A block Hankel generalized confluent Vandermonde matrix

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A R T I C L E   I N F O

A B S T R A C T

Vandermonde matrices are well known. They have a number of interesting properties and play a role in (Lagrange) interpolation problems, partial fraction expansions, and finding solutions to linear ordinary differential equations, to mention just a few applications. Usually, one takes these matrices square, $q \times q$ say, in which case the i-th column is given by $u(z_i)$, where we write $u(z) = (1, z, \ldots, z^{q-1})^\top$. If all the z_i $(i = 1, \ldots, q)$ are different, the Vandermonde matrix is non-singular, otherwise not. The latter case obviously takes place when all z_i are the same, z say, in which case one could speak of a confluent Vandermonde matrix. Non-singularity is obtained if one considers the matrix $V(z)$ whose i-th column $(i = 1, \ldots, q)$ is given by the $(i-1)$-th derivative $u^{(i-1)}(z)^\top$. We will consider generalizations of the confluent Vandermonde matrix $V(z)$ by considering matrices obtained by using as building blocks the matrices $M(z) = u(z)w(z)$, with $u(z)$ as above and $w(z) = (1, z, \ldots, z^{r-1})$, together with its derivatives $M^{(k)}(z)$. Specifically, we will look at matrices whose ij-th block is given by $M^{(i+j)}(z)$, where the indices i, j by convention have initial value zero. These in general non-square matrices exhibit a block-Hankel structure. We will answer a number of elementary questions for this matrix. What is the rank? What is the null-space? Can the latter be parametrized in a simple way? Does it depend on z? What are left or right inverses? It turns out that answers can be

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obtained by factorizing the matrix into a product of other matrix polynomials having a simple structure. The answers depend on the size of the matrix $M(z)$ and the number of derivatives $M^{(k)}(z)$ that is involved. The results are obtained by mostly elementary methods, no specific knowledge of the theory of matrix polynomials is needed.

1. Introduction and notations

1.1. Motivation

Vandermonde matrices are well known. They have a number of interesting properties and play a role in (Lagrange) interpolation problems, partial fraction expansions, and finding solutions to linear ordinary differential equations, to mention just a few applications. Usually, one takes these matrices square, $q \times q$ say, in which case the i-th column is given by $u(z_i)$, where we write $u(z) = (1, z, \dots, z^{q-1})^\top$. If all the z_i $(i = 1, \dots, q)$ are different, the Vandermonde matrix is non-singular, otherwise not. The latter case obviously takes place when all z_i are the same, z say, in which case one could speak of a confluent Vandermonde matrix. Non-singularity is obtained if one considers the matrix $V(z)$ whose i-th column $(i = 1, \dots, q)$ is given by the $(i-1)$-th derivative $u^{(i-1)}(z)^\top$, or by $u^{(i-1)}(z)^\top/(i-1)!$. In the latter case, one has $\det(V(z)) = 1$. A slightly more general situation is obtained, when one considers in general the non-square $q \times \nu$ matrix $V(z)$, with i-th column $u^{(i-1)}(z)^\top$, $i = 1, \dots, \nu - 1$. In this case one has that $V(z)$ has rank equal to $\min\{q, \nu\}$ and for $\nu > q$, the kernel of $V(z)$ is the $(\nu - q)$-dimensional subspace of $\mathbb{R}^\nu$ consisting of the vectors whose first q elements are equal to zero. Note that the building elements of the matrix V are the column vectors $u_i(z)$ and a number of its derivatives.

The observations above can be generalized in many directions. In the present paper we opt for one of them, in which we will consider generalizations of the confluent Vandermonde matrix $V(z)$ by considering matrices obtained by using as building blocks the matrices $M(z) = u(z)w(z) \in \mathbb{R}^{q \times r}$, with $u(z)$ as above and $w(z) = (1, z, \dots, z^{r-1})$, together with its derivatives $M^{(k)}$ with $0 \le k \le \nu - 1$. Note that $M(z) = V(z)$ if $r = 1$.

A special case of what follows is obtained by considering the matrix

$$\mathcal{M}^0(z) = \big(M(z), \dots, M^{(\nu-1)}(z)\big) \in \mathbb{R}^{q \times \nu r}.$$

In a recent paper [8], the kernel of the matrix $\mathcal{M}^0(z)$ (or rather, a matrix obtained by a permutation of the columns of $\mathcal{M}^0(z)$) has been studied for the case $q = r$ and $\nu \le q$. More precisely, the two aims of the cited paper were to find a right inverse and a parametrization of the kernel of the matrix $\mathcal{M}^0(z)$. The two cases $\nu = q$ or $\nu < q$

have been analyzed in detail. For these two cases two algorithms have been proposed to construct the kernel and the rank and the dimension of the kernel have been computed. This was of relevance for the characterization of a matrix polynomial equation having non-unique solutions. The origin of this problem was of a statistical nature and lied in properties of the asymptotic Fisher information matrix for estimating the parameters of an ARMAX process and a related Stein equation.

The results on the parametrization of the kernel obtained in [8] turned out to be unnecessary complicated, partly due an apparently irrelevant distinction between different cases, and not transparent. In the present paper we reconsider the problem by embedding it into a much more general approach to obtaining properties of matrix polynomials that can be viewed as generalizations of a confluent Vandermonde matrix (in a single variable) as indicated above. The solution, which is shown to exhibit a very simple and elegant structure, to the original problem of describing the kernel under consideration, now follows as a byproduct of the current analysis. These are special cases of the unified situations of Corollary 2.5 and Proposition 4.1 of the present paper. In this way we generalize in the present paper the results of [8]. Another approach to find a basis for the kernel has been followed in [7], which is also subsumed by Proposition 4.1. The results of [7] are closer in spirit to those of the present paper than the results in [8]. Both cited papers contain some examples of right inverses of $\mathcal{M}^0(z)$, that are special cases of what will be obtained in the present paper. Moreover, we will generalize the results for $\mathcal{M}^0(z)$ to results for the matrix $\mathcal{M}(z)$ that is defined by

$$
\mathcal{M}(z) = \begin{pmatrix} M(z) & \cdots & M^{(\nu-1)}(z) \\ \vdots & & \vdots \\ M^{(\mu-1)}(z) & \cdots & M^{(\mu+\nu-2)}(z) \end{pmatrix} \in \mathbb{R}^{\mu q \times \nu r},
$$

whose blocks we denote $\mathcal{M}(z)^{ij}$, $i = 0, \ldots, \mu-1$, $j = 0, \ldots, \nu-1$, so $\mathcal{M}(z)^{ij} = M^{(i+j)}(z)$. We will also consider the related matrix $\mathcal{N}(z)$ whose blocks are $\mathcal{N}(z)^{ij} = \frac{M^{(i+j)}(z)}{i!j!}$.

