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# On Stein's equation, Vandermonde matrices and Fisher's information matrix of time series processes. Part I: The autoregressive moving average process

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**Abstract**

This paper introduces several forms of relationships between Fisher's information matrix of an autoregressive-moving average or ARMA process and the solution of a corresponding Stein equation. Fisher's information matrix consists of blocks associated with the autoregressive and moving average parameters. An interconnection with a solution of Stein's equation is set forth for the block case as well as for Fisher's information matrix as a global matrix involving all parameter blocks. Both cases have their importance for the interpretation of the estimated parameters. The cases of distinct and multiple eigenvalues are addressed. The obtained links involve equations with left and right inverses, these can be expressed in terms of the inverse of appropriate Vandermonde matrices. A condition is set forth for establishing an equality between Fisher's information matrix and a solution to Stein's equation. Two examples are presented for illustrating some of the results obtained. The global and off-diagonal block case with distinct and multiple roots, respectively, are considered. © 2001 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

The main objects of study of this paper consists of investigating interconnections between Fisher's information matrix and solutions of Stein's equation. The links are verified for a univariate ARMA process, this type of structure is known both in the statistical and engineering literature, see, e.g. [2,3,10]. The ARMA process consists of autoregressive and moving average parameters which are estimated accordingly and they are respectively the coefficients of the autoregressive and moving average polynomials, in [1], some algebraic properties of ARMA process type polynomials are also studied. The quality of these estimated parameters is described by Fisher's information matrix which corresponds to the Cramer–Rao lower bound. The latter is part of an inequality which is of fundamental importance both in statistical theory and estimation in signal processing. Fisher's information matrix consists of blocks which are associated with the ARMA parameters. The purpose of this paper is also to study possible algebraic properties of statistical information which in our case is described by Fisher's information matrix. A companion matrix will be chosen for Stein's equation such that the eigenvalues of the corresponding resolvent are equivalent with the roots of the appropriate ARMA polynomial(s). This allows the link between Fisher's information matrix and a solution of Stein's equation to be established by using Cauchy's residue theorems, eventually the link is obtained through associating the common poles of both expressions. Two cases are considered: (i) one block is investigated individually and linked with a corresponding solution of Stein's equation; (ii) Fisher's information matrix where all the parameters are taken into account and expressed in terms of the Sylvester resultant matrix, is connected with an appropriate solution of Stein's equation. In both cases (i) and (ii) the situations of distinct and multiple roots are studied. The relations produced in both cases (distinct and multiple roots) contain left and right inverses. Depending on the case, each of these inverses can be expressed in terms of a Vandermonde matrix. For the case of multiple roots a generalized Vandermonde is involved. Some of the Vandermonde matrices obtained in cases (i) and (ii) are not square so that an appropriate left or right inverse is then additionally necessary for further study. The global approach and the off-diagonal block cases are illustrated by means of an example for distinct and multiple roots, respectively.

The paper is organized as follows: in Section 2 the link is established between Fisher's information matrix and a solution of Stein's equation for the case of the parameter block and for both multiple and distinct eigenvalues. In Section 3, left and right inverses obtained in Section 2 are formulated in terms of appropriate Vandermonde matrices. In Section 4 the link is set forth for the global Fisher information matrix which contains all the parameters. Section 5 analyses left and right inverses obtained in Section 4 and is followed by Section 6 which points out the analogy with the solution of Lyapunov equation. Examples are presented in Section 7.

In this section the main theorem that will be extensively used in this paper is formulated and is based on Lancaster and Rodman [6]. Some additional notational conventions concerning the companion matrix and some of its properties will be summarized .

Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $\Gamma \in \mathbb{C}^{n\times m}$ .

The Stein equation

$$S - BSA^{\mathrm{T}} = \Gamma \tag{1.1}$$

has a unique solution iff $\lambda\mu \neq 1$ for any $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$. From [6] we take

**Theorem 1.1.** *If the Stein equation* (1.1) *has a unique solution S, then*

$$S = \frac{1}{2\pi \mathrm{i}} \oint_C (\lambda I - B)^{-1}\, \Gamma\, (I - \lambda A)^{-\mathrm{T}}\, \mathrm{d}\lambda, \tag{1.2}$$

*where C is a single closed contour with $\sigma(B)$ inside C and for each nonzero $w \in \sigma(A)$, $w^{-1}$ is outside C.*

Consider the notations for the following companion matrix:

$$X = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -x_n & -x_{n-1} & \cdots & -x_1 \end{pmatrix}.$$

It is known that its characteristic polynomial $x(z)$ is given by

$$x(z) = \det(zI - X) = z^n + x_1 z^{n-1} + \cdots + x_n. \tag{1.3}$$

The reciprocal polynomial $x^*$ of $x$ is given by

$$x^*(z) = \det(I - zX) = 1 + x_1 z + \cdots + x_n z^n. \tag{1.4}$$

We will also use the Hörner polynomials $x_k(\cdot)$, recursively defined by $x_0(z) = 1$ and $x_k(z) = z x_{k-1}(z) + x_k$. Furthermore we will use the adjoint matrices

$$\mathrm{adj}(zI - X) = (zI - X)^{-1} x(z) \tag{1.5}$$

and

$$\mathrm{adj}(I - zX) = (I - zX)^{-1} x^*(z). \tag{1.6}$$

It is straightforward to verify that the adjoint matrix $\mathrm{adj}(I - zX) = z^{n-1}\,\mathrm{adj}(z^{-1}I - X)$. Given the structure of the matrix $X$, an explicit expression of the corresponding adjoint matrix can be formulated in the next proposition.

**Proposition 1.2.** *Consider a square matrix X of dimension n with the parametrization given above, the related adjoint matrix adj $(zI - X)$ is described by*

$$\operatorname{adj}(zI - X) = \sum_{k=1}^{n} x_{n-k}(z) X^{k-1}, \tag{1.7}$$

*where* $x_{n-k}(z)$ *is a polynomial defined in* (1.3) *of degree* $n - k$.

**Proof.** Just multiply the right-hand side of (1.7) with $(zI - X)$, work the product, use the recursive definition of the Hörner polynomials and Caley–Hamilton $(\sum_{k=0}^{n} x_k X^{n-k} = 0)$ to see that the result is $x(z)I$, which is what one has to prove. □

## 2. Link solution Stein's equation–Fisher's information: The parameter-block approach

### 2.1. General case

In this section the Fisher information matrix of an ARMA process will be formulated where the parameter blocks are considered, whereas in Section 4 the global form will be studied.

Depending on the situation, both cases have their importance and this is the reason why the two cases are treated separately.

Consider the ARMA process $y$ specified as the solution of

$$a^*(L)y = c^*(L)\varepsilon \tag{2.1}$$

with $L$ the lag operator and $\varepsilon$ a white noise sequence. We make the assumptions that both $a$ and $c$ have zeros inside the unit disc, $a$ and $c$ are the following monic polynomials:

$$a(z) = z^p + a_1 z^{p-1} + \cdots + a_p,$$

$$c(z) = z^q + c_1 z^{q-1} + \cdots + c_q.$$

By $a^*$ and $c^*$ we denote the reciprocal polynomials, $a^*(z) = z^p a(z^{-1})$ and $c^*(z) = z^q c(z^{-1})$. It is known that Fisher 's information matrix of (2.1) is

$$F(\theta) = \begin{pmatrix} F_{aa} & F_{ac} \\ F_{ac}^{\mathrm{T}} & F_{cc} \end{pmatrix}. \tag{2.2}$$

Define the vectors $u_k(z) = (1, z, \ldots, z^{k-1})^{\mathrm{T}}$ and $u_k^*(z) = (z^{k-1}, z^{k-1}, \ldots, 1)^{\mathrm{T}}$ and $\theta = (a_1, a_2, \ldots, a_p, c_1, c_2, \ldots, c_q)^{\mathrm{T}}$.

The matrices appearing in (2.2) can be expressed as.

$$F_{ac} = -\frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{u_p(z) u_q^{*\mathrm{T}}(z)}{c(z) a^*(z)} \,\mathrm{d}z, \tag{2.3}$$

$$F_{aa} = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{u_p(z) u_p^{*\mathrm{T}}(z)}{a(z) a^*(z)} \,\mathrm{d}z, \tag{2.4}$$

$$F_{cc} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{u_q(z) u_q^{*T}(z)}{c(z)c^*(z)} \, dz. \tag{2.5}$$

First the link between Fisher's information matrix and a solution of Stein's equation for the $(a, a)$-block is deduced, consequently an interconnection for the $(c, c)$ block can be provided. This will be followed by a corresponding link for the off-diagonal block $(a, c)$.

First the situation with eigenvalues (roots) having an arbitrary multiplicity will be developed, followed by the formulation where all the eigenvalues are distinct. Let us consider Stein's equation followed by its solution with the following matrices and the contour being $C = \{z : |z| = 1\}$. Let $A$ be the companion matrix

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -a_p & -a_{p-1} & \cdots & -a_1 \end{pmatrix},$$

and let $S_{aa}$ be the solution of

$$S_{aa} - A S_{aa} A^{T} = \Gamma^{aa}.$$

A special case of $\Gamma^{aa} = e_p e_p^{T}$, where $e_p$ is the last standard basis vector in a Euclidean space $\mathbb{R}^p$ and verifies the solution $F_{aa} - A F_{aa} A^{T} = e_p e_p^{T}$.

According to Theorem 1.1:

$$S_{aa} = \frac{1}{2\pi i} \oint_C (\lambda I - A)^{-1} \Gamma^{aa} (I - \lambda A)^{-T} \, d\lambda. \tag{2.6}$$

The resolvents can be written as $a(\lambda)(\lambda I - A)^{-1} = \operatorname{adj}(\lambda I - A)$, $a^*(\lambda)(I - \lambda A)^{-1} = \operatorname{adj}(I - \lambda A)$. Since the eigenvalues of $A$ are within the unit disc, the conditions for a unique solution of Stein's equation is fulfilled. The idea of considering block-companion matrices in such a type of equations is also suggested in [5]. However in this paper the scalar case will be studied.

We write an explicit form of the solution of Stein's equation in such a way so that its poles correspond to the poles appearing in the expression of Fisher's information matrix.

$$S_{aa} = \frac{1}{2\pi i} \oint_C \frac{\operatorname{adj}(zI - A) \Gamma^{aa} \operatorname{adj}(I - zA)^{T}}{a(z)a^*(z)} \, dz, \tag{2.7}$$

$$F_{aa} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{u_p(z) u_p^{*T}(z)}{a(z)a^*(z)} \, dz. \tag{2.8}$$

We assume the polynomial $a(z)$ having $p_0$ distinct roots, $\alpha_1, \alpha_2, \ldots, \alpha_{p_0}$, with algebraic multiplicity $n_1 + 1, n_2 + 1, \ldots, n_{p_0} + 1$, respectively, and $\sum_{i=1}^{p_0} (n_i + 1) = p$. Consequently in virtue of Cauchy's residue theorem, the solution of $F_{aa}$ can be written as

$$F_{aa} = a_1(\alpha_1) + a_2(\alpha_2) + \cdots + a_{p_0}(\alpha_{p_0}),$$

where

$$a_i(\alpha_i) = \frac{1}{n_i!}\left(\frac{\partial^{n_i}}{\partial z^{n_i}} \frac{u_p(z)u_p^{*\mathrm{T}}(z)}{\left(\prod_{j=1, j\neq i}^{p_0}(z-\alpha_j)^{n_j+1}\right)a^*(z)}\right)_{z=\alpha_i}, \quad 1 \leqslant i \leqslant p_0. \tag{2.9}$$

One way to obtain a common factor between $F_{aa}$ and $S_{aa}$ consists of a separation between numerator and denominator of the $a_i(\alpha_i)$'s by using Leibnitz's rule of the $n_i$th derivative of a product of two or more functions. Let us denote

$$\xi_i(\alpha_i) = \left(\frac{1}{\left(\prod_{j=1, j\neq i}^{p_0}(z-\alpha_j)^{n_j+1}\right)a^*(z)}\right)_{z=\alpha_i}.$$

A useful way to factorize (2.9) is by vectorizing Fisher's information matrix according to $\mathrm{vec}(ABC) = (C^{\mathrm{T}} \otimes A)\ \mathrm{vec}B$, where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times s}$, and $\otimes$ denotes the Kronecker product. We therefore have

$$\begin{aligned}\mathrm{vec}\ F_{aa} &= \mathrm{vec}\ a_1(\alpha_1) + \mathrm{vec}\ a_2(\alpha_2) + \cdots + \mathrm{vec}\ a_{p_0}(\alpha_{p_0}) \\ &= W_n(\alpha)\left(\xi_{n_1}^{\mathrm{T}}(\alpha_1),\ \xi_{n_2}^{\mathrm{T}}(\alpha_2), \ldots, \xi_{n_{p_0}}^{\mathrm{T}}(\alpha_{p_0})\right)^{\mathrm{T}},\end{aligned} \tag{2.10}$$

where

$$W_n(\alpha) = \left(W_{n_1}(\alpha_1), W_{n_2}(\alpha_2), \ldots, W_{n_{p_0}}(\alpha_{p_0})\right),$$

where the matrix $W_{n_i}(\alpha_i)$ of dimension $p^2 \times (n_i+1)$ is

$$W_{n_i}(\alpha_i) = \frac{1}{n_i!}\left(W_{n_i}^{(n_i)}(z), W_{n_i}^{(n_i-1)}(z), \ldots, W_{n_i}^{(0)}(z)\right)_{z=\alpha_i},$$

each block being

$$W_{n_i}^{(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}\left(u_p^*(z)\otimes u_p(z)\right)\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i,$$

and the matrix $\xi_{n_i}(\alpha_i)$ of dimension $(n_i+1)\times 1$ is given by

$$\xi_{n_i}(\alpha_i) = \left(\xi_i(z), \frac{\partial}{\partial z}\xi_i(z), \ldots, \frac{\partial^{n_i}}{\partial z^{n_i}}\xi_i(z)\right)^{\mathrm{T}}_{z=\alpha_i}.$$

It is according to (2.10) that the interconnection between Fisher's information matrix and the corresponding Stein solution will be established. However, we will present an alternative to (2.10) which results in an additional link with Vandermonde matrices. The prevectorization form of Fisher's information matrix for the $(a, a)$-block can be written as

$$F_{aa} = a^{(1)}(\alpha)a^{(2)}(\alpha), \tag{2.11}$$

where

$$a^{(1)}(\alpha) = \left(a_1^{(1)}(\alpha_1), a_2^{(1)}(\alpha_2), \ldots, a_{p_0}^{(1)}(\alpha_{p_0})\right)$$

with

$$a_i^{(1)}(\alpha_i) = \frac{1}{n_i!}\left(a_i^{(1),(n_i)}(z), a_i^{(1),(n_i-1)}(z), \ldots, a_i^{(1),(0)}(z)\right)_{z=\alpha_i},$$

where each block is given by

$$a_i^{(1),(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}\left(u_p(z)u_p^{*\mathrm{T}}(z)\right)\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

