}$ is top-ranked by at least $2k!$ voters, minority protection implies that x or y must be bottom-ranked by every ranking in $f(\hat{R})$. Just as in Step 1, we will show that $f(\hat{R})$ must contain all such rankings. To this end, we will first consider the outcome for the set $X \setminus \{x\}$. Since f agrees with Instant Runoff for this agenda, we will compute f_{IR}. For this, we note for every profile R^z, R^{wz}, etc. that after removing the common prefix, these profile breaks down into several copies of a completely symmetric profile. In particular, this means that such these profiles do not differentiate between the plurality scores of the remaining candidates. Now, fix any subset $X' \subseteq X$ with $x \notin X$. We investigate the plurality scores of the candidates $z \in X'$ in $\hat{R}|_{X'}$. For this, we use a case distinction with respect to whether $y \in X'$.

First, assume that $y \notin X'$. In this case, every candidate $z \in X'$ with $z \neq c$ gets $4(k-1)!$ additional top-ranks from the voters corresponding to the subprofiles R^{xz} and R^{yz}. Further, if $c \in X'$, it gets $2(k-1)!$ additional top-ranks from the voters corresponding to the profiles R^{xyc} and R^{yxc}. The voters forming the profiles R^{xy} and R^{yx} can be ignored as they reduce to a profile containing each input ranking over the remaining candidates equally often. Similarly, for any $z \notin X'$ with $z \notin \{x, y\}$, the voters in $R^z|_{X'}$ top-rank each candidate equally often. Since c is top-ranked by $2(k-1)!$ more votes than every candidate $z \in X \setminus \{x, y, c\}$ in $\hat{R}$, our analysis shows that every candidate $z \in X'$ has the same plurality score in $\hat{R}|_{X'}$.

Secondly, assume that $y \in X'$. In this case, every candidate $z \in X' \setminus \{y, z\}$ gets $2(k-1)!$ additional top ranks from the voters corresponding to the subprofile R^{xz}. Further, y gets $4(k-1)!$ additional top-ranks from the voters in R^{xy} and R^{xyc}. For all candidates $z \notin X$ and $z \neq x$, the profile $R^z|_{X'}$ can be ignored as every candidate $z \in X'$ is top-ranked equally often. Hence, in $R|_{X'}$, every candidate is top-ranked by the same number of voters. In summary, this means that $f_{\texttt{IR}}(R|_{X\setminus\{x\}}) = \mathcal{R}(X \setminus \{x\})$ because, after deleting x, every candidate is always top-ranked by the same number of voters. Further, an analogous argument shows that $f_{\texttt{IR}}(R|_{X\setminus\{y\}}) = \mathcal{R}(X \setminus \{y\})$.

Based on this insight, we get again that $v(\hat{R}|_{X\setminus\{x\}})u^{\rhd_x, \rhd'_x} = 0$ for all $\rhd_x, \rhd'_x \in \mathcal{R}(X \setminus \{x\})$ and $v(\hat{R}|_{X\setminus\{y\}})u^{\rhd_y, \rhd'_y} = 0$ for all $\rhd_y, \rhd'_y \in \mathcal{R}(X \setminus \{x\})$. By using an analogous argument as in Step 1, we can now infer a contradiction if $f(R)$ does not contain all rankings that bottom-rank x and y.

As the last point, we will show that this implies that $u_c^{\rhd,\rhd'} = 0$. To this end, we recall that $u_x^{\rhd,\rhd'} = -u_y^{\rhd,\rhd'}$ and $\sum_{z\in X\setminus\{x,y\}} u_z^{\rhd,\rhd'} = 0$. Further, because $\rhd, \rhd' \in f(R)$, we have that $v(\hat{R})u^{\rhd,\rhd'} = 0$. Hence, we conclude that

$$
\begin{aligned}
0 &= v(\hat{R})u^{\rhd,\rhd'} \\
&= 2(k+1)(k-1)! \cdot u_c^{\rhd,\rhd'} + 2k(k-1)! \sum_{z\in X\setminus\{x,y,c\}} u_z^{\rhd,\rhd'} \\
&\quad + 2(k-1)(k-1)! \cdot u_x^{\rhd,\rhd'} + 2(k-1)(k-1)! \cdot u_y^{\rhd,\rhd'} \\
&= 2(k-1)! u_c^{\rhd,\rhd'} + 2k(k-1)! \sum_{z\in X\setminus\{x,y\}} u_z^{\rhd,\rhd'} \\
&= 2(k-1)! u_c^{\rhd,\rhd'}.
\end{aligned}
$$

Since c is chosen arbitrarily, this completes the proof of this lemma. □

We are now ready to show that f is Instant Runoff.