A special case occurs for the choices of the parameters $r = 1$, $\mu = 1$ and $\nu = q$. Then we write $\mathcal{M}(z) = U_q(z)$ and $\mathcal{N}(z) = \widetilde{U}_q(z)$, ordinary (normalized) $q \times q$ confluent Vandermonde matrices. The results in the next sections (almost) reduce to trivialities for ordinary confluent Vandermonde matrices, so the contribution of the present paper originates with allowing the parameters μ, ν, q and r to be arbitrary.

Classical Vandermonde matrices and confluent Vandermonde matrices have often been studied in the literature, see [9,3] for definitions. More recent papers often focussed on finding formulas for their inverses and on efficient numerical procedures to compute them, a somewhat random choice of references are the papers [1,2,4–6,10–12]. Nevertheless it seems that the generalization $\mathcal{M}(z)$ of the confluent Vandermonde matrix is, to the best of our knowledge, unknown in the literature.

1.2. Notations and conventions

Derivatives of a function $z \mapsto f(z)$ (often matrix valued) are denoted by $f^{(k)}(z)$ or by $(\frac{d}{dz})^k f(z)$. The variable z is in principle $\mathbb{C}$-valued. If A is an $m \times n$ matrix, for notational convenience we adopt the convention to label its elements A_{ij} with $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$. We will see shortly why this a convenient convention. Entries of a matrix are also indicated both by superindices, according to what is typographically most appropriate. When dealing with matrix polynomials $A(z)$, in proving results we usually suppress the dependence on z and simply write A.

If $\mathcal{A}$ is a block matrix, we sometimes use subindexes to indicate its constituting blocks, but more often denote its blocks by superindices, so we write $\mathcal{A}^{ij}$ or A^{ij} in the latter case. Also single superindices are used and we will come across notations like A^k. These should not be confused with powers of A. The meaning of A^k will always be clear from the context.

Throughout the paper q, r, μ and ν are fixed positive integers, although often certain relations among them are supposed (e.g. $q + 1 \le \nu < q + r$). Given the integers q, r, μ and ν and the variable z, we consider a number of matrix polynomials. The first are $u(z) = (1, \ldots, z^{q-1})^\top$ and $w(z) = (1, \ldots, z^{r-1})$. With the above labelling convention, valid throughout the paper, we have for the elements of $u(z)$ the expression $u_i(z) = z^i$ for $i = 0, \ldots, q-1$. Next to these we consider the matrix polynomials

$$M(z) = u(z)w(z) \in \mathbb{R}^{q \times r},$$

$$\mathcal{M}_j^0(z) = M^{(j)}(z) = \left(\frac{d}{dz}\right)^j M(z) \in \mathbb{R}^{q \times r},$$

$$\mathcal{M}^0(z) = (\mathcal{M}_0^0(z), \ldots, \mathcal{M}_{\nu-1}^0(z)) \in \mathbb{R}^{q \times \nu r},$$

$$\mathcal{N}_j^0(z) = \frac{1}{j!} M^{(j)}(z) = \frac{1}{j!}\left(\frac{d}{dz}\right)^j M(z) \in \mathbb{R}^{q \times r},$$

$$\mathcal{N}^0(z) = (\mathcal{N}_0^0(z), \ldots, \mathcal{N}_{\nu-1}^0(z)) \in \mathbb{R}^{q \times \nu r}.$$

We also consider the $\mu q \times \nu r$ matrix polynomial $\mathcal{M}(z)$ which is defined by its ij-blocks

$$\mathcal{M}^{ij}(z) = M^{(i+j)}(z), \tag{1.1}$$

for $i = 0, \ldots, \mu-1$, $j = 0, \ldots, \nu-1$. The size of $\mathcal{M}(z)$ is equal to $\mu q \times \nu r$. If $\mu = 1$, we retrieve $\mathcal{M}^0(z)$. Note that $\mathcal{M}(z)$ has a block Hankel structure, even for the case $\mu \neq \nu$.

Along with $\mathcal{M}(z)$ we consider the matrix $\mathcal{N}(z)$ that has a block structure with blocks

$$\mathcal{N}^{ij}(z) = \frac{1}{i!j!} M^{(i+j)}(z),$$

for $i = 0, \ldots, \mu - 1$ and $j = 0, \ldots, \nu - 1$. Note that $\mathcal{N}(z)$ also has dimensions $\mu q \times \nu r$ and that $\mathcal{N}(z)$ reduces to $\mathcal{N}^0(z)$ for $\mu = 1$. Unlike $\mathcal{M}(z)$, $\mathcal{N}(z)$ doesn't exhibit a block Hankel structure. The following obvious relation holds.

$$\mathcal{M}(z) = (D_\mu \otimes I_q)\mathcal{N}(z)(D_\nu \otimes I_r), \tag{1.2}$$

where D_μ is the diagonal matrix with entries $D_{ii} = i!$, $i = 0, \ldots, \mu - 1$ and D_ν likewise.

Let us now introduce the matrices $U_q(z)$ (of size $q \times q$) and $W_r(z)$ (of size $r \times r$) by

$$U_q(z) = \big(u(z), \ldots, u^{(q-1)}(z)\big)$$

and

$$W_r(z)^\top = \big(w(z)^\top, \ldots, w^{(r-1)}(z)^\top\big).$$

Along with the matrices $U_q(z)$ and $W_r(z)$, we introduce the matrices

$$\widetilde{U}_q(z) = \big(\widetilde{u}_0(z), \ldots, \widetilde{u}_{q-1}(z)\big), \tag{1.3}$$

with

$$\widetilde{u}_j(z) = \frac{1}{j!} u^{(j)}(z), \quad j = 0, \ldots, q-1,$$

and $\widetilde{W}_r(z)$ given by

$$\widetilde{W}_r(z)^\top = \big(\widetilde{w}_0(z), \ldots, \widetilde{w}_{r-1}(z)\big), \tag{1.4}$$

with

$$\widetilde{w}_j(z) = \frac{1}{j!} w^{(j)}(z), \quad j = 0, \ldots, r.$$

We have the obvious relations

$$U_q(z) = \widetilde{U}_q(z)D_q, \qquad W_r(z) = D_r\widetilde{W}_r(z),$$

where D_q and D_r are diagonal matrices, similarly defined as D_μ. One easily verifies that $\widetilde{U}_q(z)$ has elements $\widetilde{U}_q^{ij}(z) = z^{i-j}\binom{i}{j}$ and by a simple computation that $\widetilde{U}_q(z)^{-1} = \widetilde{U}_q(-z)$. Similar properties hold for $\widetilde{W}_r(z)$. Later on we will also come across the matrices $U_\mu(z)$ and $W_\nu(z)$, which have the same structure as $U_q(z)$ and $W_r(z)$ and only differ in size (unless $\mu = q$ and $\nu = r$).