We also have

$$a^{(2)}(\alpha) = \left(\left(a_1^{(2)}(\alpha_1)\right)^{\mathrm{T}}, \left(a_2^{(2)}(\alpha_2)\right)^{\mathrm{T}}, \ldots, \left(a_{p_0}^{(2)}(\alpha_{p_0})\right)^{\mathrm{T}}\right)^{\mathrm{T}}$$

with

$$a_i^{(2)}(\alpha_i) = \left(\xi_i(z)I_p, \frac{\partial}{\partial z}\xi_i(z)I_p, \ldots, \frac{\partial^{n_i}}{\partial^{n_i}}\xi_i(z)I_p\right)^{\mathrm{T}}_{z=\alpha_i}.$$

The solution of Eq. (2.7), obtained by means of Cauchy's residue theorem, can now be expressed as follows:

$$S_{aa} = A_1(\alpha_1) + A_2(\alpha_2) + \cdots + A_{p_0}(\alpha_{p_0}),$$

where

$$A_i(\alpha_i) = \frac{1}{n_i!}\left(\frac{\partial^{n_i}}{\partial z^{n_i}}\frac{\operatorname{adj}(zI - A)\Gamma^{aa}\operatorname{adj}(I - zA)^{\mathrm{T}}}{\left(\prod_{j=1, j\neq i}^{p_0}(z-\alpha_j)^{n_j+1}\right)a^*(z)}\right)_{z=\alpha_i}.$$

A similar factorization as in (2.10) is used here and we obtain

$$S_{aa} = M_n(\alpha)\left(\xi_{n_1}^{\mathrm{T}}(\alpha_1)\otimes I_p, \xi_{n_2}^{\mathrm{T}}(\alpha_2)\otimes I_p, \ldots, \xi_{n_{p_0}}^{\mathrm{T}}(\alpha_{p_0})\otimes I_p\right)^{\mathrm{T}} \tag{2.12}$$

with

$$M_n(\alpha) = \left(M_{n_1}(\alpha_1), M_{n_2}(\alpha_2), \ldots, M_{n_{p_0}}(\alpha_{p_0})\right),$$

where $M_{n_i}(\alpha_i)$ is the $p \times (n_i + 1)$ $p$-dimensional matrix given by

$$M_{n_i}(\alpha_i) = \frac{1}{n_i!}\left(M_{n_i}^{(n_i)}(z), M_{n_i}^{(n_i-1)}(z), \ldots, M_{n_i}^{(0)}(z)\right)_{z=\alpha_i}$$

and

$$M_{n_i}^{(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}\left(\operatorname{adj}(zI - A)\Gamma^{aa}\operatorname{adj}(I - zA)^{\mathrm{T}}\right)\right)_{z=\alpha_i},$$
$$0 \leqslant k \leqslant n_i.$$

Representing the second term in (2.10) by $\xi_\alpha$, allows (2.12) to be written as

$$S_{aa} = M_n(\alpha)\left(\xi_\alpha \otimes I_p\right). \tag{2.13}$$

From the relations obtained for $F_{aa}$ and $S_{aa}$ in (2.10) and (2.13), respectively, one can deduce the link between Fisher's information matrix and the solution of Stein's equation. Therefore we have:

**Theorem 2.1.** *The matrices $S_{aa}$ and $F_{aa}$ are linked through*

$$S_{aa} = M_n(\alpha)\left\{\left(W_n(\alpha)_L^- \mathit{vec} F_{aa}\right) \otimes I_p\right\}. \tag{2.14}$$

*where $(\cdot)_L^-$ is any left inverse.*

Analogously for $F_{cc}$ which consists of $q_0$ distinct roots $\gamma_1, \gamma_2, \ldots, \gamma_{q_0}$ with algebraic multiplicity $m_1+1, m_2+1, \ldots, m_{q_0}+1$, respectively, and $\sum_{i=1}^{q_0}(m_i+1) = q$. The companion matrix used in Stein's equation is

$$C = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -c_q & -c_{q-1} & \cdots & -c_1 \end{pmatrix}.$$

To establish a link between $F_{cc}$ and the solution of $S_{cc} - CS_{cc}C^{\mathrm{T}} = \Gamma^{cc}$ we introduce some notation in the same spirit as we use above. Define

$$N_m(\gamma) = \left(N_{m_1}(\gamma_1), N_{m_2}(\gamma_2), \ldots, N_{m_{q_0}}(\gamma_{q_0})\right),$$

where

$$N_{m_i}(\gamma_i) = \frac{1}{m_i!}\left(N_{m_i}^{(m_i)}(z), N_{m_i}^{(m_i-1)}(z), \ldots, N_{m_i}^{(0)}(z)\right)_{z=\gamma_i}$$

with

$$N_{m_i}^{(m_i-k)}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}}\left(\mathrm{adj}(zI-C)\Gamma^{cc}\mathrm{adj}(I-zC)^{\mathrm{T}}\right)\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.$$

We also put

$$W_m(\gamma) = \left(W_{m_1}(\gamma_1), W_{m_2}(\gamma_2), \ldots, W_{m_{q_0}}(\gamma_{q_0})\right),$$

where

$$W_{m_i}(\gamma_i) = \frac{1}{m_i!}\left(W_{m_i}^{(m_i)}(z), W_{m_i}^{(m_i-1)}(z), \ldots, W_{m_i}^{(0)}(z)\right)_{z=\gamma_i}$$

with

$$W_{m_i}^{(m_i-k)}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}}\left(u_q^*(z) \otimes u_q(z)\right)\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.$$

**Proposition 2.2.** *For the* $(c,c)$*-block we have*

$$S_{cc} = N_m(\gamma)\left\{\left(W_m(\gamma)_L^{-}\mathrm{vec}F_{cc}\right)\otimes I_q\right\}.$$

Next a link involving $F_{ac}$ and the solution of $S_{ca} - CS_{ca}A^{\mathrm{T}} = \Gamma^{ca}$ will be set forth for $p > q$. From (2.3) we obtain

$$F_{ac} = c_1(\gamma_1) + c_2(\gamma_2) + \cdots + c_{q_0}(\gamma_{p_0}),$$

where

$$c_i(\gamma_i) = -\frac{1}{m_i!}\left(\frac{\partial^{m_i}}{\partial z^{m_i}}\left(u_p(z)\zeta_i(z)u_q^{*\mathrm{T}}(z)\right)\right)_{z=\gamma_i}, \quad 1 \leqslant i \leqslant q_0,$$

with

$$\zeta_i(z) = \frac{1}{\left(\prod_{j=1,j\neq i}^{q_0}(z-\gamma_j)^{n_j+1}\right)a^*(z)}.$$

Vectorization of $F_{ac}$ results in

$$\begin{aligned}\mathrm{vec}F_{ac} &= \mathrm{vec}c_1(\gamma_1) + \mathrm{vec}c_2(\gamma_2) + \cdots + \mathrm{vec}c_{q_0}(\gamma_{q_0})\\ &= -V_m(\gamma)\left(\zeta_{m_1}^{\mathrm{T}}(\gamma_1), \zeta_{m_2}^{\mathrm{T}}(\gamma_2), \ldots, \zeta_{m_{q_0}}^{\mathrm{T}}(\gamma_{q_0})\right)^{\mathrm{T}},\end{aligned} \tag{2.15}$$

where

$$V_m(\gamma) = \left(V_{m_1}(\gamma_1), V_{m_2}(\gamma_2), \ldots, V_{m_{q_0}}(\gamma_{q_0})\right)$$

with

$$V_{m_i}(\gamma_i) = \frac{1}{m_i!}\left(V_{m_i}^{(m_i)}(z), V_{m_i}^{(m_i-1)}(z), \ldots, V_{m_i}^{(0)}(z)\right)_{z=\gamma_i}$$

with each block

$$V_{m_i}^{(m_i-k)}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}}\left(u_q^*(z)\otimes u_p(z)\right)\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i,$$

and

$$\zeta_{m_i}(\gamma_i) = \left(\zeta_i(z), \frac{\partial}{\partial z}\zeta_i(z), \ldots, \frac{\partial^{m_i}}{\partial z^{m_i}}\zeta_i(z)\right)_{z=\gamma_i}^{\mathrm{T}}.$$

A similar form as (2.15) can now be obtained for $F_{ca}$. We have

$$\mathrm{vec}F_{ca} = -Q_m(\gamma)\left(\zeta_{m_1}^{\mathrm{T}}(\gamma_1), \zeta_{m_2}^{\mathrm{T}}(\gamma_2), \ldots, \zeta_{m_{q_0}}^{\mathrm{T}}(\gamma_{q_0})\right)^{\mathrm{T}}, \tag{2.16}$$

where $Q_m(\gamma)$ has the same structure as $V_m(\gamma)$ but where the blocks have the form given by

$$Q_{m_i}^{(m_i-k)}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}}\left(u_p(z)\otimes u_q^*(z)\right)\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.$$

The corresponding Stein solution yields

$$
\begin{aligned}
S_{ca} &= \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\operatorname{adj}(zI - C)\Gamma^{ca} \operatorname{adj}(I - zA)^{\mathrm{T}}}{c(z)a^{*}(z)}\, \mathrm{d}z, \\
&= C_1(\gamma_1) + C_2(\gamma_2) + \cdots + C_{q_0}(\gamma_{q_0}),
\end{aligned}
$$

where

$$
C_i(\gamma_i) = \frac{1}{m_i!} \left( \frac{\partial^{m_i}}{\partial z^{m_i}} \left( \operatorname{adj}(zI - C)\Gamma^{ca} \operatorname{adj}(I - zA)^{\mathrm{T}} \right) \zeta_i(z) \right)_{z=\gamma_i}.
$$

Stein's solution can be written as

$$
S_{ca} = E_m(\gamma) \left( \zeta^{\mathrm{T}}_{m_1}(\gamma_1) \otimes I_p, \zeta^{\mathrm{T}}_{m_2}(\gamma_2) \otimes I_p, \ldots, \zeta^{\mathrm{T}}_{m_{q_0}}(\gamma_{q_0}) \otimes I_p \right)^{\mathrm{T}},
$$

where

$$
E_m(\gamma) = \left( E_{m_1}(\gamma_1), E_{m_2}(\gamma_2), \ldots, E_{m_{q_0}}(\gamma_{q_0}) \right),
$$

and the $q \times (m_i + 1)p$ matrix

$$
E_{m_i}(\gamma_i) = \frac{1}{m_i!} \left( E^{(m_i)}_{m_i}(z), E^{(m_i-1)}_{m_i}(z), \ldots, E^{(0)}_{m_i}(z) \right)_{z=\gamma_i}
$$

with the blocks

$$
E^{(m_i-k)}_{m_i}(\gamma_i) = \binom{m_i}{k} \left( \frac{\partial^{m_i-k}}{\partial z^{m_i-k}} \left( \operatorname{adj}(zI - C)\Gamma^{ca} \operatorname{adj}(I - zA)^{\mathrm{T}} \right) \right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.
$$

The interconnections are given in the following lemma. Its proof is similar to the proof of Theorem 2.1.

**Lemma 2.3.** *The off-diagonal blocks yield the connections*

$$
S_{ca} = -E_m(\gamma) \left\{ \left( V_m(\gamma)^{-}_{L} \operatorname{vec} F_{ac} \right) \otimes I_p \right\} \tag{2.17}
$$

*and*

$$
S_{ca} = -E_m(\gamma) \left\{ \left( Q_m(\gamma)^{-}_{L} \operatorname{vec} F_{ca} \right) \otimes I_p \right\}. \tag{2.18}
$$

Some results for the case $p \leqslant q$ will be outlined in Section 3.1.

### 2.2. *Special case*

In this section, the case in which all the eigenvalues are assumed to be distinct will be considered. As a result the preceeding formulas take a simpler form.

The $F_{aa}$ in Fisher's information matrix is given by

$$
\begin{aligned}
F_{aa} = {} & u_p(\alpha_1) u^{*\mathrm{T}}_p(\alpha_1) r_1(\alpha_1) + u_p(\alpha_2) u^{*\mathrm{T}}_p(\alpha_2) r_2(\alpha_2) \\
& + \cdots + u_p(\alpha_p) u^{*\mathrm{T}}_p(\alpha_p) r_p(\alpha_p),
\end{aligned} \tag{2.19}
$$

where

$$r_i(\alpha_i) = \left( \frac{1}{\prod_{j=1, j \neq i}^{p} (z - \alpha_j) a^*(z)} \right)_{z=\alpha_i} \qquad \text{with } 1 \leqslant i \leqslant p.$$

An appropriate factorization of (2.19) yields

$$\begin{aligned} F_{aa} &= \left(u_p(\alpha_1), u_p(\alpha_2), ..., u_p(\alpha_p)\right) \operatorname{diag}\left(r_1(\alpha_1), r_2(\alpha_2), \ldots, r_p(\alpha_p)\right) \\ &\quad \times \left(u_p^*(\alpha_1), u_p^*(\alpha_2), \ldots, u_p^*(\alpha_p)\right)^{\mathrm{T}}. \end{aligned} \tag{2.20}$$

The first and last term of (2.20) are Vandermonde (type) matrices which will be denoted by $V_\alpha$ and $V_\alpha^*$, respectively.

$$V_\alpha = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_p \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_p^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^{p-1} & \alpha_2^{p-1} & \cdots & \alpha_p^{p-1} \end{pmatrix}$$

and

$$V_\alpha^* = \begin{pmatrix} \alpha_1^{p-1} & \alpha_1^{p-2} & \cdots & \alpha_1 & 1 \\ \alpha_2^{p-1} & \alpha_2^{p-2} & \cdots & \alpha_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_p^{p-1} & \alpha_p^{p-2} & \cdots & \alpha_p & 1 \end{pmatrix}.$$

The corresponding Stein solution can be factorized analogously.

$$S_{aa} = A_1 \left(I_p \otimes \Gamma^{aa}\right) \left\{\operatorname{diag}\left(r_1(\alpha_1), r_2(\alpha_2), \ldots, r_p(\alpha_p)\right) \otimes I_p\right\} A_2^{\mathrm{T}} \tag{2.21}$$

with

$$A_1 = \left(\operatorname{adj}(\alpha_1 I - A), \operatorname{adj}(\alpha_2 I - A), \ldots, \operatorname{adj}(\alpha_p I - A)\right)$$

and

$$A_2 = \left(\operatorname{adj}(I - \alpha_1 A), \operatorname{adj}(I - \alpha_2 A), \ldots, \operatorname{adj}(I - \alpha_p A)\right).$$

**Proposition 2.4.** *Combining* (2.20) *and* (2.21) *links* $F_{aa}$ *and* $S_{aa}$ *by*

$$S_{aa} = A_1 \left(V_\alpha^{-1} F_{aa} V_\alpha^{-*} \otimes \Gamma^{aa}\right) A_2^{\mathrm{T}}. \tag{2.22}$$

We used the shorthand notation $V_\alpha^{-*} = \left(V_\alpha^*\right)^{-1}$.