Theorem 1. *Instant Runoff is the only SPF that satisfies standardness, quasi-reinforcement, bottom IIA, and minority protection.*

Proof. Fix a profile R over an agenda X. When $|X| = 2$, f is Instant Runoff by minority protection and Lemma 11. We hence suppose that $|X| \geq 3$ and assume inductively that $f(R|_Y) = f_{\texttt{IR}}(R|_Y)$ for every $Y \subsetneq X$. We will first show that $f(R|_X) \subseteq f_{\texttt{IR}}(R|_X)$. By bottom IIA and our induction hypothesis, it suffices to prove that f only bottom-ranks candidates that are top-ranked by the least voters. Thus, let x denote a candidate with minimal plurality score in R, and let $\rhd$ be a ranking that bottom-ranks a candidate y that is top-ranked by strictly more voters than x. Let $\rhd'$ be the ranking derived from $\rhd$ by exchanging the positions of x and y. We observe that $\rhd$ and $\rhd'$ are prefix-consistent. Lemma 14 hence shows that $v(R)u^{\rhd,\rhd'} = -s_{PL}(R, y) + s_{PL}(R, x) < 0$, so $v(R) \notin \bar{D}_{\rhd}$. This suffices to prove that $\rhd \notin f(R)$. By this argument, f indeed only can choose ranking that bottom-rank the candidate with the fewest first-place votes. By bottom IIA and our induction hypothesis, this implies that $f(R) \subseteq f_{\texttt{IR}}(R)$.

Finally, suppose for contradiction that $f(R) \subsetneq f_{\texttt{IR}}(R)$. Let $\rhd$ be a ranking in $f_{\texttt{IR}}(R) \setminus f(R)$. Since Instant Runoff is non-imposing, there is a profile R' such that $f_{\texttt{IR}}(R') = \{\rhd\}$. By our previous set inclusion, this also means that $f(R') = \{\rhd\}$. Next, continuity requires that there is $\lambda \in \mathbb{N}$ such that $f(\lambda R + R') \subseteq f(R)$. However, by quasi-reinforcement, we know that $f_{\texttt{IR}}(\lambda R + R') = \{\rhd\}$ for all $\lambda \in \mathbb{N}$. This contradicts that $f(R) \subseteq f_{\texttt{IR}}(R)$ for all profiles R, so our assumption that $f(R) \subsetneq f_{\texttt{IR}}(R)$ is wrong. Hence, $f(R) = f_{\texttt{IR}}(R)$ and f is indeed Instant Runoff. □

A.6 The Role of Anonymity in Theorem 1

As the last point, we will show that anonymity is not necessary for the proof of Theorem 1. In more detail, we will prove that minority protection, bottom-IIA, reinforcement, and continuity imply anonymity. To this end, we first note that the definition of continuity in the main body only works for anonymous SPFs, because it is otherwise unclear to which profile λR refers to. To address this issue, we present next a definition of continuity for non-anonymous SPFs.

Continuity. An SPF f is continuous if, for all profiles $R, R' \in \mathcal{R}^*$ with $N_R \cap N_{R'} = \emptyset$ and $C_R = C_{R'}$, there is $\lambda \in \mathbb{N}$ and another profile $R'' \in \mathcal{R}''$ such that

(1) $C_{R''} = C_R$, $|N_{R''}| = \lambda |N_R|$, and $N_{R''} \cap N_{R'} = \emptyset$,

(2) there is a mapping ϕ from $N_{R''}$ to N_R such that $\succ''_i = \succ_{\phi(i)}$ for all $i \in N_{R''}$ and $|\{i \in N_{R''} : \phi(i) = j\}| = \lambda$ for each $j \in N_R$,

(3) $f(R'' + R') \subseteq f(R)$.