Finally we introduce the often used square shift matrices $S_k \in \mathbb{R}^{k \times k}$ (k arbitrary), defined by its ij-elements $\delta_{i+1,j}$ (Kronecker deltas). Other matrices will be introduced along the way.

The structure of the remainder of the paper is as follows. After having fixed some notation and other conventions, in Section 2 we shall introduce an auxiliary matrix polynomial $\mathcal{A}(z)$ that is instrumental in deriving properties of $\mathcal{M}(z)$ and $\mathcal{N}(z)$. Among them are factorizations, which will be treated in Section 3. These lead to establishing the rank of $\mathcal{M}(z)$, which is shown to be independent of z. A major issue in the present paper is to find a simple and transparent parametrization of the kernels of the matrices $\mathcal{M}(z)$ and $\mathcal{N}(z)$. This is done first in Section 4 for the special case of $\mathcal{N}^0(z)$. The results of that section will be used in Section 5 to characterize the kernel of $\mathcal{N}(z)$. The results of Section 6 form the cornerstone of finding right inverses of $\mathcal{M}(z)$, especially those that are not dependent on z, which is a main topic of the final Section 7.

2. The matrix $\mathcal{A}(z)$

This section is devoted to the matrix $\mathcal{A}(z)$, to be defined below, that is instrumental in deriving properties of $\mathcal{M}(z)$ and $\mathcal{N}(z)$. In particular it is used for obtaining useful factorizations in Section 3. The matrix $\mathcal{A}(z)$, also of block Hankel type and of size $\mu q \times \nu r$ is defined by its blocks $\mathcal{A}_{ij}(z) := A^{i+j}(z)$ (here $i + j$ is used as a super index) of size $q \times r$, for $i = 0, \ldots, \mu - 1$, $j = 0, \ldots, \nu - 1$. The matrices $A^k(z)$ for $k \geq 0$ are the $q \times r$ quasi-symmetric matrix polynomials (this matrix polynomial is only square for $q = r$) given by their elements $(i, j = 0, \ldots, q - 1)$

$$
A_{ij}^k(z) := \frac{1}{i!j!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{i+j} z^k
$$

$$
= \binom{k}{i+j} \binom{i+j}{i} z^{k-i-j} = \begin{cases} \frac{k!}{i!\,j!\,(k-i-j)!} z^{k-i-j} & \text{if } i+j \leq k \\ 0 & \text{else.} \end{cases} \tag{2.1}
$$

Note that k in $A^k(z)$ is used as a super index and in z^k as a power. The block Hankel matrix $\mathcal{A}(z)$ of size $\mu q \times \nu r$ is then defined by its blocks $\mathcal{A}^{ij}(z)$ which are equal to $A^{i+j}(z)$, for $i = 0, \ldots, \mu - 1$ and $j = 0, \ldots, \nu - 1$. One easily verifies that (with $A_{-1,j}^{k-1} = A_{i,-1}^{k-1} = 0$) for $k \geq 1$ it holds that

$$
A_{ij}^k(z) = z A_{ij}^{k-1}(z) + A_{i-1,j}^{k-1}(z) + A_{i,j-1}^{k-1}(z).
$$

In matrix notation this relation becomes

$$
A^k(z) = z A^{k-1}(z) + A^{k-1}(z) S_r + S_q^\top A^{k-1}(z). \tag{2.2}
$$

Let $J_q(z) = z I_q + S_q^\top$. Then we can rewrite (2.2) as

$$
A^k(z) = J_q(z) A^{k-1}(z) + A^{k-1}(z) S_r, \tag{2.3}
$$

or as

$$A^k(z) = S_q^\top A^{k-1}(z) + A^{k-1}(z)J_r(z)^\top. \tag{2.4}$$

By induction one easily proves that the recursion (2.3) leads to the explicit expressions, involving powers of $S_q^\top$ and S_r,

$$A^k(z) = \sum_{i=0}^{k} \binom{k}{i} J_q(z)^i A^0 S_r^{k-i}$$

and

$$A^{k+l}(z) = \sum_{i=0}^{k} \binom{k}{i} J_q(z)^i A^l(z) S_r^{k-i}. \tag{2.5}$$

Likewise one shows that (2.4) leads to

$$A^k(z) = \sum_{i=0}^{k} \binom{k}{i} (S_q^\top)^i A^0 (J_r(z)^\top)^{k-i}. \tag{2.6}$$

On the other hand, the recursion (2.2) has solution

$$A^k(z) = \sum_{j=0}^{k} \binom{k}{j} z^{k-j} \sum_{i=0}^{j} \binom{j}{i} (S_q^\top)^i A^0 S_r^{j-i}. \tag{2.7}$$

It also follows that

$$A^k(0) = \sum_{i=0}^{k} \binom{k}{i} (S_q^\top)^i A^0 S_r^{k-i}.$$

Look back at the $A^k(z)$ as defined in (2.1). One computes for $k \geq 0$

$$A_{ij}^k(0) = \begin{cases} \binom{k}{i} & \text{if } i + j = k \\ 0 & \text{else.} \end{cases} \tag{2.8}$$

Henceforth we shall write A^k instead of $A^k(0)$. Note that A^k, which is in general not square, has only nonzero elements on one 'anti-diagonal', the one with ij-elements having sum $i + j = k$. We have the following result.

Proposition 2.1. *The matrices $A^k = A^k(0)$ satisfy the recursion for $k \geq 0$*

$$A^{k+1} = A^k S_r + S_q^\top A^k,$$

and hence

$$A^k = \sum_{j=0}^{k} \binom{k}{j} (S_q^\top)^j A^0 S_r^{k-j}.$$

Moreover, we also have the polynomial expansion

$$A^k(z) = \sum_{j=0}^{k} \binom{k}{j} z^{k-j} A^j.$$

Proof. The recursion follows by taking $z = 0$ in (2.2). The expansion follows from Proposition 2.1 and Eq. (2.7). □

Two more matrices will be introduced next. First, let $\bar{A}$ be the $\mu q \times \nu r$ block-matrix whose ij-th block (of size $q \times r$) is given by $\bar{A}^{ij} = (S_q^\top)^j A^0 S_r^i$ for $i = 0, \ldots, \mu - 1$ and $j = 0, \ldots, \nu - 1$.