A similar link is established for the $(c, c)$ block. We introduce the notations

$$C_1 = \left(\operatorname{adj}(\gamma_1 I - C), \operatorname{adj}(\gamma_2 I - C), \ldots, \operatorname{adj}(\gamma_q I - C)\right)$$

and

$$C_2 = \left(\operatorname{adj}(I - \gamma_1 C), \operatorname{adj}(I - \gamma_2 C), \ldots, \operatorname{adj}(I - \gamma_q C)\right).$$

**Proposition 2.5.** *The matrices $S_{cc}$ and $F_{cc}$ are connected via*

$$S_{cc} = C_1 \left( V_\gamma^{-1} F_{cc} V_\gamma^{-*} \otimes \Gamma^{cc} \right) C_2^{\mathrm{T}}.$$

Next we consider the link between the off-diagonal blocks and Stein's solution for $p > q$.

$$F_{ac} = -\left(u_p(\gamma_1), u_p(\gamma_2), \ldots, u_p(\gamma_q)\right) \operatorname{diag}\left(s_1(\gamma_1), s_2(\gamma_2), \ldots, s_q(\gamma_q)\right) \times \left(u_q^*(\gamma_1), u_q^*(\gamma_2), \ldots, u_q^*(\gamma_q)\right)^{\mathrm{T}} \tag{2.23}$$

with

$$s_i(\gamma_i) = \left( \frac{1}{\prod_{j=1, j \neq i}^{q} (z - \gamma_j) a^*(z)} \right)_{z=\gamma_i} \qquad \text{with } 1 \leqslant i \leqslant q$$

and the first and third matrix in (2.23) are Vandermonde matrices which are denoted by $\bar{V}_\gamma$ and $V_\gamma^*$, respectively.

$$\bar{V}_\gamma = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_q \\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_q^2 \\ \vdots & \vdots & & \vdots \\ \gamma_1^{p-1} & \gamma_2^{p-1} & \cdots & \gamma_q^{p-1} \end{pmatrix}$$

and

$$V_\gamma^* = \begin{pmatrix} \gamma_1^{q-1} & \gamma_1^{q-2} & \cdots & \gamma_1 & 1 \\ \gamma_2^{q-1} & \gamma_2^{q-2} & \cdots & \gamma_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_q^{q-1} & \gamma_q^{q-2} & \cdots & \gamma_q & 1 \end{pmatrix}.$$

The appropriate Stein solution is

$$S_{ca} = C_1 \left(I_q \otimes \Gamma^{ca}\right) \left\{\operatorname{diag}\left(s_1(\gamma_1), s_2(\gamma_2), \ldots, s_q(\gamma_q)\right) \otimes I_p\right\} A_3^{\mathrm{T}} \tag{2.24}$$

with

$$A_3 = \left(\operatorname{adj}(I - \gamma_1 A), \operatorname{adj}(I - \gamma_2 A), \ldots, \operatorname{adj}(I - \gamma_q A)\right).$$

As for the general case Fisher's information for the $(c, a)$-block is also considered, this is formulated in the following proposition.

**Proposition 2.6.** *Combining* (2.23) *and* (2.24) *yields*

$$S_{ca} = -C_1 \left( \bar{V}_{\gamma,L}^{-1} F_{ac} V_{\gamma}^{-*} \otimes \Gamma^{ca} \right) A_3^{\mathrm{T}} \tag{2.25}$$

*and*

$$S_{ca} = -C_1 \left( V_{\gamma}^{-*\mathrm{T}} F_{ca} \bar{V}_{\gamma,R}^{-\mathrm{T}} \otimes \Gamma^{ca} \right) A_3^{\mathrm{T}}, \tag{2.26}$$

*where* $\bar{V}_{\gamma,L}^{-}$ *and* $\bar{V}_{\gamma,R}^{-\mathrm{T}}$ *are left and right inverses of* $\bar{V}_{\gamma}$ *and* $\bar{V}_{\gamma}^{\mathrm{T}}$, *respectively.*

The appearance of Vandermonde matrices in this and subsequent sections is of course not a coincidence. It can be explained as follows. Consider again Eq. (1.1). Let $T_B$ be the matrix that brings $B$ on its Jordan form: $J_B = T_B B T_B^{-1}$ and likewise $J_A = T_A A T_A^{-1}$. Let $\hat{S} = T_B S T_A^{\mathrm{T}}$ and $\hat{\Gamma} = T_B \Gamma T_A^{\mathrm{T}}$. Then Eq. (1.1) transforms into

$$\hat{S} - J_B\, \hat{S} J_A^{\mathrm{T}} = \hat{\Gamma}. \tag{2.27}$$

As soon as the matrices $A$ and $B$ are of companion type, the matrices $T_A$ and $T_B$ are just the inverses of (generalized) Vandermonde matrices. More precisely, if the matrix $A$ has the following companion form:

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}, \tag{2.28}$$

then $J_A = V_A^{-1} A V_A$, where $V_A$ is given as follows. Let $\pi_A(z) = \prod_{i=1}^{s} (z - \alpha_i)^{m_i} = \sum_{j=0}^{n} a_j z^{n-j}$ be the characteristic polynomial of $A$ with $m_1 + m_2 + \cdots + m_s = n$. Let then $U_i(z)$ be the $n \times m_i$ matrix with $k$th column equal to $(1/(k-1)!) u^{(k-1)}(z)$ and write $U_i = U_i(\alpha_i)$. Then $V_A = (U_1, \ldots, U_s)$.

If moroever $S_{m_i}$ denotes the $m_i \times m_i$ shift matrix (its $i, j$ element is the Kronecker $\delta_{ij}$), then $J_A$ is the block diagonal matrix with entries $\alpha_i I_{m_i} + S_{m_i}^{\mathrm{T}}$, so $V_A^{-1} A V_A = J_A$.

## 3. Left and right inverses: Blocks of the Fisher information matrix

In this section we will derive explicit formulas for certain right and left inverses that are used in Section 2.1. Some of these matrices are of fundamental importance for a successful realization of an interconnection between Stein's solution and Fisher's information matrix as set forth in this paper.

### 3.1. *General case*

A left inverse involved in Theorem 2.1 has the properties summarized in the following lemma. The following notations are first introduced. Let the $p \times p$ generalized Vandermonde matrix be

$$\widetilde{W}_{\alpha} = \left(\widetilde{W}_{n_1}(\alpha_1), \widetilde{W}_{n_2}(\alpha_2), \ldots, \widetilde{W}_{n_{p_0}}(\alpha_{p_0})\right),$$

where

$$\widetilde{W}_{n_i}(\alpha_i) = \frac{1}{n_i!}\left(\widetilde{W}_{n_i}^{(n_i)}(z), \widetilde{W}_{n_i}^{(n_i-1)}(z), \ldots, \widetilde{W}_{n_i}^{(0)}(z)\right)_{z=\alpha_i}$$

with

$$\widetilde{W}_{n_i}^{(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}} u_p(z)\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

**Lemma 3.1.** *The next relations hold true*:

$$\left(0_{p\times p(p-1)}, I_p\right) W_n(\alpha) = \widetilde{W}_{\alpha} \quad \textit{and} \quad \left(0_{p\times p(p-1)}, \widetilde{W}_{\alpha}^{-1}\right) W_n(\alpha) = I_p.$$

*So* $\left(0_{p\times p(p-1)}, \widetilde{W}_{\alpha}^{-1}\right)$ *is a left inverse of* $W_n(\alpha)$.

**Proof.** Straightforward computation completes the proof. □

Some property of (2.11) is now set forth in the next lemma.

**Lemma 3.2.** *The following relations are verified for the matrix* $a^{(1)}(\alpha)$ *in Eq.* (2.11):

$$a^{(1)}(\alpha)\left(I_p \otimes e_p\right) = \widetilde{W}_{\alpha} \quad \textit{and} \quad a^{(1)}(\alpha)\left(\widetilde{W}_{\alpha}^{-1} \otimes e_p\right) = I_p,$$

*where* $e_p$ *is the last standard basis vector of a Euclidean space* $\mathbb{R}^p$, *it can be concluded that a right inverse of* $a^{(1)}(\alpha)$ *is given by*

$$\left(0_{p\times(p-1)}, v_1^{\mathrm{T}}, 0_{p\times(p-1)}, v_2^{\mathrm{T}}, \ldots, 0_{p\times(p-1)}, v_p^{\mathrm{T}}\right)^{\mathrm{T}},$$

*where* $v_1, v_2, \ldots, v_p$ *are the rows of* $\widetilde{W}_{\alpha}^{-1}$.

**Proof.** Straightforward. □

In order to formulate left inverses appearing in (2.17) and (2.18), the following notations are introduced. Let the $q \times q$ generalized Vandermonde matrices be

$$\widetilde{W}_{\gamma} = \left(\widetilde{W}_{m_1}(\gamma_1), \widetilde{W}_{m_2}(\gamma_2), \ldots, \widetilde{W}_{m_{q_0}}(\gamma_{q_0})\right),$$

where

$$\widetilde{W}_{m_i}(\gamma_i) = \frac{1}{m_i!}\left(\widetilde{W}_{m_i}^{(m_i)}(z), \widetilde{W}_{m_i}^{(m_i-1)}(z), \ldots, \widetilde{W}_{m_i}^{(0)}(z)\right)_{z=\gamma_i}$$

with

$$\widetilde{W}_{m_i}^{(m_i-k)}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}} u_q(z)\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.$$

We also put

$$\widetilde{W}^*_\gamma = \left(\widetilde{W}^*_{m_1}(\gamma_1), \widetilde{W}^*_{m_2}(\gamma_2), \ldots, \widetilde{W}^*_{m_{q_0}}(\gamma_{q_0})\right),$$

where

$$\widetilde{W}^*_{m_i}(\gamma_i) = \frac{1}{m_i!}\left(\widetilde{W}^{*(m_i)}_{m_i}(z), \widetilde{W}^{*(m_i-1)}_{m_i}(z), \ldots, \widetilde{W}^{*(0)}_{m_i}(z)\right)_{z=\gamma_i}$$

with

$$\widetilde{W}^{*(m_i-k)}_{m_i}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}} u^*_q(z)\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.$$

A lemma specifing left inverses needed in (2.17) and (2.18) can now be set forth.

**Lemma 3.3.** *It can be verified that for* $p > q$

$$\left(0_{q\times p(q-1)}, I_q, 0_{q\times(p-q)}\right) V_m(\gamma) = \widetilde{W}_\gamma$$

*and*

$$\left(0_{q\times p(q-1)}, \widetilde{W}^{-1}_\gamma, 0_{q\times(p-q)}\right) V_m(\gamma) = I_q.$$

*Similarly*,

$$\left(I_q, 0_{q\times q(p-1)}\right) Q_m(\gamma) = \widetilde{W}^*_\gamma$$

*and*

$$\left(\widetilde{W}^{-*}_\gamma, 0_{q\times q(p-1)}\right) Q_m(\gamma) = I_q.$$

*We may take* $(0_{q\times p(q-1)}, \widetilde{W}^{-1}_\gamma, 0_{q\times(p-q)})$ *and* $(\widetilde{W}^{-*}_\gamma, 0_{q\times q(p-1)})$ *for* $V_m(\gamma)^-_L$ *and* $Q_m(\gamma)^-_L$, *respectively.*

**Proof.** Straightforward. □

As mentioned before, for the case $p \leqslant q$ an interconnection between Stein's solution and Fisher's information matrix will not be envisaged since an appropriate left inverse is not directly available, however some attractive properties are worth considering.

The following notations are introduced. Consider the $p \times q$ generalized Vandermonde matrix

$$\bar{W}_p(\gamma) = \left(\bar{W}_p(\gamma_1), \bar{W}_p(\gamma_2), \ldots, \bar{W}_p(\gamma_{q_0})\right),$$

where

$$\bar{W}_p(\gamma_i) = \frac{1}{m_i!}\left(\bar{W}^{(m_i)}_p(z), \bar{W}^{(m_i-1)}_p(z), \ldots, \bar{W}^{(0)}_p(z)\right)_{z=\gamma_i}$$

with

$$\bar{W}_p^{(m_i-k)}(\gamma_i) = \binom{m_i}{k} \left( \frac{\partial^{m_i-k}}{\partial z^{m_i-k}} u_p(z) \right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i.$$

**Lemma 3.4.** *The following relation holds true for $p \leqslant q$:*

$$\left( 0_{q\times p(q-1)}, \begin{pmatrix} I_p \\ 0_{(q-p)\times p} \end{pmatrix} \right) V_m(\gamma) = \begin{pmatrix} \bar{W}_p(\gamma) \\ 0_{(q-p)\times q} \end{pmatrix}.$$

**Proof.** Straightforward. □

For $p \leqslant q$ the matrix $\bar{W}_p(\gamma)$ has a right inverse, e.g. as specified in the next corollary.

**Corollary 3.5.** *Let $\gamma_p$ be the $p$th root of polynomial $c(z)$ and $f_1, f_2, \ldots, f_p$ the rows of $V_{p,\gamma}^{-1}$, where the $p \times p$ Vandermonde matrix is*

$$V_{p,\gamma} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_p \\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_p^2 \\ \vdots & \vdots & & \vdots \\ \gamma_1^{p-1} & \gamma_2^{p-1} & \cdots & \gamma_p^{p-1} \end{pmatrix}.$$

*Then for $p \leqslant q$*

$$\bar{W}_p(\gamma) \left( 0_{p\times m_1}, e_1, 0_{p\times m_2}, e_2, \ldots, 0_{p\times m_{q_0}}, e_p \right)^{\mathrm{T}} = V_{p,\gamma}$$

*and*

$$\bar{W}_p(\gamma) \left( 0_{p\times m_1}, f_1^{\mathrm{T}}, 0_{p\times m_2}, f_2^{\mathrm{T}}, \ldots, 0_{p\times m_{q_0}}, f_p^{\mathrm{T}} \right)^{\mathrm{T}} = I_p$$

*and*

$$\left( 0_{q\times p(q-1)}, \begin{pmatrix} I_p \\ 0_{(q-p)\times p} \end{pmatrix} \right) V_m(\gamma) \left( \bar{W}_{p,R}^{-}(\gamma)\, 0_{q\times(q-p)} \right)$$
$$= \begin{pmatrix} I_p & 0_{p\times(q-p)} \\ 0_{(q-p)\times p} & 0_{(q-p)\times(q-p)} \end{pmatrix}.$$

**Proof.** Straightforward. □

An appropriate factorization of the first term in the right-hand side of (2.17) and (2.18) is proposed and Vandermonde matrices are detected.