Less formally, this definition says that for every two profiles R and R', we can find a profile R'' that contains λ copies of every voter in R and is voter-disjoint with R' such that $f(R'' + R')$ is a subset of $f(R)$. When the SPF f is anonymous, this condition is equivalent to the definition given in the main body.

As we will show next, this non-anonymous definition of continuity, together with bottom IIA, quasi-reinforcement, and minority protection implies anonymity. The proof of the following proposition is inspired by Lemma 1 of Young (1974). Moreover, Step 2 in our proof shows that an analogous result holds for all SPFs that satisfy quasi-reinforcement and return all rankings for the profile where every ranking is present.

Proposition 2. *If an SPF f satisfies minority protection, bottom IIA, quasi-reinforcement, and continuity, it is also anonymous.*

Proof. Fix an SPF f that satisfies our conditions. We will prove our proposition in two steps. First, we show that

$f(R) = \mathcal{R}(C_R)$ for all profiles R where each input ranking in $\mathcal{R}(C_R)$ is reported once. Based on this insight, we show in the second step that f is anonymous.

Step 1: Fix an agenda X with $k = |X|$ and let R be a profile where every ranking in $\mathcal{R}(X)$ is reported once. We assume for contradiction that $f(R) \neq \mathcal{R}(X)$ and let $\rhd$ be a ranking in $\mathcal{R}(X) \setminus f(R)$. Further, let $\succ^*$ denote the reverse ranking of $\rhd$, and let R' be a profile where every ranking in $\mathcal{R}(X) \setminus \{\succ^*\}$ is reported once. As reasoned in Lemma 9, it holds that $f(R) = \{\rhd\}$ by bottom IIA and minority protection. Finally, by continuity, there is $\lambda \in \mathbb{N}$ and a profile R that, intuitively, consists of λ copies of R such that $f(R'' + R') \subseteq f(R)$. In particular, this means that $\rhd \notin f(R'' + R')$. However, for every agenda $Y \subseteq X$, it holds that each candidate but $b(\rhd|_Y) = t(\succ^*|_Y)$ is top-ranked by $\frac{(\lambda+1)\cdot k!}{|Y|}$ voters in $R'' + R'$, whereas $b(\rhd|_Y)$ is only top-ranked by $\frac{(\lambda+1)\cdot k!}{|Y|} - 1$ voters. Since $\frac{(\lambda+1)\cdot k!}{|Y|} > \frac{(\lambda+1)\cdot k!-1}{|Y|}$, minority protection requires that every ranking in $f((R''+R')|_Y)$ bottom-ranks $b(\rhd|_Y)$. Finally, by repeatedly applying this insight and pairing it with bottom IIA, it follows that $f(R'' + R') = \{\rhd\}$, which contradicts our previous insight.

Step 2: Fix two profiles R and R' over an agenda X, and a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $R' = \pi(R)$. We will show that $f(R) = f(R')$. To this end, let R'' denote a profile that is voter-disjoint from both R and R' such that each ranking in $\mathcal{R}(X)$ is reported by the same number of voters in $R+R''$. Since R' is merely a permutation of R, this means also that every ranking in $\mathcal{R}(X)$ is reported by the same number of voters in $R' + R''$. By Step 1 and quasi-reinforcement, it follows that $f(R + R'') = f(R' + R'') = \mathcal{R}(X)$. Now, if R and R' are voter disjoint, quasi-reinforcement shows that $f(R) = f(R + R' + R'')$ and $f(R') = f(R + R' + R'')$ because $f(R) \subseteq f(R' + R'')$ and $f(R') \subseteq f(R + R'')$. On the other hand, if R and R' are not voter disjoint, we can consider a fourth profile $\hat{R}$ that is voter-disjoint from R, R', and R'' and a permutation of R. We can always find such a profile because we have infinitely many potential voters, whereas profiles are only defined for a finite number of voters. By repeating our previous argument for $\hat{R}$, we find that $f(R) = f(\hat{R})$ and $f(\hat{R}) = f(R')$, so $f(R) = f(R')$ as desired. This proves that f is indeed anonymous. $\square$