Second, we introduce the matrix $\mathcal{L}_{\mu,q}(z)$ of size $\mu q \times \mu q$ with blocks $\mathcal{L}_{\mu,q}(z)_{ij} = \binom{i}{j} J_q(z)^{i-j}$ for $i \geq j$ and zero else, $i, j = 0, \ldots, \mu - 1$. Since $\mathcal{L}_{\mu,q}(z)$ is block lower diagonal with identity matrix as diagonal blocks, it follows that $\mathcal{L}_{\mu,q}(z)$ is invertible and that its inverse has ij-block equal to $\binom{i}{j} J_q(z)^{i-j}(-1)^{i-j}$. Likewise one defines the matrix $\mathcal{L}_{\nu,r}(z)$ of size $\nu r \times \nu r$.

Theorem 2.2. *The factorization*

$$\mathcal{A}(z) = \mathcal{L}_{\mu,q}(z) \bar{A} \mathcal{L}_{\nu,r}(z)^\top \tag{2.9}$$

holds true. For $z = 0$ and with $\mathcal{A} = \mathcal{A}(0)$ and $\mathcal{L}_{\mu,q} = \mathcal{L}_{\mu,q}(0)$ and $\mathcal{L}_{\nu,r} = \mathcal{L}_{\nu,r}(0)$, this becomes

$$\mathcal{A} = \mathcal{L}_{\mu,q} \bar{A} \mathcal{L}_{\nu,r}^\top. \tag{2.10}$$

Moreover, $\det(\mathcal{A}(z)) = \det(\bar{A})$, when both matrices are square.

Proof. We compute the ij-block on the right hand side of (2.9). Using the definitions of $\mathcal{L}_{\mu,q}(z)$, $\mathcal{L}_{\nu,r}(z)$ and $\bar{A}$, we obtain (see the explanation below)

$$\left(\mathcal{L}_{\mu,q}(z) \bar{A} \mathcal{L}_{\nu,r}(z)^\top\right)_{ij} = \sum_{k,l} \mathcal{L}_{\mu,q}(z)_{ik} \bar{A}^{kl} \mathcal{L}_{\nu,r}(z)_{lj}^\top$$

$$= \sum_{k=0}^{i} \sum_{l=0}^{j} \binom{i}{k} J_q(z)^{i-k} (S_q^\top)^l A^0 S_r^k \binom{j}{l} (J_r(z)^\top)^{j-l}$$

$$= \sum_{k=0}^{i} \binom{i}{k} J_q(z)^{i-k} \left(\sum_{l=0}^{j} (S_q^\top)^l A^0 \binom{j}{l} (J_r(z)^\top)^{j-l} \right) S_r^k$$

$$= \sum_{k=0}^{i} \binom{i}{k} J_q(z)^{i-k} A^j(z) S_r^k$$

$$= A^{i+j}(z).$$

In the third equality above we used that $J_r(z)^\top$ and S_r commute, in the fourth equality we used (2.6) with an appropriate change of the indices, whereas the last equality similarly follows from (2.5). The relation between the determinants follows from $\det(\mathcal{L}_{\mu,q}(z)) = 1$. □

Let e_i be the i-th standard column basis vector of $\mathbb{R}^q$, $i = 0, \ldots, q-1$ and let f_i be the i-th standard column basis vector of $\mathbb{R}^r$, $i = 0, \ldots, r-1$. For convenience of notation, we put $e_i = 0$ for $i \geq q$ and $f_j = 0$ for $j \geq r$. With this convention, we always have for example $(S_q^\top)^i e_0 = e_i$.

Theorem 2.3. *It holds that the block $\bar{\mathcal{A}}^{ij} = e_j f_i^\top$, the rank of $\bar{\mathcal{A}}$ is equal to $\min\{\mu, r\} \times \min\{\nu, q\}$. If $\bar{\mathcal{A}}$ is a fat matrix, i.e. $\mu q \leq \nu r$, then $\bar{\mathcal{A}}$ has full (row) rank iff $\mu \leq r$ and $\nu \geq q$. If $\bar{\mathcal{A}}$ is a tall matrix, so $\mu q \geq \nu r$, then it has full (column) rank iff $\mu \leq r$ and $\nu \leq q$. In the special case that $\bar{\mathcal{A}}$ is square, so $\mu q = \nu r$, we get that $\bar{\mathcal{A}}$ has full rank, and it is then invertible, iff $\mu = r$ and $\nu = q$, in which case $\bar{\mathcal{A}}$ is a matrix of size $qr \times qr$. In this case we have for the inverse $(\bar{\mathcal{A}})^{-1} = (\bar{\mathcal{A}})^\top$ and $\det(\bar{\mathcal{A}}) = (-1)^{\frac{1}{4} qr(q-1)(r-1)}$. Specializing even more to $q = r$, we get $\det(\bar{\mathcal{A}}) = (-1)^{\frac{1}{2} q(q-1)}$.*

Proof. Since $A^0 = e_0 f_0^\top$ and $\bar{\mathcal{A}}^{ij} = (S_q^\top)^j A^0 S_r^i$, it follows that $\bar{\mathcal{A}}^{ij} = f_j e_i^\top$.

The rank of $\bar{\mathcal{A}}$ is equal to the rank of $\bar{\mathcal{A}}(\bar{\mathcal{A}})^\top$, which is easy to compute. We get for its ij-th block $\sum_{l=0}^{\nu-1} e_l f_i^\top f_j e_l^\top = f_i^\top f_j \sum_{l=0}^{q} e_l e_l^\top =: f_i^\top f_j I_{q,\nu-1}$, where $I_{q,\nu-1} = \sum_{l=0}^{\nu-1} e_l e_l^\top$. It follows that $\text{rank}(I_{q,\nu-1}) = \min\{q, \nu\}$. Furthermore we have $f_i^\top f_j = 0$ for $i \neq j$ and $f_i^\top f_i = 1$ iff $i \leq r - 1$. We conclude that $\bar{\mathcal{A}}(\bar{\mathcal{A}})^\top$ is block diagonal, where the diagonal ii-blocks are equal to $I_{q,\nu-1}$ for $i \leq r - 1$ and zero otherwise. The number of nonzero diagonal blocks is equal to $\min\{\mu, r\}$, hence the rank of $\bar{\mathcal{A}}(\bar{\mathcal{A}})^\top$ is equal to $\min\{\mu, r\} \times \min\{\nu, q\}$.

Assume that $\bar{\mathcal{A}}$ is fat and that $\mu \leq r$ and $\nu \geq q$. Then the rank of $\bar{\mathcal{A}}$ equals μq, the number of rows of $\bar{\mathcal{A}}$. For the converse statement we assume that $\mu > r$ (the case $\nu < q$ can be treated similarly). In this case the rank becomes $r \times \min\{\nu, q\}$, which is strictly less than μq, the number of rows of $\bar{\mathcal{A}}$, which then has rank deficiency. The dual statements for a tall $\bar{\mathcal{A}}$ follow by symmetry.