The block element $E_{m_i}(\gamma_i)$ of the matrix $E_m(\gamma)$ can be factorized as follows:

$$E_{m_i}(\gamma_i) = \frac{1}{m_i!} E_C^{(i)}(\gamma_i) D^{(i)} E_A^{(i)}(\gamma_i), \tag{3.1}$$

where

$$E_C^{(i)}(\gamma_i) = \left(E_C^{(m_i)}(\gamma_i), E_C^{(m_i-1)}(\gamma_i), \ldots, E_C^{(0)}(\gamma_i)\right)$$

with each block

$$E_C^{(m_i-k)}(\gamma_i) = \binom{m_i}{k}\left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}}(\mathrm{adj}(zI-C))\right)_{z=\gamma_i}, \quad 0 \leqslant k \leqslant m_i$$

and

$$D^{(i)} = \left(I_{(m_i+1)(m_i+2)/2} \otimes \Gamma^{ca}\right)$$

and

$$E_A^{(i)}(\gamma_i) = \mathrm{diag}\left(E_A^{(0)}(\gamma_i), E_A^{(1)}(\gamma_i), \ldots, E_A^{(m_i)}(\gamma_i)\right),$$

where each block is

$$E_A^{(m_i-k)}(\gamma_i) = \left(\frac{\partial^{m_i-k}}{\partial z^{m_i-k}}\left(\mathrm{adj}\,(I-zA)^{\mathrm{T}}\right)\right)_{z=\gamma_i}^{\mathrm{T}}$$
$$\text{with } k = m_i, m_i-1, \ldots, 0.$$

The first, second and third matrix in (3.1) have the following size, $q\times\, q(m_i+1)(m_i+2)/2$, $q(m_i+1)(m_i+2)/2 \times p(m_i+1)(m_i+2)/2$ and $p(m_i+1)(m_i+2)/2 \times p(m_i+1)$, respectively. The first term in the right-hand side of (2.17) and (2.18) can now be factorized accordingly as

$$\begin{aligned}&\left(\frac{1}{m_1!}E_C^{(1)}(\gamma_1), \frac{1}{m_2!}E_C^{(2)}(\gamma_2), \ldots, \frac{1}{m_{q_0}!}E_C^{(q_0)}(\gamma_{q_0})\right)\\ &\quad\times \mathrm{diag}\left(D^{(1)}, D^{(2)}, \ldots, D^{(q_0)}\right)\\ &\quad\times \mathrm{diag}\left(E_A^{(1)}(\gamma_1), E_A^{(2)}(\gamma_2), \ldots, E_A^{(q_0)}(\gamma_{q_0})\right).\end{aligned} \tag{3.2}$$

Observe the dimensions of the first, second and third term in ( 3.2) being

$$q \times q\left(\sum_{i=1}^{q_0}\frac{(m_i+1)(m_i+2)}{2}\right),$$

$$q\left(\sum_{i=1}^{q_0}\frac{(m_i+1)(m_i+2)}{2}\right) \times p\left(\sum_{i=1}^{q_0}\frac{(m_i+1)(m_i+2)}{2}\right)$$

and

$$p\left(\sum_{i=1}^{q_0}\frac{(m_i+1)(m_i+2)}{2}\right) \times pq,$$

respectively.

The presence of Vandermonde matrices in the first term of (3.2) can be detected in the following lemma by exploiting the property that the last column of $\mathrm{adj}(zI-C)$ is $u_q(z)$. For typographical brevity we set

$$q\frac{(m_i+1)(m_i+2)}{2}-1=\delta_i.$$

We introduce

$$E_C(\gamma)=\left(E_C^{(1)}(\gamma_1),E_C^{(2)}(\gamma_2),\ldots,E_C^{(q_0)}(\gamma_{q_0})\right)$$

and the Vandermonde matrices

$$\bar{V}_{\gamma,q_0}=\begin{pmatrix}1 & 1 & \cdots & 1\\ \gamma_1 & \gamma_2 & \cdots & \gamma_{q_0}\\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_{q_0}^2\\ \vdots & \vdots & & \vdots\\ \gamma_1^{q-1} & \gamma_2^{q-1} & \cdots & \gamma_{q_0}^{q-1}\end{pmatrix}$$

and

$$V_{\gamma,q_0}=\begin{pmatrix}1 & 1 & \cdots & 1\\ \gamma_1 & \gamma_2 & \cdots & \gamma_{q_0}\\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_{q_0}^2\\ \vdots & \vdots & & \vdots\\ \gamma_1^{q_0-1} & \gamma_2^{q_0-1} & \cdots & \gamma_{q_0}^{q_0-1}\end{pmatrix}.$$

**Lemma 3.6.** *The following relations hold true*:

$$E_C(\gamma)\left(0_{q_0\times\delta_1},e_1,0_{q_0\times\delta_2},e_2,\ldots,0_{q_0\times\delta_{q_0}},e_{q_0}\right)^{\mathrm{T}}=\bar{V}_{\gamma,q_0},$$

*where* $e_1,e_2,\ldots,e_{q_0}$ *are standard basis vectors in a Euclidean space* $\mathbb{R}^{q_0}$. *Furthermore*

$$\left(I_{q_0},0_{q_0\times(q-q_0)}\right)\bar{V}_{\gamma,q_0}=V_{\gamma,q_0}\quad\textit{and}\quad\left(V_{\gamma,q_0}^{-1},0_{q_0\times(q-q_0)}\right)\bar{V}_{\gamma,q_0}=I_{q_0}$$

*and*

$$\left(V_{\gamma,q_0}^{-1},0_{q_0\times(q-q_0)}\right)E_C(\gamma)\left(0_{q_0\times\delta_1},e_1,0_{q_0\times\delta_2},e_2,\ldots,0_{q_0\times\delta_{q_0}},e_{q_0}\right)^{\mathrm{T}}=I_{q_0}.$$

**Proof.** Straightforward. □

For formulating some results for the third term of (3.1) we focus on $E_A^{(i)}(\gamma_i)$.

**Lemma 3.7.** *The next holds true.*

$$\begin{pmatrix}0_{(m_i+1)\times p(m_i+1)-1},e_1,0_{(m_i+1)\times(pm_i)-1},e_2,\ldots,0_{(m_i+1)\times p-1},e_{m_i+1}\\ 0_{(m_i+1)(p-1)\times p(m_i+1)(m_i+2)/2}\end{pmatrix}$$
$$\times E_A^{(i)}(\gamma_i)=\begin{pmatrix}U_p^*(\gamma_i)\\ 0_{(m_i+1)(p-1)\times p(m_i+1)}\end{pmatrix},$$

*where*

$$U_p^*(\gamma_i) = \operatorname{diag}\left(\frac{\partial^{m_i}}{\partial z^{m_i}} u_p^{*\mathrm{T}}(z), \frac{\partial^{m_i-1}}{\partial z^{m_i}} u_p^{*T}(z), \ldots, u_p^{*\mathrm{T}}(z)\right)_{z=\gamma_i}.$$

**Proof.** Straightforward since the last row of $\operatorname{adj}(I - zA)^{\mathrm{T}}$ is $u_p^{*\mathrm{T}}(z)$. □

Since the upper block $U_p^*(\gamma_i)$ is of dimension $(m_i + 1) \times p(m_i + 1)$, we further proceed with a choice for a right inverse.

**Lemma 3.8.** *One has*

$$U_p^*(\gamma_i)\left(\iota_{m_i+1} \otimes I_p\right) = \tilde{\tilde{V}}_{\gamma_i}^*,$$

*where* $\iota_{m_i+1}$ *is an* $(m_i + 1)$ *column vector consisting of ones and the* $(m_i + 1) \times p$ *Vandermonde matrix*

$$\tilde{\tilde{V}}_{\gamma_i}^* = \left(\frac{\partial^{m_i}}{\partial z^{m_i}} u_p^*(z), \frac{\partial^{m_i-1}}{\partial z^{m_i}} u_p^*(z), \ldots, u_p^*(z)\right)_{z=\gamma_i}^{\mathrm{T}}.$$

**Proof.** Straightforward. □

A right inverse of $\tilde{\tilde{V}}_{\gamma_i}^*$ and consequently of $U_p^*(\gamma_i)$ can now be deduced and summarized in the next lemma.

**Lemma 3.9.** *In virtue of the previous lemma we have*

$$\tilde{\tilde{V}}_{\gamma_i}^* \begin{pmatrix} 0_{p-(m_i+1)\times(m_i+1)} \\ I_{m_i+1} \end{pmatrix} = \tilde{V}_{\gamma_i}^*$$

*and*

$$U_p^*(\gamma_i)\left(\iota_{m_i+1} \otimes I_p\right) \begin{pmatrix} 0_{p-(m_i+1)\times(m_i+1)} \\ \tilde{V}_{\gamma_i}^{-*} \end{pmatrix} = I_{m_i+1},$$

*where* $\tilde{V}_{\gamma_i}^*$ *is the* $(m_i + 1) \times (m_i + 1)$ *Vandermonde matrix*

$$\tilde{V}_{\gamma_i}^* = \left(\frac{\partial^{m_i}}{\partial z^{m_i}} u_{m_i+1}^*(z), \frac{\partial^{m_i-1}}{\partial z^{m_i}} u_{m_i+1}^*(z), \ldots, u_{m_i+1}^*(z)\right)_{z=\gamma_i}^{\mathrm{T}}.$$

*A right inverse of* $U_p^*(\gamma_i)$ *is then*

$$U_p^*(\gamma_i)_R^- = \left(\iota_{m_i+1} \otimes I_p\right) \begin{pmatrix} 0_{p-(m_i+1)\times(m_i+1)} \\ \tilde{V}_{\gamma_i}^{-*} \end{pmatrix}.$$

**Proof.** Straightforward. □

**Lemma 3.10.** *The following is easily verified*:

$$\begin{pmatrix} 0_{(m_i+1)\times p(m_i+1)-1}, e_1, 0_{(m_i+1)\times(pm_i)-1}, e_2, \ldots, 0_{(m_i+1)\times p-1}, e_{m_i+1} \\ 0_{(m_i+1)(p-1)\times p(m_i+1)(m_i+2)/2} \end{pmatrix}$$
$$E_A^{(i)}(\gamma_i)\left(U_p^*(\gamma_i)_R^-, 0_{p(m_i+1)\times(m_i+1)(p-1)}\right)$$
$$= \begin{pmatrix} I_{m_i+1} & 0_{(m_i+1)\times(m_i+1)(p-1)} \\ 0_{(m_i+1)(p-1)\times(m_i+1)} & 0_{(m_i+1)(p-1)\times(m_i+1)(p-1)} \end{pmatrix}. \tag{3.3}$$

**Proof.** Straightforward. □

Let us denote the first and third term of the left-hand side of (3.3) by $\mathscr{A}_{m_i+1}$ and $\mathscr{B}_{m_i+1}$, respectively and the term on the right-hand side of (3.3) by $\mathscr{R}_{m_i+1}$. Eq. (3.3) can now be used for establishing an interconnection involving the last term of ( 3.2).

**Corollary 3.11.** *The next holds*:

$$\operatorname{diag}\left(\mathscr{A}_{m_1+1}, \mathscr{A}_{m_2+1}, \ldots, \mathscr{A}_{m_{q_0}+1}\right) \operatorname{diag}\left(E_A^{(1)}(\gamma_1), E_A^{(2)}(\gamma_2), \ldots, E_A^{(q_0)}(\gamma_{q_0})\right)$$
$$\times \operatorname{diag}\left(\mathscr{B}_{m_1+1}, \mathscr{B}_{m_2+1}, \ldots, \mathscr{B}_{m_{q_0}+1}\right) = \operatorname{diag}\left(\mathscr{R}_{m_1+1}, \mathscr{R}_{m_2+1}, \ldots, \mathscr{R}_{m_{q_0}+1}\right).$$

**Proof.** Straightforward. □

The results formulated in the Lemmas 3.6–3.10 and Corollary 3.11 allow us to detect Vandermonde matrices in the factorization proposed for $E_m(\gamma)$. The dependence of a generalized Vandermonde matrix can also be detected in the following structure:

$$G_n(\alpha) = \left(G_{n_1}(\alpha_1), G_{n_2}(\alpha_2), \ldots, G_{n_{p_0}}(\alpha_{p_0})\right),$$

where

$$G_{n_i}(\alpha_i) = \frac{1}{n_i!}\left(G_{n_i}^{(n_i)}(z), G_{n_i}^{(n_i-1)}(z), \ldots, G_{n_i}^{(0)}(z)\right)_{z=\alpha_i}$$

with each block

$$G_{n_i}^{(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}(\operatorname{adj}(zI-A))\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

Since the last column of $\operatorname{adj}(zI-A)$ is the vector $u_p(z)$, a connection with the generalized Vandermonde matrix $\widetilde{W}_\alpha$ can then be established. This will be formulated in the following lemma.

**Lemma 3.12.** *The following equations are easily verified*:

$$G_n(\alpha)(I_p \otimes e_p) = \widetilde{W}_\alpha \quad \textit{and} \quad G_n(\alpha)(\widetilde{W}_\alpha^{-1} \otimes e_p) = I_p,$$

*where* $\widetilde{W}_\alpha$ *is the generalized Vandermonde matrix defined in Lemma* 3.1.

**Proof.** Straightforward. □

### 3.2. *Special case*

We have the case where all the eigenvalues are distinct which are presented in this section. Eq. (2.20) can also be factorized according to

$$F_{aa} = \left(u_p(\alpha_1)u_p^{*\mathrm{T}}(\alpha_1), u_p(\alpha_2)u_p^{*\mathrm{T}}(\alpha_2), \ldots, u_p(\alpha_p)u_p^{*\mathrm{T}}(\alpha_p)\right) \times \left(r_1(\alpha_1)I_p, r_2(\alpha_2)I_p, \ldots, r_p(\alpha_p)I_p\right)^{\mathrm{T}}.$$

A right inverse of the first term of this factorization is provided in the next lemma and can be considered as a special case of Lemma 3.2.

**Lemma 3.13.** *The next relations hold true*:

$$\left(u_p(\alpha_1)u_p^{*\mathrm{T}}(\alpha_1), u_p(\alpha_2)u_p^{*\mathrm{T}}(\alpha_2), \ldots, u_p(\alpha_p)u_p^{*\mathrm{T}}(\alpha_p)\right)(I_p \otimes e_p) = V_\alpha$$

*and*

$$\left(u_p(\alpha_1)u_p^{*\mathrm{T}}(\alpha_1), u_p(\alpha_2)u_p^{*\mathrm{T}}(\alpha_2), \ldots, u_p(\alpha_p)u_p^{*\mathrm{T}}(\alpha_p)\right)(V_\alpha^{-1} \otimes e_p) = I_p,$$

*where $V_\alpha$ is the Vandermonde matrix defined in* (2.20). *Hence $(V_\alpha^{-1} \otimes e_p)$ is a right inverse of $(u_p(\alpha_1)u_p^{*\mathrm{T}}(\alpha_1), u_p(\alpha_2)u_p^{*\mathrm{T}}(\alpha_2), \ldots, u_p(\alpha_p)u_p^{*\mathrm{T}}(\alpha_p))$.*

**Proof.** Straightforward. □

The first and third term of (2.21) also involve Vandermonde matrices, this can be summarized in the following lemma which can be seen as a special case of Lemma 3.12.

**Lemma 3.14.** *The following equations may be verified*:

$$A_1(I_p \otimes e_p) = V_\alpha \quad \textit{and} \quad A_1(V_\alpha^{-1} \otimes e_p) = I_p$$

*as well as*

$$(I_p \otimes e_p^{\mathrm{T}})A_2^{\mathrm{T}} = V_\alpha^* \quad \textit{and} \quad (V_\alpha^{-*} \otimes e_p^{\mathrm{T}})A_2^{\mathrm{T}} = I_p,$$

*where the Vandermonde matrix $V_\alpha^*$ is defined in* (2.20).

**Proof.** Straightforward. □

We first specify $\bar{V}_{\gamma,L}^{-}$ and $\bar{V}_{\gamma,L}^{-\mathrm{T}}$ extracted from (2.25) and (2.26), respectively, for linking $S_{ca}$ with $F_{ac}$ and $F_{ca}$, followed by the appropriate inverses of the first terms of (2.24). A special case of Lemma 3.3 is summarized in the following lemma.

**Lemma 3.15.** *Since $p > q$,*

$$(I_q, 0_{q\times(p-q)})\bar{V}_\gamma = V_\gamma \quad \textit{and} \quad (V_\gamma^{-1}, 0_{q\times(p-q)})\bar{V}_\gamma = I_q.$$

*Taking the transpose yields a right inverse necessary for linking $S_{ca}$ with $F_{ca}$:*

$$\bar{V}_\gamma^{\mathrm{T}}\begin{pmatrix} I_q \\ 0_{(p-q)\times q}\end{pmatrix} = V_\gamma^{\mathrm{T}} \quad \textit{and} \quad \bar{V}_\gamma^{\mathrm{T}}\begin{pmatrix} V_\gamma^{-\mathrm{T}} \\ 0_{(p-q)\times q}\end{pmatrix} = I_q,$$

$$C_1(I_q \otimes e_q) = V_\gamma \quad \textit{and} \quad C_1(V_\gamma^{-1} \otimes e_q) = I_q,$$

$$(I_q \otimes e_q^{\mathrm{T}})A_3^{\mathrm{T}} = \bar{V}_\gamma^*$$

*with*

$$\bar{V}_\gamma^* = \begin{pmatrix} \gamma_1^{p-1} & \gamma_1^{p-2} & \cdots & \gamma_1 & 1 \\ \gamma_2^{p-1} & \gamma_2^{p-2} & \cdots & \gamma_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_q^{p-1} & \gamma_q^{p-2} & \cdots & \gamma_q & 1 \end{pmatrix}$$

*and*

$$V_\gamma = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_q \\ \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_q^2 \\ \vdots & \vdots & & \vdots \\ \gamma_1^{q-1} & \gamma_2^{q-1} & \cdots & \gamma_q^{q-1} \end{pmatrix}.$$

**Proof.** Straightforward. □

**Lemma 3.16.** *A right inverse of $\bar{V}_\gamma^*$ is given by*

$$\bar{V}_\gamma^*\begin{pmatrix} 0_{(p-q)\times q} \\ I_q \end{pmatrix} = V_\gamma^* \quad \textit{and} \quad \bar{V}_\gamma^*\begin{pmatrix} 0_{(p-q)\times q} \\ V_\gamma^{-*} \end{pmatrix} = I_q.$$

*The next relation then holds true:*

$$\left(I_q \otimes e_q^{\mathrm{T}}\right) A_3^{\mathrm{T}}\begin{pmatrix} 0_{(p-q)\times q} \\ V_\gamma^{-*} \end{pmatrix} = I_q.$$

**Proof.** Straightforward. □

**Lemma 3.17.** *For $p \leqslant q$ we have*

$$\left(u_p(\gamma_1), u_p(\gamma_2), \ldots, u_p(\gamma_q)\right)\begin{pmatrix} I_p \\ 0_{(q-p)\times p}\end{pmatrix} = V_{p,\gamma}$$

*and*

$$\big(u_p(\gamma_1), u_p(\gamma_2), \ldots, u_p(\gamma_q)\big)\begin{pmatrix} V_{p,\gamma}^{-1} \\ 0_{(q-p)\times p}\end{pmatrix} = I_p,$$

*where* $V_{p,\gamma}$ *is a* $(p \times p)$ *Vandermonde matrix defined in Corollary* 3.5.

**Proof.** Straightforward. □

## 4. Link solution Stein equation–Fisher information: The global approach

### *4.1. General case*

In this section an extension of previous sections is implemented by constructing interconnections where the entire Fisher information matrix, not decomposed, is taken as one block. Fisher's information matrix will be interconnected not only with the corresponding Stein solution but also with Sylvester's resultant, see [4], where the following property is established.

$$F(\theta) = S(c, -a)P(\theta)S^{\mathrm{T}}(c, -a), \tag{4.1}$$

where $S(c, -a)$ is the $(p + q) \times (p + q)$ Sylvester resultant defined as

$$S(a, c) = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_p & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \ddots & \\ 0 & & 1 & a_1 & a_2 & \cdots & a_p \\ 1 & c_1 & c_2 & \cdots & c_q & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \ddots & \\ 0 & & 1 & c_1 & c_2 & \cdots & c_q \end{pmatrix},$$

and

$$P(\theta) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{u_{p+q}(z)u^*_{p+q}(z)^{\mathrm{T}}}{a(z)c(z)a^*(z)c^*(z)}\, \mathrm{d}z. \tag{4.2}$$

An equivalent formulation of equation (4.1) was already given by McLeod in [8] and more explicitly in [9].

The interconnection with a corresponding Stein solution will first be constructed with $P(\theta)$. Applying Cauchy's residue theorem to (4.2)

$$\begin{aligned} P(\theta) = {} & g_1(\alpha_1) + g_2(\alpha_2) + \cdots + g_{p_0}(\alpha_{p_0}) \\ & + h_1(\gamma_1) + h_2(\gamma_2) + \cdots + h_{q_0}(\gamma_{q_0}), \end{aligned}$$

where

$$g_i(\alpha_i) = \frac{1}{n_i!}\frac{\partial^{n_i}}{\partial z^{n_i}}\left(\frac{u_{p+q}(z)u^{*\mathrm{T}}_{p+q}(z)}{\left(\prod_{j=1,\,j\neq i}^{p_0}(z-\alpha_j)^{n_j+1}\right)\left(\prod_{l=1}^{q_0}(z-\gamma_l)^{m_l+1}\right)a^*(z)c^*(z)}\right)_{z=\alpha_i},$$
$$1 \leqslant i \leqslant p_0,$$
$$h_j(\gamma_j) = \frac{1}{m_j!}\frac{\partial^{m_j}}{\partial z^{m_j}}\left(\frac{u_{p+q}(z)u^{*\mathrm{T}}_{p+q}(z)}{\left(\prod_{r=1}^{p_0}(z-\alpha_r)^{n_r+1}\right)\left(\prod_{l=1,l\neq j}^{q_0}(z-\gamma_l)^{m_l+1}\right)a^*(z)c^*(z)}\right)_{z=\gamma_j},$$
$$1 \leqslant j \leqslant q_0.$$

A similar approach will be used as in the previous sections, namely a separation between numerator and denominator in order to achieve an appropriate connection. By setting

$$\mu_i(\alpha_i) = \left(\frac{1}{\left(\prod_{j=1,\,j\neq i}^{p_0}(z-\alpha_j)^{n_j+1}\right)\left(\prod_{l=1}^{q_0}(z-\gamma_l)^{m_l+1}\right)a^*(z)c^*(z)}\right)_{z=\alpha_i},$$
$$\nu_j(\gamma_j) = \left(\frac{1}{\left(\prod_{r=1}^{p_0}(z-\alpha_r)^{n_r+1}\right)\left(\prod_{l=1,l\neq j}^{q_0}(z-\gamma_l)^{m_l+1}\right)a^*(z)c^*(z)}\right)_{z=\gamma_j},$$

vectorization of $P(\theta)$ yields

$$\begin{aligned}\mathrm{vec}P(\theta) &= \mathrm{vec}\ g_1(\alpha_1) + \mathrm{vec}\ g_2(\alpha_2) + \cdots + \mathrm{vec}\ g_{p_0}(\alpha_{p_0}) + \mathrm{vec}\ h_1(\gamma_1)\\ &\quad + \mathrm{vec}\ h_2(\gamma_2) + \cdots + \mathrm{vec}\ h_{q_0}(\gamma_{q_0})\\ &= \left(\bar{W}_n(\alpha), \bar{V}_m(\gamma)\right)\Big(\mu^{\mathrm{T}}_{n_1}(\alpha_1), \mu^{\mathrm{T}}_{n_2}(\alpha_2), \ldots, \mu^{\mathrm{T}}_{n_{p_0}}(\alpha_{p_0}),\\ &\qquad \nu^{\mathrm{T}}_{m_1}(\gamma_1), \nu^{\mathrm{T}}_{m_2}(\gamma_2), \ldots, \nu^{\mathrm{T}}_{m_{q_0}}(\gamma_{q_0})\Big)^{\mathrm{T}},\end{aligned} \tag{4.3}$$

where

$$\bar{W}_n(\alpha) = \left(\bar{W}_{n_1}(\alpha_1), \bar{W}_{n_2}(\alpha_2), \ldots, \bar{W}_{n_{p_0}}(\alpha_{p_0})\right)$$

with the matrices $\bar{W}_{n_i}(\alpha_i)$ given by

$$\bar{W}_{n_i}(\alpha_i) = \frac{1}{n_i!}\left(\bar{W}^{(n_i)}_{n_i}(z), \bar{W}^{(n_i-1)}_{n_i}(z), \ldots, \bar{W}^{(0)}_{n_i}(z)\right)_{z=\alpha_i}$$

each block being

$$\bar{W}^{(n_i-k)}_{n_i}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}\left(u^*_{p+q}(z)\otimes u_{p+q}(z)\right)\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

We further have

$$\bar{V}_m(\gamma) = \left(\bar{V}_{m_1}(\gamma_1), \bar{V}_{m_2}(\gamma_2), \ldots, \bar{V}_{m_{q_0}}(\gamma_{q_0})\right)$$

and $\bar{V}_{m_j}(\gamma_j)$ is given by

$$\bar{V}_{m_j}(\gamma_j) = \frac{1}{m_j!}\left(\bar{V}_{m_j}^{(m_j)}(z), \bar{V}_{m_j}^{(m_j-1)}(z), \ldots, \bar{V}_{m_j}^{(0)}(z)\right)_{z=\gamma_j},$$

each block has the following form:

$$\bar{V}_{m_j}^{(m_j-k)}(\gamma_j) = \binom{m_j}{k}\left(\frac{\partial^{m_j-k}}{\partial z^{m_j-k}}\left(u_{p+q}^*(z) \otimes u_{p+q}(z)\right)\right)_{z=\gamma_j}, \quad 0 \leqslant k \leqslant m_j.$$

The remaining terms are

$$\mu_{n_i}(\alpha_i) = \left(\mu_i(z), \frac{\partial}{\partial z}\mu_i(z), \ldots, \frac{\partial^{n_i}}{\partial z^{n_i}}\mu_i(z)\right)_{z=\alpha_i}^{\mathrm{T}},$$

$$\nu_{m_j}(\gamma_j) = \left(\nu_j(z), \frac{\partial}{\partial z}\nu_j(z), \ldots, \frac{\partial^{m_j}}{\partial z^{m_j}}\nu_j(z)\right)_{z=\gamma_j}^{\mathrm{T}}.$$

As in (2.11) an alternative representation will be developed for (4.2).

$$P(\theta) = \left(b^{(1)}(\alpha), d^{(1)}(\gamma)\right)\left(\left(b^{(2)}(\alpha)\right)^{\mathrm{T}}, \left(d^{(2)}(\gamma)\right)^{\mathrm{T}}\right)^{\mathrm{T}} \tag{4.4}$$

with

$$b^{(1)}(\alpha) = \left(b_1^{(1)}(\alpha_1), b_2^{(1)}(\alpha_2), \ldots, b_{p_0}^{(1)}(\alpha_{p_0})\right),$$

where

$$b_i^{(1)}(\alpha_i) = \frac{1}{n_i!}\left(b_i^{(1),(n_i)}(z), b_i^{(1),(n_i-1)}(z), \ldots, b_i^{(1),(0)}(z)\right)_{z=\alpha_i},$$

each block is given by

$$b_i^{(1),(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}u_{p+q}(z)\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

Additionally we have

$$d^{(1)}(\gamma) = \left(d_1^{(1)}(\gamma_1), d_2^{(1)}(\gamma_2), \ldots, d_{q_0}^{(1)}(\gamma_{q_0})\right)$$

with

$$d_j^{(1)}(\gamma_j) = \frac{1}{m_j!}\left(d_j^{(1),(m_j)}(z), d_j^{(1),(m_j-1)}(z), \ldots, d_j^{(1),(0)}(z)\right)_{z=\gamma_j}.$$

The structure of each block is

$$d_j^{(1),(m_j-k)}(\gamma_j) = \binom{m_j}{k}\left(\frac{\partial^{m_j-k}}{\partial z^{m_j-k}}u_{p+q}(z)\right)_{z=\gamma_j}, \quad 0 \leqslant k \leqslant m_j.$$

We further have

$$\left(b^{(2)}(\alpha)\right)^T = \left(\left(b_1^{(2)}(\alpha_1)\right)^{\mathrm{T}}, \left(b_2^{(2)}(\alpha_2)\right)^{\mathrm{T}}, \ldots, \left(b_{p_0}^{(2)}(\alpha_{p_0})\right)^{\mathrm{T}}\right),$$
$$\left(d^{(2)}(\gamma)\right)^{\mathrm{T}} = \left(\left(d_1^{(2)}(\gamma_1)\right)^{\mathrm{T}}, \left(d_2^{(2)}(\gamma_2)\right)^{\mathrm{T}}, \ldots, \left(d_{q_0}^{(2)}(\gamma_{q_0})\right)^{\mathrm{T}}\right).$$