Assume next that the matrix $\bar{\mathcal{A}}$ is square. It has full rank iff the two sets of conditions for the tall and fat case hold simultaneously, which yields the assertion on invertibility. Assume then that $\mu = r$ and $\nu = q$. By the computations in the first part of the proof we see that the diagonal blocks of $\bar{\mathcal{A}}(\bar{\mathcal{A}})^\top$ are all equal to I_q. Since there are now r of these blocks, we obtain that $\bar{\mathcal{A}}(\bar{\mathcal{A}})^\top$ is the $qr \times qr$ identity matrix.

To compute the determinant for this case, we observe that the columns of $\bar{\mathcal{A}}$ consist of all the basis vectors of $\mathbb{R}^{qr}$, but in a permuted order. Therefore, the determinant is

equal to plus or minus one. To establish the value of the sign, we compute the number of inversions of the permutation. This turns out to be equal to $\frac{1}{4}qr(q-1)(r-1)$, which results in a determinant equal to $+1$ iff this number is even, and -1 in the other case. If $r = q$, then $(-1)^{\frac{1}{4}q^2(q-1)^2} = (-1)^{\frac{1}{2}q(q-1)}$. A way of computing the number of inversions is to write the order of the numbering of the column basis vectors in rectangular array. Decomposing every number $x \in \{0, 1, \ldots, qr - 1\}$ in a unique way as $x = nq + m$ with $m \in \{0, \ldots, q-1\}$ and $n \in \{0, \ldots, r-1\}$, we can identify every such x with a pair (m, n). An inversion $i(x, y)$ occurs when $x < y$, but in the permuted order x is preceded by y. For every x the number of inversions $i(x, y)$ is the number of elements in the rectangle strictly to the South-West of x in the rectangular array. So, if x corresponds to (m, n) then the number of inversions $i(x, y)$ is equal to $(q - 1 - m)n$. Summing these numbers for $m = 0, \ldots, q - 1$ and $n = 0, \ldots, r - 1$ yields the total number of inversions. $\square$

Remark 2.4. In the case were $\bar{A}$ is square and invertible, it is the permutation matrix having the property $\mathrm{vec}(X^\top) = \bar{A}\,\mathrm{vec}(X)$, for any $X \in \mathbb{R}^{q \times r}$.

Corollary 2.5. *The rank of the matrix $\mathcal{A}(z)$ is for all $z \in \mathbb{C}$ equal to $\min\{\mu, r\} \times \min\{\nu, q\}$. If $\mu q = \nu r$, then $\mathcal{A}(z)$ is square and invertible and $\det(\mathcal{A}(z)) = (-1)^{\frac{1}{4}qr(q-1)(r-1)}$.*

Proof. The assertion on the rank follows from Theorems 2.2 and 2.3 upon noting that the matrices $\mathcal{L}_{\mu,q}(z)$ and $\mathcal{L}_{\nu,r}(z)$ are invertible. Since $\det(\mathcal{L}_{\mu,q}(z)) = 1$, also the assertion about the determinant follows. $\square$

3. Factorizations of the matrices $\mathcal{M}(z)$ and $\mathcal{N}(z)$

In the present section we obtain a factorization of the matrix polynomial $\mathcal{M}(z)$, from which a factorization of the matrix $\mathcal{N}(z)$ follows as a simple corollary. In the next proposition we use the Kronecker symbol $\otimes$ to denote tensor products.

Proposition 3.1. *The factorization*

$$\mathcal{M}(z) = \big(I_\mu \otimes U_q(z)\big)\mathcal{A}\big(I_\nu \otimes W_r(z)\big) \tag{3.1}$$

holds true. Moreover, $\mathcal{M}(z)$ and $\mathcal{A}$ have the same rank equal to $\min\{\mu, r\} \times \min\{\nu, r\}$.

Proof. Computation of the product $(I_q \otimes U_q(z))\mathcal{A}(I_q \otimes W_r(z))$ by using the definition of $\mathcal{A}$ as block matrix, yields a matrix that consists of blocks

$$U_q(z)A^k W_r(z) = \sum_{i=0}^{q-1}\sum_{j=0}^{r-1} A_{ij}^k u^{(i)}(z)w^{(j)}(z).$$

Using Eq. (2.8), we get that this expression reduces to

$$\sum_{i=0}^{q-1}\sum_{j=0}^{r-1}\binom{k}{i}u^{(i)}(z)w^{(j)}(z)\delta_{j,k-i} = \sum_{i=0\vee(k-r+1)}^{(q-1)\wedge k}\binom{k}{i}u^{(i)}(z)w^{(k-i)}(z)$$

$$= \sum_{i=0}^{k}\binom{k}{i}u^{(i)}(z)w^{(k-i)}(z),$$

since $u^{(i)}(z) = 0$ for $i \geq q$ and $w^{(k-i)}(z) = 0$ for $i \leq k - r$. Recall that the blocks $\mathcal{M}(z)$ consist of the derivatives $\frac{d^k}{dz^k}(u(z)w(z))$. The product rule for differentiation then yields (3.1). The statement concerning the rank immediately follows from Theorem 2.3, since $U_q(z)$ and $W_r(z)$ are invertible. $\square$

Remark 3.2. The factorization in Proposition 3.1 exhibits a nice form of symmetry. Therefore it would be nice if also the matrix $\mathcal{A}$ could be factorized in some symmetric way. There doesn't seem to be an easy way to do this. Consider a truly symmetric case, for instance $\mu = \nu = q = r = 2$. In this case $\mathcal{A}$ is non-singular and we have

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}.$$

A first reasonable factorization in this symmetric case would be of the form $\mathcal{A} = AA^{\top}$, which would imply that $\mathcal{A}$ is positive definite. One easily sees that this is not the case. As a next attempt, one could try to use the singular value decomposition of $\mathcal{A}$, but the eigenvalues of $\mathcal{A}$ are not particularly nice, so this look as a dead end too.