The elements $(b^{(2)}(\alpha))^{\mathrm{T}}$ and $(d^{(2)}(\gamma))^{\mathrm{T}}$ consist of terms with the following structures: $b_i^{(2)}(\alpha_i) = \mu_{n_i}(\alpha_i) \otimes I_{p+q}$ and $d_j^{(2)}(\gamma_j) = \nu_{m_j}(\gamma_j) \otimes I_{p+q}$. In order to apply Theorem 1.1 we introduce the following $(p+q) \times (p+q)$ companion matrix:

$$\bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 1 \\ -g_{p+q} & -g_{p+q-1} & \cdots & -g_1 \end{pmatrix},$$

where the entries $g_i$ are given by $z^{p+q} + \sum_{i=1}^{p+q} g_i z^{p+q-i} = a(z)c(z) = g(z)$. The condition for uniqueness of the solution of Stein's equation is verified. Stein's equation and its solution are, respectively,

$$S - \bar{A}S\bar{A}^{\mathrm{T}} = \Gamma,$$
$$S = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} (zI - \bar{A})^{-1}\Gamma(I - z\bar{A})^{-\mathrm{T}}\,\mathrm{d}z,$$
$$= \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{\operatorname{adj}(zI - \bar{A})\Gamma \operatorname{adj}(I - z\bar{A})^{\mathrm{T}}}{a(z)c(z)a^*(z)c^*(z)}\,\mathrm{d}z. \tag{4.5}$$

Applying Cauchy's residue theorem to (4.5) yields

$$S = \bar{A}_1(\alpha_1) + \bar{A}_2(\alpha_2) + \cdots + \bar{A}_{p_0}(\alpha_{p_0}) + \bar{B}_1(\gamma_1) + \bar{B}_2(\gamma_2) + \cdots + \bar{B}_{q_0}(\gamma_{q_0}),$$
$$\bar{A}_i(\alpha_i) = \frac{1}{n_i!}\left(\frac{\partial^{n_i}}{\partial z^{n_i}} \frac{\operatorname{adj}(zI - \bar{A})\Gamma \operatorname{adj}(I - z\bar{A})^{\mathrm{T}}}{\left(\prod_{j=1, j\neq i}^{p_0}(z-\alpha_j)^{n_j+1}\right)\left(\prod_{l=1}^{q_0}(z-\gamma_l)^{m_l+1}\right)a^*(z)c^*(z)}\right)_{z=\alpha_i},$$
$$\bar{B}_j(\gamma_j) = \frac{1}{m_j!}\left(\frac{\partial^{m_j}}{\partial z^{m_j}} \frac{\operatorname{adj}(zI - \bar{A})\Gamma \operatorname{adj}(I - z\bar{A})^{\mathrm{T}}}{\left(\prod_{r=1}^{p_0}(z-\alpha_r)^{n_r+1}\right)\left(\prod_{l=1, l\neq j}^{q_0}(z-\gamma_l)^{m_l+1}\right)a^*(z)c^*(z)}\right)_{z=\gamma_j}.$$

A similar factorization as in (2.12) gives

$$S = \left(\bar{M}_n(\alpha), \bar{N}_m(\gamma)\right)\left(\left(b^{(2)}(\alpha)\right)^{\mathrm{T}}, \left(d^{(2)}(\gamma)\right)^{\mathrm{T}}\right)^{\mathrm{T}} \tag{4.6}$$

with

$$\bar{M}_n(\alpha) = \left(\bar{M}_{n_1}(\alpha_1), \bar{M}_{n_2}(\alpha_2), \ldots, \bar{M}_{n_{p_0}}(\alpha_{p_0})\right),$$

where the $(p+q) \times (n_i+1)(p+q)$ matrix

$$\bar{M}_{n_i}(\alpha_i) = \frac{1}{n_i!}\left(\bar{M}_{n_i}^{(n_i)}(z), \bar{M}_{n_i}^{(n_i-1)}(z), \ldots, \bar{M}_{n_i}^{(0)}(z)\right)_{z=\alpha_i}$$

has blocks

$$\bar{M}_{n_i}^{(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}(\operatorname{adj}(zI-\bar{A})\Gamma\operatorname{adj}(I-z\bar{A})^{\mathrm{T}})\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

We further have

$$\bar{N}_m(\gamma) = \left(\bar{N}_{m_1}(\gamma_1), \bar{N}_{m_2}(\gamma_2), \ldots, \bar{N}_{m_{q_0}}(\gamma_{q_0})\right),$$

where the $(p+q) \times (m_j+1)(p+q)$ matrix is given by

$$\bar{N}_{m_j}(\gamma_j) = \frac{1}{m_j!}\left(\bar{N}_{m_j}^{(m_j)}(z), \bar{N}_{m_j}^{(m_j-1)}(z), \ldots, \bar{N}_{m_j}^{(0)}(z)\right)_{z=\gamma_j}$$

with each block

$$\bar{N}_{m_j}^{(m_j-k)}(\gamma_j) = \binom{m_j}{k}\left(\frac{\partial^{m_j-k}}{\partial z^{m_j-k}}(\operatorname{adj}(zI-\bar{A})\Gamma\operatorname{adj}(I-z\bar{A})^{\mathrm{T}})\right)_{z=\gamma_j}, \quad 0 \leqslant k \leqslant m_j.$$

Combination of (4.3) and (4.6) proves the following theorem.

**Theorem 4.1.** *The relation holds true*:

$$S = \left(\bar{M}_n(\alpha), \bar{N}_m(\gamma)\right)\left\{\left(\left(\bar{W}_n(\alpha), \bar{V}_m(\gamma)\right)_L^{-} \operatorname{vec} P(\theta)\right) \otimes I_{p+q}\right\}.$$

From (4.1) can be seen that vec $P(\theta) = (S(c,-a) \otimes S(c,-a))^{-1}\operatorname{vec} F(\theta)$. The Sylvester resultant matrix is nonsingular if no common roots are assumed to exist between the polynomials $a(z)$ and $c(z)$. Theorem 4.1 combined with this property interconnects Stein's solution, Fisher's information matrix, Sylvester's resultant and a generalized Vandermonde matrix which is hidden in the left inverse. In [4] it has been verified that the following equality holds:

$$P(\theta) - \bar{A}P(\theta)\bar{A}^{\mathrm{T}} = \Gamma \tag{4.7}$$

for $\Gamma = e_{p+q}e_{p+q}^{\mathrm{T}}$, where $e_{p+q}$ is the last basis vector of the Euclidean space $\mathbb{R}^{p+q}$.

An equation for the Fisher matrix itself, similar to (4.7), is given in [4]. It can also be derived from (4.7) by using the expression given on p. 208 of [9], that relates the Fisher matrix for the ARMA case to the one of a corresponding AR process.

### *4.2. Special case*

In this section interconnections are also established when Fisher's information matrix is considered as a global matrix but with distinct roots for $a(z)$ and $c(z)$. Starting from (4.2) with these assumptions and applying a similar factorization as in (2.20) results in a factorization of $P(\theta)$ as the product of the following three terms.

$$P(\theta) = UDU^*, \tag{4.8}$$

where

$$\begin{aligned} U = \big(&u_{p+q}(\alpha_1), u_{p+q}(\alpha_2), \ldots, u_{p+q}(\alpha_p),\\ &u_{p+q}(\gamma_1), u_{p+q}(\gamma_2), \ldots, u_{p+q}(\gamma_q)\big), \end{aligned}$$

$$D = \operatorname{diag}\big(\varphi_1(\alpha_1), \varphi_2(\alpha_2), \ldots, \varphi_p(\alpha_p), \psi_1(\gamma_1), \psi_2(\gamma_2), \ldots, \psi_q(\gamma_q)\big)$$

and

$$\begin{aligned} U^* = \Big(&u^*_{p+q}(\alpha_1), u^*_{p+q}(\alpha_2), \ldots, u^*_{p+q}(\alpha_p),\\ &u^*_{p+q}(\gamma_1), u^*_{p+q}(\gamma_2), \ldots, u^*_{p+q}(\gamma_q)\Big)^{\mathrm{T}} \end{aligned}$$

with

$$\varphi_i(\alpha_i) = \left(\frac{1}{\Big(\prod_{j=1, j\neq i}^{p}(z-\alpha_j)\Big)\Big(\prod_{l=1}^{q}(z-\gamma_l)\Big)a^*(z)c^*(z)}\right)_{z=\alpha_i}$$

and

$$\psi_j(\gamma_j) = \left(\frac{1}{\Big(\prod_{r=1}^{p}(z-\alpha_r)\Big)\Big(\prod_{l=1, l\neq j}^{q}(z-\gamma_l)\Big)a^*(z)c^*(z)}\right)_{z=\gamma_j}.$$

The first and third term of (4.8) are Vandermonde matrices $V_{\alpha\gamma}$ and $V^*_{\alpha\gamma}$. Respectively

$$V_{\alpha\gamma} = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{p+q-1}\\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{p+q-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \alpha_p & \alpha_p^2 & \cdots & \alpha_p^{p+q-1}\\ 1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^{p+q-1}\\ 1 & \gamma_2 & \gamma_2^2 & \cdots & \gamma_2^{p+q-1}\\ \vdots & \vdots & \vdots & \vdots & \\ 1 & \gamma_q & \gamma_q^2 & \cdots & \gamma_q^{p+q-1} \end{pmatrix}^{\mathrm{T}}$$

and

$$V_{\alpha\gamma}^{*}=\begin{pmatrix}\alpha_1^{p+q-1} & \alpha_1^{p+q-2} & \cdots & \alpha_1 & 1\\ \alpha_2^{p+q-1} & \alpha_2^{p+q-2} & \cdots & \alpha_2 & 1\\ \vdots & \vdots & & \vdots & \vdots\\ \alpha_p^{p+q-1} & \alpha_p^{p+q-2} & \cdots & \alpha_p & 1\\ \gamma_1^{p+q-1} & \gamma_1^{p+q-2} & \cdots & \gamma_1 & 1\\ \gamma_2^{p+q-1} & \gamma_2^{p+q-2} & \cdots & \gamma_2 & 1\\ \vdots & \vdots & & \vdots & \vdots\\ \gamma_q^{p+q-1} & \gamma_q^{p+q-2} & \cdots & \gamma_q & 1\end{pmatrix}.$$

We now also have a factorization of Stein's solution

$$S=\left(\bar{A}_1,\bar{A}_2\right)\left(I_{p+q}\otimes\Gamma\right)\left(D\otimes I_{p+q}\right)\left(\bar{A}_3,\bar{A}_4\right)^{\mathrm{T}}, \tag{4.9}$$

where

$$\begin{aligned}\bar{A}_1&=\left(\operatorname{adj}(\alpha_1 I-\bar{A}),\operatorname{adj}(\alpha_2 I-\bar{A}),\ldots,\operatorname{adj}(\alpha_p I-\bar{A})\right),\\ \bar{A}_2&=\left(\operatorname{adj}(\gamma_1 I-\bar{A}),\operatorname{adj}(\gamma_2 I-\bar{A}),\ldots,\operatorname{adj}(\gamma_q I-\bar{A})\right),\\ \bar{A}_3&=\left(\operatorname{adj}(I-\alpha_1\bar{A}),\operatorname{adj}(I-\alpha_2\bar{A}),\ldots,\operatorname{adj}(I-\alpha_p\bar{A})\right),\\ \bar{A}_4&=\left(\operatorname{adj}(I-\gamma_1\bar{A}),\operatorname{adj}(I-\gamma_2\bar{A}),\ldots,\operatorname{adj}(I-\gamma_q\bar{A})\right)\end{aligned}$$

and $D$ as above.

The combination of (4.8) and (4.9) produces an interconnection as summarized in the following lemma.

**Lemma 4.2.** *The following equation holds true*:

$$S=\left(\bar{A}_1,\bar{A}_2\right)\left(V_{\alpha\gamma}^{-1}P(\theta)V_{\alpha\gamma}^{-*}\otimes\Gamma\right)\left(\bar{A}_3,\bar{A}_4\right)^{\mathrm{T}}. \tag{4.10}$$

In virtue of (4.1) and (4.10) an interrelationship between Stein's solution, Fisher's information matrix, Sylvester's resultant and Vandermonde matrices is set forth for the case of distinct roots. As in (2.23) an alternative factorization can be considered.

$$\begin{aligned}P(\theta)=&\Big(u_{p+q}(\alpha_1)u_{p+q}^{*\mathrm{T}}(\alpha_1),\ldots,u_{p+q}(\alpha_p)u_{p+q}^{*\mathrm{T}}(\alpha_p),u_{p+q}(\gamma_1)u_{p+q}^{*\mathrm{T}}(\gamma_1)\\ &\quad,\ldots,u_{p+q}(\gamma_q)u_{p+q}^{*\mathrm{T}}(\gamma_q)\Big)\\ &\times\left(\varphi_1(\alpha_1)I_{p+q},\varphi_2(\alpha_2)I_{p+q},\ldots,\varphi_p(\alpha_p)I_{p+q},\psi_1(\gamma_1)I_{p+q},\psi_2(\gamma_2)I_{p+q},\right.\\ &\quad\left.\ldots,\psi_q(\gamma_q)I_{p+q}\right)^{\mathrm{T}}.\end{aligned} \tag{4.11}$$

## 5. Left and right inverses: Global approach

As in Section 3 left and right inverses will be presented.