Remark 3.3. Instead of the matrices $U_q(z)$ and $W_r(z)$, one can also use the matrices $\widetilde{U}_q(z)$ (see (1.3)) and $\widetilde{W}_r(z)$ (see (1.4)) to get a factorization of $\mathcal{M}(z)$. One then has to replace the matrix $\mathcal{A}$ with $\widetilde{\mathcal{A}}$, whose blocks $\widetilde{\mathcal{A}}_{ij}$ are equal to $\widetilde{A}^{i+j}$, where the matrices $\widetilde{A}^k$ are specified by their elements

$$\widetilde{A}_{ij}^k = \begin{cases} k! & \text{if } i + j = k \\ 0 & \text{else.} \end{cases}$$

One then gets

$$\mathcal{M}(z) = \bigl(I_\mu \otimes \widetilde{U}_q(z)\bigr)\widetilde{\mathcal{A}}\bigl(I_\nu \otimes \widetilde{W}_r(z)\bigr), \tag{3.2}$$

which can be proved in the same way as (3.1), or by using this identity and the relation

$$\widetilde{A}^k = D_q A^k D_r. \tag{3.3}$$

Next we derive a factorization of the matrix $\mathcal{N}(z)$. Let the matrix $\hat{\mathcal{A}}$ be defined by its $\mu \times \nu$ blocks $\hat{A}^{ij}$, for $i = 0, \ldots, \mu - 1$, $j = 0, \ldots, \nu - 1$, where the matrices $\hat{A}^{ij} \in \mathbb{R}^{q \times r}$ have elements

$$\hat{A}^{ij}_{kl} = \begin{cases} \binom{i+j}{i} & \text{if } k + l = i + j \\ 0 & \text{else,} \end{cases}$$

for $k = 0, \ldots, q - 1$, $l = 0, \ldots, r - 1$.

Proposition 3.4. *The factorization*

$$\mathcal{N}(z) = \big(I_\mu \otimes \widetilde{U}_q(z)\big)\hat{\mathcal{A}}\big(I_\nu \otimes \widetilde{W}_r(z)\big) \tag{3.4}$$

holds true. The matrices $\mathcal{A}$ and $\hat{\mathcal{A}}$ are related through

$$(D_\mu \otimes I_q)\hat{\mathcal{A}}(D_\nu \otimes I_r) = (I_\mu \otimes D_q)\mathcal{A}(I_\nu \otimes D_r). \tag{3.5}$$

Further relations between $\mathcal{A}$, $\widetilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are

$$\widetilde{\mathcal{A}} = (D_\mu \otimes I_q)\hat{\mathcal{A}}(D_\nu \otimes I_r) \tag{3.6}$$

$$= (I_\mu \otimes D_q)\mathcal{A}(I_\nu \otimes D_r) \tag{3.7}$$

Proof. The proof of the factorization (3.4) is similar to the proof of (3.1). Relation (3.5) follows by an elementary computation. Likewise one proves (3.6) and (3.7). $\square$

Proposition 3.5. *Additionally we have the following factorizations of $\mathcal{M}(z)$ and $\mathcal{N}(z)$.*

$$\mathcal{A}(z) = \big(\widetilde{U}_\mu(z) \otimes I_q\big)\mathcal{A}(0)\big(\widetilde{W}_\nu(z) \otimes I_r\big) \tag{3.8}$$

$$\mathcal{M}(z) = \big(I_\mu \otimes \widetilde{U}_q(z)\big)\mathcal{M}(0)\big(I_\nu \otimes \widetilde{W}_r(z)\big) \tag{3.9}$$

$$\mathcal{M}(z) = \big(\widetilde{U}_\mu(-z) \otimes U_q(z)\big)\mathcal{A}(z)\big(\widetilde{W}_\nu(-z) \otimes W_r(z)\big) \tag{3.10}$$

$$\mathcal{N}(z) = \big(U_\mu(z)^{-1} \otimes U_q(z)\big)\mathcal{A}(z)\big(W_\nu(z)^{-1} \otimes W_r(z)\big) \tag{3.11}$$

$$\mathcal{N}(z) = \big(I_\mu \otimes \widetilde{U}_q(z)\big)\mathcal{N}(0)\big(I_\nu \otimes \widetilde{W}_r(z)\big) \tag{3.12}$$

Proof. First we show an auxiliary result. Let $\mathcal{P}_{\mu,q}(z) = \mathcal{L}_{\mu,q}(z)\mathcal{L}_{\mu,q}(0)^{-1}$. Then

$$\mathcal{P}_{\mu,q}(z) = \widetilde{U}_\mu(z) \otimes I_q. \tag{3.13}$$

Indeed, by direct computation starting from the definition of $\mathcal{L}_{\mu,q}(z)$, one finds that the ij-block $\mathcal{P}_{\mu,q}(z)^{ij}$ equals $\binom{i}{j}z^{i-j}I_q$ for $0 \leq j \leq i \leq \mu - 1$. Since $\widetilde{U}_\mu(z)_{ij} = \binom{i}{j}z^{i-j}$, the result follows.

We now proceed to prove the indentities. Use (2.9), both for arbitrary z and $z = 0$, and Eq. (3.13) to get (3.8). To obtain (3.9), in the same vain we use (3.1) twice. Then

(3.10) follows upon combining (3.8) and (3.1). Finally, (3.11) follows from (3.10) and (1.2) and (3.12) follows from (3.9) and (1.2). □

In the situation where $\mathcal{M}(z)$ and $\mathcal{N}(z)$ are square, we are able to compute their determinants.

Corollary 3.6. *Let* $\mu q = \nu r$. *Then all relevant matrices are square and we have the following expressions for the determinants of* $\mathcal{M}(z)$ *and* $\mathcal{N}(z)$.

$$\det\big(\mathcal{M}(z)\big) = \left(\prod_{i=0}^{q-1} i!\right)^{\mu} \left(\prod_{j=0}^{r-1} j!\right)^{\nu} \det(\mathcal{A}),$$

$$\det\big(\mathcal{N}(z)\big) = \det(\hat{\mathcal{A}}),$$

$$= \left(\prod_{i=0}^{\mu-1} i!\right)^{q} \left(\prod_{j=0}^{\nu-1} j!\right)^{r} \det(\mathcal{A}),$$

with $\det(\mathcal{A})$ *given in Corollary 2.5. Hence the matrices* $\mathcal{M}(z)$ *and* $\mathcal{N}(z)$ *are unimodular.*

Proof. The matrix $I_\mu \otimes \widetilde{U}_q(z)$ has determinant one, which gives the first equation in view of (3.4). Use (3.2), (3.3) and its consequence

$$\det(\widetilde{\mathcal{A}}) = \big(\det(D_q)\big)^{\mu} \big(\det(D_r)\big)^{\nu} \det(\mathcal{A})$$

to get the formula for $\det(\mathcal{M}(z))$. Then the last equation follows from (3.5). □

In the next sections we investigate and characterize the kernel of the matrix $\mathcal{N}(z)$ for different values of the parameters. It turns out that characterizing the kernel of $\mathcal{N}(z)$ yields more elegant results than characterizing the kernel of $\mathcal{M}(z)$. On the other hand, a factorization of $\mathcal{M}(z)$ leads to more elegant expressions than factorizations of $\mathcal{N}(z)$. Of course, in view of (1.2), results for one of the two can easily be transformed into results for the other. We will focus on the kernel of $\mathcal{N}(z)$ only, simple because the obtained results have a more elegant appearance. As it turns out, it is instrumental to consider the kernel of $\mathcal{N}^0(z)$ (which is $\mathcal{N}(z)$ for $\mu = 1$) first, since the results obtained for this case, serve as building blocks for the kernel of $\mathcal{N}(z)$.