### 5.1. General case

The notations that will be used are introduced. The $(p+q)\times(p+q)$ generalized Vandermonde matrix is given by

$$\tilde{\tilde{\bar{W}}}_{\alpha\gamma}=\Big(\tilde{\tilde{\bar{W}}}_{n_1}(\alpha_1),\tilde{\tilde{\bar{W}}}_{n_2}(\alpha_2),\ldots,\tilde{\tilde{\bar{W}}}_{n_{p_0}}(\alpha_{p_0}),\\ \tilde{\tilde{\bar{V}}}_{m_1}(\gamma_1),\tilde{\tilde{\bar{V}}}_{m_2}(\gamma_2),\ldots,\tilde{\tilde{\bar{V}}}_{m_{q_0}}(\gamma_{q_0})\Big),$$

where

$$\tilde{\tilde{\bar{W}}}_{n_i}(\alpha_i)=\frac{1}{n_i!}\left(\tilde{\tilde{\bar{W}}}^{(n_i)}_{n_i}(z),\tilde{\tilde{\bar{W}}}^{(n_i-1)}_{n_i}(z),\ldots,\tilde{\tilde{\bar{W}}}^{(0)}_{n_i}(z)\right)_{z=\alpha_i}$$

with

$$\tilde{\tilde{\bar{W}}}^{(n_i-k)}_{n_i}(\alpha_i)=\binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}u_{p+q}(z)\right)_{z=\alpha_i},\quad 0\leqslant k\leqslant n_i.$$

The matrix

$$\tilde{\tilde{\bar{V}}}_{m_j}(\gamma_j)=\frac{1}{m_j!}\left(\tilde{\tilde{\bar{V}}}^{(m_j)}_{m_j}(z),\tilde{\tilde{\bar{V}}}^{(m_j-1)}_{m_j}(z),\ldots,\tilde{\tilde{\bar{V}}}^{(0)}_{m_j}(z)\right)_{z=\gamma_j}$$

with

$$\tilde{\tilde{\bar{V}}}^{(m_j-k)}_{m_j}(\gamma_j)=\binom{m_j}{k}\left(\frac{\partial^{m_j-k}}{\partial z^{m_j-k}}u_{p+q}(z)\right)_{z=\gamma_j},\quad 0\leqslant k\leqslant m_j.$$

**Lemma 5.1.** *The following relations are verified*:

$$\left(0_{p+q\times(p+q)(p+q-1)},I_{p+q}\right)\left(\bar{W}_n(\alpha)\;\bar{V}_m(\gamma)\right)=\tilde{\tilde{\bar{W}}}_{\alpha\gamma},$$

$$\left(0_{p+q\times(p+q)(p+q-1)},\tilde{\tilde{\bar{W}}}^{-1}_{\alpha\gamma}\right)\left(\bar{W}_n(\alpha)\;\bar{V}_m(\gamma)\right)=I_{p+q}.$$

**Proof.** Straightforward. □

**Lemma 5.2.** *The first term of* (4.4) *verifies*

$$\left(b^{(1)}(\alpha),d^{(1)}(\gamma)\right)\left(I_{p+q}\otimes e_{p+q}\right)=\tilde{\tilde{\bar{W}}}_{\alpha\gamma},$$

$$\left(b^{(1)}(\alpha),d^{(1)}(\gamma)\right)\left(\tilde{\tilde{\bar{W}}}^{-1}_{\alpha\gamma}\otimes e_{p+q}\right)=I_{p+q}.$$

**Proof.** Straightforward. □

An extension of Lemma 3.12 allows us to consider $(K_n(\alpha), L_m(\gamma))$. With

$$K_n(\alpha) = \Big(K_{n_1}(\alpha_1), K_{n_2}(\alpha_2), \ldots, K_{n_{p_0}}(\alpha_{p_0})\Big)$$

and

$$K_{n_i}(\alpha_i) = \frac{1}{n_i!}\Big(K_{n_i}^{(n_i)}(z), K_{n_i}^{(n_i-1)}(z), \ldots, K_{n_i}^{(0)}(z)\Big)_{z=\alpha_i},$$

where each block is described as

$$K_{n_i}^{(n_i-k)}(\alpha_i) = \binom{n_i}{k}\left(\frac{\partial^{n_i-k}}{\partial z^{n_i-k}}\big(\operatorname{adj}(zI-\bar{A})\big)\right)_{z=\alpha_i}, \quad 0 \leqslant k \leqslant n_i.$$

Next we have

$$L_m(\gamma) = \Big(L_{m_1}(\gamma_1), L_{m_2}(\gamma_2), \ldots, L_{m_{q_0}}(\gamma_{q_0})\Big)$$

with

$$L_{m_j}(\gamma_j) = \frac{1}{m_j!}\Big(L_{m_j}^{(m_j)}(z), L_{m_j}^{(m_j-1)}(z), \ldots, L_{m_j}^{(0)}(z)\Big)_{z=\gamma_j},$$

where each block is

$$L_{m_j}^{(m_j-k)}(\gamma_j) = \binom{m_j}{k}\left(\frac{\partial^{m_j-k}}{\partial z^{m_j-k}}\big(\operatorname{adj}(zI-\bar{A})\big)\right)_{z=\gamma_j}, \quad 0 \leqslant k \leqslant m_j.$$

Then we have:

**Lemma 5.3.** *The following equations hold true*:

$$(K_n(\alpha), L_m(\gamma))\big(I_{p+q} \otimes e_{p+q}\big) = \tilde{\tilde{\bar{W}}}_{\alpha\gamma},$$

$$(K_n(\alpha), L_m(\gamma))\left(\tilde{\tilde{\bar{W}}}{}_{\alpha\gamma}^{-1} \otimes e_{p+q}\right) = I_{p+q}.$$

**Proof.** Straightforward. □

## *5.2. Special case*

A right inverse of the first term of (4.11) has the property summarized in the next lemma.

**Lemma 5.4.** *The following equations hold*:

$$\Big(u_{p+q}(\alpha_1)u_{p+q}^{*\mathrm{T}}(\alpha_1), \ldots, u_{p+q}(\alpha_p)u_{p+q}^{*\mathrm{T}}(\alpha_p), u_{p+q}(\gamma_1)u_{p+q}^{*\mathrm{T}}(\gamma_1)$$
$$, \ldots, u_{p+q}(\gamma_q)u_{p+q}^{*\mathrm{T}}(\gamma_q), \Big)\big(I_{p+q} \otimes e_{p+q}\big) = V_{\alpha\gamma},$$

$$\Big(u_{p+q}(\alpha_1)u^{*\mathrm{T}}_{p+q}(\alpha_1),\ldots,u_{p+q}(\alpha_p)u^{*\mathrm{T}}_{p+q}(\alpha_p),u_{p+q}(\gamma_1)u^{*\mathrm{T}}_{p+q}(\gamma_1)$$
$$,\ldots,u_{p+q}(\gamma_q)u^{*\mathrm{T}}_{p+q}(\gamma_q),\Big)\left(V^{-1}_{\alpha\gamma}\otimes e_{p+q}\right)=I_{p+q}.$$

**Proof.** Straightforward. □

The first and third term of (4.10) also involve Vandermonde matrices.

**Lemma 5.5.** *One has*

$$\left(\bar{A}_1,\bar{A}_2\right)\left(I_{p+q}\otimes e_{p+q}\right)=V_{\alpha\gamma}\quad\textit{and}\quad\left(\bar{A}_1,\bar{A}_2\right)\left(V^{-1}_{\alpha\gamma}\otimes e_{p+q}\right)=I_{p+q},$$

$$\left(I_{p+q}\otimes e^{\mathrm{T}}_{p+q}\right)\left(\bar{A}_3,\bar{A}_4\right)^{\mathrm{T}}=V^{*}_{\alpha\gamma}\quad\textit{and}\quad\left(V^{-*}_{\alpha\gamma}\otimes e^{\mathrm{T}}_{p+q}\right)\left(\bar{A}_3,\bar{A}_4\right)^{\mathrm{T}}=I_{p+q}.$$

**Proof.** Straightforward. □

## 6. Connection Lyapunov equation–Fisher's information

The purpose of this section consists of showing that possible interconnections between the solution of Lyapunov's equation and Fisher's information matrix are similarly obtained as for Stein's equation. We therefore illustrate the $(a,a)$ block with distinct roots, the results for other blocks are similar. First we recall the following theorem from [6]. Let $A\in\mathbb{C}^{m\times m}$, $B\in\mathbb{C}^{n\times n}$ and $\Omega\in\mathbb{C}^{n\times m}$. The Lyapunov equation

$$LA-BL=\Omega \tag{6.1}$$

has a unique solution iff $A$ and $B$ have no eigenvalues in common.

**Theorem 6.1.** *If the Lyapunov equation* (6.1) *has a unique solution* $L$, *then*

$$L=\frac{1}{2\pi\mathrm{i}}\oint_D(\lambda I-B)^{-1}\,\Omega\,(\lambda I-A)^{-1}\,\mathrm{d}\lambda, \tag{6.2}$$

*where $D$ is a single closed contour with $\sigma(A)$ inside $D$ and $\sigma(B)$ outside $D$.*

Use (2.20) and compare with an appropriate form of (6.2), the choice of a companion matrix form for $A$ is similar with the one used in (2.6), whereas $B$ is associated with the monic polynomial $b(z)=1+a_1z+\cdots+a_{p-1}z^{p-1}+z^p$, denote the companion matrix $B$

$$B=\begin{pmatrix}0&1&\cdots&0\\ \vdots&&\ddots&\vdots\\ 0&&&1\\ -1&-a_1&\cdots&-a_{p-1}\end{pmatrix}.$$

We will apply Cauchy's residue theorem to (6.2) and combine with $F_{aa}$ as developed in Section 2.2. First the following notations are introduced.

$$B_1 = \left(\mathrm{adj}(\alpha_1 I - B), \mathrm{adj}(\alpha_2 I - B), \ldots, \mathrm{adj}(\alpha_p I - B)\right)$$

$$b_L(\alpha) = \mathrm{diag}\left(b_1(\alpha_1), b_2(\alpha_2), \ldots, b_p(\alpha_p)\right)$$

with $b_i(\alpha_i) = (\det(zI - B))^{-1}_{z=\alpha_i} = (b(z))^{-1}_{z=\alpha_i}$. The interconnection with $F_{aa}$ is summarized in the following lemma.

**Lemma 6.2.** *The next link holds true*:

$$L_{aa} = B_1 \left(V_\alpha^{-1} F_{aa} V_\alpha^{-*} \otimes \Omega\right) \left(b_L(\alpha) \otimes I_p\right) A_2^{\mathrm{T}}. \tag{6.3}$$

Note that the last column of $\mathrm{adj}(zI - B)$ is the vector $u_p(z)$ so that Vandermonde matrices can be detected in a similar way as in previous sections by searching right and left inverses.

## 7. Examples

In this section we illustrate by means of two examples how a solution of Stein's equation can be computed through Fisher's information matrix. Mathematica 3.0 is used.

### 7.1. *Distinct roots and global approach*

The first case to be considered is the ARMA time series process with $p = 1$ and $q = 1$, first without specifying the structure of $\Gamma$ and then for an explicit $\Gamma$. Fisher's information matrix is taken as one block, the assumption of distinct eigenvalues is in force and the link (4.10) is evaluated. Fisher's information matrix of the ARMA process, described by

$$y(t) + ay(t-1) = \varepsilon(t) + c\varepsilon(t-1)$$

is evaluated according to (2.3)–(2.5) and is given by

$$F(\theta) = \begin{pmatrix} \frac{1}{1-a^2} & -\frac{1}{1-ac} \\ -\frac{1}{1-ac} & \frac{1}{1-c^2} \end{pmatrix}.$$

We will use this expression in order to find a solution to Stein's equation and for that purpose we give the main terms involved in the equation linking the solution of Stein'sequation with $P(\theta)$. The roots of the autoregressive and moving average polynomials are $\alpha = -a$ and $\gamma = -c$, respectively, we assume that $a \neq c$. The Sylvester resultant is

$$S(c, -a) = \begin{pmatrix} 1 & c \\ -1 & -a \end{pmatrix}$$

and $P(\theta)$ is according to (4.1)

$$P(\theta) = S^{-1}(-c, a)F(\theta)S^{-\mathrm{T}}(-c, a)$$
$$= \frac{1}{(1-a^2)(ac-1)(c^2-1)}\begin{pmatrix} 1+ac & -(a+c) \\ -(a+c) & 1+ac \end{pmatrix}.$$

The Vandermonde matrices $V_{ac}$ and $V_{ac}^*$ are,

$$V_{ac} = \begin{pmatrix} 1 & 1 \\ -a & -c \end{pmatrix} \quad \text{and} \quad V_{ac}^* = \begin{pmatrix} -a & 1 \\ -c & 1 \end{pmatrix}.$$

$$V_{ac}^{-1}P(\theta)V_{ac}^{-*} = \begin{pmatrix} \frac{1}{(1-a^2)(1-ac)(c-a)} & 0 \\ 0 & \frac{1}{(1-c^2)(1-ac)(a-c)} \end{pmatrix}.$$

Introduce

$$\bar{A}_1 = \operatorname{adj}(-aI - \bar{A}), \quad \bar{A}_2 = \operatorname{adj}(-cI - \bar{A}),$$
$$\bar{A}_3 = \operatorname{adj}(I + a\bar{A}), \quad \bar{A}_4 = \operatorname{adj}(I + c\bar{A}).$$

with

$$\left(\bar{A}_1, \bar{A}_2\right) = \begin{pmatrix} c & 1 & a & 1 \\ -ac & -a & -ac & -c \end{pmatrix},$$
$$\bar{A} = \begin{pmatrix} 0 & 1 \\ -ac & -(a+c) \end{pmatrix}$$
$$\left(\bar{A}_3, \bar{A}_4\right)^{\mathrm{T}} = \begin{pmatrix} 1-a^2-ac & -a & 1-c^2-ac & -c \\ a^2c & 1 & ac^2 & 1 \end{pmatrix}^{\mathrm{T}},$$
$$\Gamma = \begin{pmatrix} \Gamma^{11} & \Gamma^{12} \\ \Gamma^{21} & \Gamma^{22} \end{pmatrix}.$$

Stein's equation and its corresponding solution expressed in terms of $P(\theta)$ can now be formulated, the latter according to (4.10)

$$S - \bar{A}S\bar{A}^{\mathrm{T}} = \Gamma,$$
$$S = \left(\bar{A}_1, \bar{A}_2\right)\left(V_{ac}^{-1}P(\theta)V_{ac}^{-*} \otimes \Gamma\right)\left(\bar{A}_3, \bar{A}_4\right)^{\mathrm{T}},$$
$$= \frac{1}{(1-a^2)(1-c^2)(-1+ac)}\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix} \tag{7.1}$$

with

$$S^{11} = -\Big\{(1 + a^3c - c^2 + ac(c^2-1) + a^2(2c^2-1))\Gamma^{11} + a^2c(\Gamma^{12} + \Gamma^{21})$$
$$+ \Gamma^{22} + ac(c\Gamma^{12} + c\Gamma^{21} + \Gamma^{22})\Big\},$$

$$
\begin{aligned}
S^{12} &= a^3c^2\Gamma^{11} + (-1+c^2)\Gamma^{12} + a^2(c^3\Gamma^{11} + \Gamma^{12} + c^2\Gamma^{21}) + c\Gamma^{22} \\
&\quad + a(c\Gamma^{12} + c\Gamma^{21} + \Gamma^{22}),
\end{aligned}
$$

$$
\begin{aligned}
S^{21} &= ac\Gamma^{12} - \Gamma^{21} + c^2\Gamma^{21} + a^2(ac^2\Gamma^{11} + \Gamma^{21}) + c\Gamma^{22} \\
&\quad + a(ac^2(c\Gamma^{11} + \Gamma^{12}) + c\Gamma^{21} + \Gamma^{22}),
\end{aligned}
$$

$$
\begin{aligned}
S^{22} &= -\Big\{ac^2\Gamma^{12} + a^2c(ac^2\Gamma^{11} + \Gamma^{21}) + \Gamma^{22} + a(ac(c\Gamma^{11} + \Gamma^{12}) \\
&\quad + c(c\Gamma^{21} + \Gamma^{22}))\Big\}.
\end{aligned}
$$

It is now straightforward to verify that by choosing

$$
\Gamma = e_2 e_2^{\mathrm{T}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
$$

in (7.1), equality between the solution of Stein's equation and $P(\theta)$ is obtained or equivalently, it can be checked that the Stein equation $P(\theta) - \bar{A}P(\theta)\bar{A}^{\mathrm{T}} = e_2 e_2^{\mathrm{T}}$ holds true.