4. Characterization of the kernel of $\mathcal{N}^0(z)$

We will investigate the kernel of $\mathcal{N}^0(z)$ for different values of ν, q and r. This is in different notation the problem alluded to in the introduction and earlier investigated in [8] for a special case. Notice that $\mathcal{N}^0(z)$ is $q \times r\nu$-dimensional. Furthermore for $\nu \leq q$ all derivatives $u^{(k)}(z)$ ($k \leq \nu - 1$) are nonzero and moreover linearly independent vectors, whereas $u^{(k)}(z) \equiv 0$ for $k \geq q$. It follows that the set $\text{span}\{u(z), \ldots, u^{\nu-1}(z)\}$ has

dimension equal to $\min\{\nu, q\}$. It is then an easy exercise to directly see that the rank of $\mathcal{N}^0(z)$ is also equal to $\min\{\nu, q\}$. Of course, this also follows from Proposition 3.4. In order to characterize the kernel of $\mathcal{N}^0(z)$, we have to discern three different cases. These are $\nu \leq q$, $q + 1 \leq \nu < q + r$ and $\nu \geq q + r$, each case being treated in a separate section. In all cases, our aim is to find *simple, transparent* parametrizations of the kernels.

4.1. The case $\nu \leq q$

Let $F(z)$ be the $r \times (r-1)$ matrix given by

$$
F(z) = \begin{pmatrix}
-z & 0 & \cdots & \cdots & 0 \\
1 & -z & 0 & \cdots & 0 \\
0 & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & -z \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}.
$$

Observe the trivial but crucial property, that the columns of $F(z)$ span the $(r-1)$-dimensional null space of $w(z)$.

Next we consider the matrix $K(z)$ that has the following block structure. It consists of matrices $K_{ij}(z)$ $(i = 0, \ldots, \nu - 1,\ j = 0, \ldots, \nu - 1)$ where each $K_{ij}(z)$ has size $r \times (r-1)$ and is given by

$$
K_{ii}(z) = F(z),
$$
$$
K_{i,i+1}(z) = -F'(z),
$$

whereas all the other blocks are equal to zero. For $r = 1$ the matrix $K(z)$ is empty and by convention we say that the columns of $K(z)$ in this case span the vector space $\{0\}$. Note that $K(z)$ has dimensions $\nu r \times \nu(r-1)$.

The matrix $K(z)$ looks as follows, where we suppress the dependence on z and omit zero blocks.

$$
K = I_\nu \otimes F + S_\nu \otimes F' = \begin{pmatrix}
F & F' & & & \\
 & F & F' & & \\
 & & \ddots & \ddots & \\
 & & & \ddots & F' \\
 & & & & F
\end{pmatrix}, \tag{4.1}
$$

where I_ν is the ν-dimensional unit matrix and S_ν the ν-dimensional shift matrix, with elements $S_{ij} = \delta_{i+1,j}$.

Proposition 4.1. *Let $\nu \leq q$. The $\nu r \times \nu(r-1)$ matrix $K(z)$ has rank $\nu(r-1)$ and is such that $\mathcal{N}^0(z)K(z) \equiv 0$. In other words, the columns of $K(z)$ form a basis for the kernel of $\mathcal{N}^0(z)$.*

Proof. Pick the j-th block column of $K(z)$, $K^j(z)$ say ($j \in \{0, \ldots, \nu-1\}$). We compute $\mathcal{N}^0(z)K^j(z)$. For $j = 0$, this reduces to $M(z)F(z)$, which is zero, since $w(z)F(z) = 0$. Then, for $j \geq 1$ we have $K_{j-1,j}(z) = F'(z)$. Hence, for $j \geq 1$ we get (using that $F^{(k)}(z) = 0$ for $k > 1$),

$$
\begin{aligned}
\mathcal{N}^0(z)K^j(z) &= \sum_{i=0}^{\nu-1} \mathcal{N}_i^0(z)K_{ij}(z) \\
&= \frac{1}{j!}M^{(j)}(z)F(z) + \frac{1}{(j-1)!}M^{(j-1)}(z)F'(z) \\
&= \frac{1}{j!}\big(M^{(j)}(z)F(z) + jM^{(j-1)}(z)F'(z)\big) \\
&= \frac{1}{j!}\big(M(z)F(z)\big)^{(j)} \\
&= 0.
\end{aligned}
$$

Hence $\mathcal{N}^0(z)K(z) = 0$, so all columns of $K(z)$ belong to $\ker \mathcal{N}^0(z)$. Since we know that the rank of $\mathcal{N}^0(z)$ is equal to ν, we get that $\dim \ker \mathcal{N}^0(z) = r\nu - \nu = (r-1)\nu$, which equals the number of columns of $K(z)$. Since $K(z)$ is upper triangular with the full rank matrices $F(z)$ on the block-diagonal, it has full rank. Therefore, the columns of $K(z)$ exactly span the null space of $\mathcal{N}^0(z)$. $\quad\square$

Remark 4.2. A nice feature of the matrix $K(z)$ in Proposition 4.1 is that it is a matrix polynomial of degree 1, only whereas the matrix polynomial $\mathcal{N}^0(z)$ has degree $q + r - 2$.

4.2. The case $q + 1 \leq \nu < q + r$

Next we consider what happens if $q + 1 \leq \nu \leq q + r - 1$. Note that this case is void if $r = 1$. Henceforth we assume that $r \geq 2$. Let the matrix $G(z) \in \mathbb{R}^{r \times r}$ have elements $G_{ij}(z) = z^{j-i-1}$ for $j > i$ and zero else. With $S = S_r$ the r-dimensional shift matrix, we have the compact expression $G(z) = S \sum_{k=0}^{\infty}(zS)^k$, since $S^k = 0$ for $k \geq r$. We now give some auxiliary results.