### *7.2. Multiple roots and one block*

In this example a case of multiple poles is considered for the $(a, c)$ block of Fisher's information matrix, $p = 3$ and $q = 2$, the roots are $\alpha$ and $\gamma$ for the autoregressive and moving average polynomial respectively with corresponding multiplicities $n + 1 = 3$ and $m + 1 = 2$. Link (2.17) is considered. The block $F_{ac}$ is computed according to (2.3) with $u_p(z) = (1\, z\, z^2)^{\mathrm{T}}$ and $u_q^*(z) = (z\, 1)^{\mathrm{T}}$.

$$
\begin{aligned}
F_{ac} &= -\frac{1}{\left(1 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3\right)^2} \\
&\quad \times \begin{pmatrix} 1 - a_2\gamma^2 - 2a_3\gamma^3 & -(a_1 + 2a_2\gamma + 3a_3\gamma^2) \\ 2\gamma + a_1\gamma^2 - a_3\gamma^4 & 1 - a_2\gamma^2 - 2a_3\gamma^3 \\ 3\gamma^2 + 2a_1\gamma^3 + a_2\gamma^4 & 2\gamma + a_1\gamma^2 - a_3\gamma^4 \end{pmatrix}.
\end{aligned}
$$

We shall write the matrices necessary for establishing (2.17).

$$
\begin{aligned}
V_m(\gamma) &= \left(\frac{\partial}{\partial z}\left(u_q^*(z) \otimes u_p(z)\right), u_q^*(z) \otimes u_p(z)\right)_{z=\gamma} \\
&= \begin{pmatrix} 1 & 2\gamma & 3\gamma^2 & 0 & 1 & 2\gamma \\ \gamma & \gamma^2 & \gamma^3 & 1 & \gamma & \gamma^2 \end{pmatrix}^{\mathrm{T}}.
\end{aligned}
$$

A left inverse as presented in Lemma 3.3 is set forth

$$
(0_{2\times 3}\ I_2\ 0_{2\times 1})\, V_m(\gamma) = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix} = \widetilde{W}_\gamma,
$$

$$V_m(\gamma)_L^- = \begin{pmatrix} 0 & 0 & 0 & -\gamma & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The companion matrices associated with the monic polynomials $a(z)$ and $c(z)$, respectively, are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 \\ -c_2 & -c_1 \end{pmatrix}$$

and $a^*(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3$, $c(z) = z^2 + c_1 z + c_2$ . For writing out the first term of (2.17) appropriate adjoint matrices are first given followed by $\Gamma^{ca}$.

$$\operatorname{adj}(I - zA) = \begin{pmatrix} 1 + a_1 z + a_2 z^2 & z + a_1 z^2 & z^2 \\ -a_3 z^2 & 1 + a_1 z & z \\ -a_3 z & -a_2 z - a_3 z^2 & 1 \end{pmatrix},$$

$$\operatorname{adj}(zI - C) = \begin{pmatrix} c_1 + z & 1 \\ -c_2 & z \end{pmatrix},$$

$$\Gamma^{ca} = \begin{pmatrix} \Gamma^{11} & \Gamma^{12} & \Gamma^{13} \\ \Gamma^{21} & \Gamma^{22} & \Gamma^{23} \end{pmatrix}.$$

The first term of (2.17) is

$$E_m(\gamma) = \left( \frac{\partial}{\partial z} (\operatorname{adj}(zI - C)\Gamma^{ca} \operatorname{adj}(I - zA)^{\mathrm{T}}), \operatorname{adj}(zI - C)\Gamma^{ca} \operatorname{adj}(I - zA)^{\mathrm{T}} \right)_{z=\gamma}.$$

Since it is clear that computing $E_m(\gamma)$ under its present form is quite cumbersome, a factorization proposed in Eqs. (3.1)–(3.2) subsection 3.1 shall be applied here.

$$\begin{aligned}
E_m(\gamma) &= E_C(\gamma) D E_A(\gamma), \\
E_C(\gamma) &= \left( E_C^{(1)}(\gamma) \ E_C^{(0)}(\gamma) \right), \\
E_C^{(1)}(\gamma) &= \left( \frac{\partial}{\partial z} \operatorname{adj}(zI - C), \operatorname{adj}(zI - C) \right)_{z=\gamma}, \\
E_C^{(0)}(\gamma) &= (\operatorname{adj}(zI - C))_{z=\gamma}, \\
E_A(\gamma) &= \operatorname{diag}\left( E_A^{(1)}(\gamma) \ E_A^{(0)}(\gamma) \right), \\
E_A^{(1)}(\gamma) &= \left( \operatorname{adj}(I - zA), \frac{\partial}{\partial z} \operatorname{adj}(I - zA) \right)_{z=\gamma}^{\mathrm{T}}, \\
E_A^{(0)}(\gamma) &= (\operatorname{adj}(I - zA))_{z=\gamma}^{\mathrm{T}}.
\end{aligned}$$

This results in the form

$$E_C(\gamma) = \begin{pmatrix} 1 & 0 & c_1 + z & 1 & c_1 + z & 1 \\ 0 & 1 & -c_2 & z & -c_2 & z \end{pmatrix}_{z=\gamma},$$

$$D = \operatorname{diag}\left( \Gamma^{ca} \ \Gamma^{ca} \ \Gamma^{ca} \right),$$

$$E_A(\gamma) = \left(\begin{array}{ccc:ccc} 1+a_1z+a_2z^2 & -a_3z^2 & -a_3z & 0 & 0 & 0 \\ z+a_1z^2 & 1+a_1z & -a_2z-a_3z^2 & 0 & 0 & 0 \\ z^2 & z & 1 & 0 & 0 & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_1+2a_2z & -2a_3z & -a_3 & 0 & 0 & 0 \\ 1+2a_1z & a_1 & -a_2-2a_3z & 0 & 0 & 0 \\ 2z & 1 & 0 & 0 & 0 & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0 & 0 & 1+a_1z+a_2z^2 & -a_3z^2 & -a_3z \\ 0 & 0 & 0 & z+a_1z^2 & 1+a_1z & -a_2z-a_3z^2 \\ 0 & 0 & 0 & z^2 & z & 1 \end{array}\right)_{z=\gamma}.$$

All the matrices necessary for linking Stein's solution with the corresponding block of Fisher's information are now evaluated. Consequently Stein's solution can be set forth through Fisher's information matrix according to (2.17). However, Lemmas 3.6 and 3.10 shall first be verified. With

$$\bar{V}_{\gamma,q_0} = \bar{V}_{\gamma,1} = (1\ \gamma)^{\mathrm{T}} \quad \text{and} \quad \bar{V}^{-}_{\gamma,1,L} = (1\ 0),$$

Lemma 3.6 becomes

$$(1 \quad 0) E_C(\gamma)(0_{1\times 5} \quad e_1)^{\mathrm{T}} = 1,$$

whereas

$$\widetilde{V}^{*}_{\gamma} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \quad \text{so that } \widetilde{V}^{-*}_{\gamma} = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}$$

and

$$U^{*}_{3}(\gamma_i)^{-}_{R} = (I_3\ I_3)^{\mathrm{T}} \begin{pmatrix} 0_{1\times 2} \\ \widetilde{V}^{-*}_{\gamma} \end{pmatrix}.$$

Lemma 3.10 is confirmed through the following equation

$$\begin{pmatrix} 0_{2\times 5} & e_1 & 0_{2\times 2} & e_2 \\ & 0_{4\times 9} & & \end{pmatrix} E_A(\gamma) \left(U^{*}_{3}(\gamma_i)^{-}_{R}\ 0_{6\times 4}\right) = \begin{pmatrix} I_2 & 0_{2\times 4} \\ 0_{4\times 2} & 0_{4\times 4} \end{pmatrix}.$$

Stein's equation followed by its solution is now given as

$$S_{ca} - C S_{ca} A^{\mathrm{T}} = \Gamma^{ca},$$

$$\begin{aligned} S_{ca} &= -E_m(\gamma)\left\{\left(V_m(\gamma)^{-}_{L}\mathrm{vec}\, F_{ac}\right) \otimes I_3\right\}, \\ &= \frac{1}{(1 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3)^2} \begin{pmatrix} S^{11}_{ca} & S^{12}_{ca} & S^{13}_{ca} \\ S^{21}_{ca} & S^{22}_{ca} & S^{23}_{ca} \end{pmatrix}. \end{aligned}$$

More explicitly we have

$$\begin{aligned} S^{11}_{ca} = {} & (1 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3)(\Gamma^{13}\gamma^2 + \Gamma^{12}\gamma(1 + a_1\gamma) + \Gamma^{11}(1 + a_1\gamma + a_2\gamma^2) \\ & + (a_1 + 2a_2\gamma)(\Gamma^{12} + \Gamma^{11}(c_1 + \gamma) + (1 + 2a_1\gamma)(\Gamma^{22} + \Gamma^{12}(c_1 + \gamma)) \\ & + 2\gamma(\Gamma^{23} + \Gamma^{13}(c_1 + \gamma))) - (a_1 + \gamma(2a_2 + 3a_3\gamma))((1 + a_1\gamma + a_2\gamma^2) \\ & \times (\Gamma^{21} + \Gamma^{11}(c_1 + \gamma)) + \gamma(1 + a_1\gamma)(\Gamma^{22} + \Gamma^{12}(c_1 + \gamma)) \\ & + \gamma^2(\Gamma^{23} + \Gamma^{13}(c_1 + \gamma))) \\ S^{12}_{ca} = {} & (\Gamma^{12} + a_1c_1\Gamma^{12} + c_1\Gamma^{13} + a_1\Gamma^{22} + \Gamma^{23} - 2a_3c_1\Gamma^{11}\gamma + 2a_1\Gamma^{12}\gamma \\ & + 2\Gamma^{13}\gamma - 2a_3\Gamma^{21}\gamma - 3a_3\Gamma^{11}\gamma^2)(1 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3) \\ & - (a_1 + \gamma(2a_2 + 3a_3\gamma))(-a_3\gamma^2(\Gamma^{21} + \Gamma^{11}(c_1 + \gamma)) + (1 + a_1\gamma) \\ & \times (\Gamma^{22} + \Gamma^{12}(c_1 + \gamma)) + \gamma(\Gamma^{23} + \Gamma^{13}(c_1 + \gamma))) \\ S^{13}_{ca} = {} & -(a_1 + \gamma(2a_2 + 3a_3\gamma))(\Gamma^{23} + \Gamma^{13}(c_1 + \gamma) - \gamma(a_2 + a_3\gamma)(c_1\Gamma^{12} \\ & + \Gamma^{22} + \gamma\Gamma^{12}) - a_3\gamma(\Gamma^{21} + \Gamma^{11}(c_1 + \gamma))) - (1 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3) \end{aligned}$$

$$
\begin{aligned}
&\times (-\Gamma^{13} + a_2(c_1\Gamma^{12} + \Gamma^{22} + 2\gamma\Gamma^{12}) + a_3(\Gamma^{21} + 2\Gamma^{11}\gamma + 2\gamma\Gamma^{22} \\
&+ 3\Gamma^{12}\gamma^2 + c_1(\Gamma^{11} + 2\Gamma^{12}\gamma)))
\end{aligned}
$$

$$
\begin{aligned}
S_{ca}^{21} = &-(a_1 + \gamma(2a_2 + 3a_3\gamma))(\gamma(1 + a_1\gamma)(-c_2\Gamma^{12} + \Gamma^{22}\gamma) + \gamma^2(-c_2\Gamma^{13} \\
&+ \Gamma^{23}\gamma) + (-c_2\Gamma^{11} + \Gamma^{21}\gamma)(1 + a_1\gamma + a_2\gamma^2)) + (1 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3) \\
&\times (\Gamma^{21} + 2\Gamma^{22}\gamma + 3a_2\gamma^2\Gamma^{21} + 3\Gamma^{23}\gamma^2 - c_2(\Gamma^{12} + 2a_2\gamma\Gamma^{11} + 2\gamma\Gamma^{13}) \\
&+ a_1(-c_2(\Gamma^{11} + 2\Gamma^{12}\gamma) + \gamma(2\Gamma^{21} + 3\Gamma^{22}\gamma)))
\end{aligned}
$$

$$
\begin{aligned}
S_{ca}^{22} = &-\gamma(-2\Gamma^{23} + 3a_3\Gamma^{21}\gamma - a_1\gamma\Gamma^{23} + 2a_1a_3\Gamma^{21}\gamma^2 + a_2a_3\Gamma^{21}\gamma^3 \\
&+ a_3\Gamma^{23}\gamma^3) - \Gamma^{22}(-1 - 2a_1\gamma - a_1^2\gamma^2 + a_2\gamma^2 + 2a_3\gamma^3 + a_1a_3\gamma^4) \\
&+ c_2(\Gamma^{13}(-1 + a_2\gamma^2 + 2a_3\gamma^3) + \gamma(2a_3\Gamma^{11} + 2a_2\Gamma^{12} + a_1a_3\Gamma^{11}\gamma \\
&+ a_1a_2\Gamma^{12}\gamma + 3a_3\Gamma^{12}\gamma + 2a_1a_3\Gamma^{12}\gamma^2 - a_3^2\Gamma^{11}\gamma^3))
\end{aligned}
$$

$$
\begin{aligned}
S_{ca}^{23} = &+a_1c_2\Gamma^{13} + \Gamma^{23} - a_2^2c_2\Gamma^{12}\gamma^2 - a_3^2\gamma^3(2c_2\Gamma^{11} + c_2\Gamma^{12}\gamma - \Gamma^{21}\gamma) \\
&- a_2(-c_2\Gamma^{12} - 2c_2\Gamma^{13}\gamma + 2\Gamma^{22}\gamma + a_1\Gamma^{22}\gamma^2 + \Gamma^{23}\gamma^2) - a_3(\gamma(2\Gamma^{21} \\
&+ a_1\Gamma^{21}\gamma + 3\Gamma^{22}\gamma + 2a_1\Gamma^{22}\gamma^2 + 2\Gamma^{23}\gamma^2) + c_2(-\Gamma^{11} - 2\Gamma^{12}\gamma \\
&+ a_2\Gamma^{11}\gamma^2 - a_1\Gamma^{12}\gamma^2 - 3\Gamma^{13}\gamma^2 + 2a_2\Gamma^{12}\gamma^3)).
\end{aligned}
$$

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