Lemma 4.3. *Let $G(z)$ be as above and let S be the shifted $r \times r$ identity matrix, $S_{ij} = \delta_{i+1,j}$. It holds that*

$$G(z) = S(I - zS)^{-1} \tag{4.2}$$

$$\frac{1}{j!}G^{(j)}(z) = S^{j+1}(I - zS)^{-j-1} \tag{4.3}$$

$$\frac{1}{j!}G^{(j)}(z) = G(z)^{j+1} \tag{4.4}$$

$$\frac{1}{(j+1)!}G^{(j+1)}(z) = \frac{1}{j!}G^{(j)}(z)G(z), \tag{4.5}$$

and the matrices $G^{(j)}(z)$ and $G^{(i)}(z)$ commute for all $i, j \geq 0$. One also has the following two equivalent properties

$$\frac{1}{j!}w^{(j)}(z) = w(z)G(z)^j, \tag{4.6}$$

$$\frac{1}{(j+l)!}w^{(j+l)}(z) = \frac{1}{j!}w^{(j)}(z)G(z)^l. \tag{4.7}$$

Moreover,

$$\bigl(w(z)G(z)\bigr)^{(k)} = (k+1)!\, w(z)G(z)^{k+1}. \tag{4.8}$$

Proof. To prove (4.2), we recall that $S^j = 0$ for $j \geq r$. Hence

$$G(z) = S\sum_{k=0}^{\infty}(zS)^k = S(I - zS)^{-1}.$$

Then (4.3) simply follows by differentiation, which immediately yields (4.4), from which (4.5) also follows.

We next prove (4.6) for $j = 1$. This is then equivalent to $w'(z)(I - zS) = w(z)S$, which can be verified by elementary calculations. We proceed by induction and assume that (4.6) holds. Then we have

$$\begin{aligned}
\frac{1}{(j+1)!}w^{(j+1)}(z) &= \frac{1}{(j+1)!}\frac{\mathrm{d}}{\mathrm{d}z}w^{(j)}(z) \\
&= \frac{j!}{(j+1)!}\frac{\mathrm{d}}{\mathrm{d}z}\bigl(w(z)G(z)^j\bigr) \\
&= \frac{j!}{(j+1)!}\bigl(w'(z)G(z)^j + jw(z)G(z)^{j-1}G'(z)\bigr) \\
&= \frac{j!}{(j+1)!}\bigl(w(z)G(z)^{j+1} + jw(z)G(z)^{j+1}\bigr) \\
&= w(z)G(z)^{j+1}.
\end{aligned}$$

This proves (4.6), which is easily seen to be equivalent to (4.7). Eq. (4.8) follows by induction, using (4.6). $\quad\square$

Remark 4.4. In fact, above statements about $G(z)$ also follow from the parallel ones concerning $w(z)$ and the relation

$$G(z) = \begin{pmatrix} w(z)S \\ \vdots \\ w(z)S^r \end{pmatrix}.$$

Indeed, since the matrices $G(z)$ and S commute, one has for instance

$$\frac{1}{j!}w^{(j)}(z)S = w(z)SG(z)^j,$$

which together with similar relations leads to (4.4).

Lemma 4.5. *One has for $j \geq 1$*

$$\frac{1}{j!}\left(M^{(j)}(z) - u^{(j)}(z)w(z)\right) = \frac{1}{(j-1)!}M^{(j-1)}(z)G(z). \tag{4.9}$$

For $j \geq q$, this reduces to

$$\frac{1}{j!}M^{(j)}(z) = \frac{1}{(j-1)!}M^{(j-1)}(z)G(z). \tag{4.10}$$

More generally, one has for $m, p \geq 0$

$$\frac{M^{(q-1+p+m)}(z)}{(q-1+p+m)!} = \frac{M^{(q-1+m)}(z)}{(q-1+m)!}G(z)^p = \frac{M^{(q-1)}(z)}{(q-1)!}G(z)^{p+m}. \tag{4.11}$$

Proof. Recall that $M(z) = u(z)w(z)$. Below we use (4.7) to compute

$$\frac{1}{j!}M^{(j)}(z) = \frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}u^{(j-i)}(z)w^{(i)}(z)$$

$$= \sum_{i=0}^{j}\frac{u^{(j-i)}(z)}{(j-i)!}\frac{w^{(i)}(z)}{i!}$$

$$= \sum_{i=1}^{j}\frac{u^{(j-i)}(z)}{(j-i)!}\frac{w^{(i-1)}(z)}{(i-1)!}G(z) + \frac{u^{(j)}(z)}{j!}w(z)$$

$$= \frac{1}{(j-1)!}\sum_{i=1}^{j}\binom{j-1}{i-1}u^{(j-i)}(z)w^{(i-1)}(z)G(z) + \frac{u^{(j)}(z)}{j!}w(z)$$

$$= \frac{1}{(j-1)!}\sum_{l=0}^{j-1}\binom{j-1}{l}u^{(j-1-l)}(z)w^{(l)}(z)G(z) + \frac{u^{(j)}(z)}{j!}w(z)$$

$$= \frac{1}{(j-1)!}M^{(j-1)}(z)G(z) + \frac{u^{(j)}(z)}{j!}w(z).$$

Eq. (4.10) follows from (4.9), because $u^{(j)} = 0$ for $j \geq q$. Finally, (4.11) follows by iteration of (4.10). $\square$

We consider the matrix polynomial $\bar{K}(z)$ which for the present case has the following structure. For $i, j = 0, \ldots, q - 1$ it has blocks $\bar{K}_{ij}(z)$ of size $r \times (r - 1)$ such that $\bar{K}_{jj}(z) = F(z)$ and $\bar{K}_{j-1,j}(z) = F'(z)$, $F(z)$ as before. For $i, j = q, \ldots, \nu - 1$ the blocks are $\bar{K}_{jj}(z) = I_r$ and $\bar{K}_{j-1,j}(z) = -G(z)$, all of size $r \times r$. Finally, we have that $\bar{K}_{q-1,q}(z) = -G(z)$. All other blocks are equal to zero. Notice that $\bar{K}(z)$ is of size $\nu r \times (\nu r - q)$. One easily verifies that $\bar{K}(z)$ has full column rank.

The matrix $\bar{K}(z)$ looks as follows, where again we suppress the dependence on z and omit zero blocks.

$$
\bar{K} = \left(\begin{array}{ccccc:ccccc}
F & F' & & & & & & & & \\
 & F & F' & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & & \ddots & F' & & & & & \\
 & & & & F & -G & & & & \\
\hdashline
 & & & & & I_r & -G & & & \\
 & & & & & & I_r & -G & & \\
 & & & & & & & \ddots & \ddots & \\
 & & & & & & & & \ddots & -G \\
 & & & & & & & & & I_r
\end{array}\right).
\tag{4.12}
$$

A compact expression for $\bar{K}$ is

$$
\bar{K} = \begin{pmatrix} I_q \otimes F + S_q \otimes F' & -\ell_q \mathrm{f}_{\nu-q}^\top \otimes G \\ 0 & I_{\nu-q} \otimes I_r - S_{\nu-q} \otimes G \end{pmatrix},
\tag{4.13}
$$

where $\mathrm{f}_{\nu-q}$ is the first standard basis vector of $\mathbb{R}^{\nu-q}$ and ℓ_q the last standard basis vector of $\mathbb{R}^q$.

Proposition 4.6. *Let $q + 1 \leq \nu \leq q + r - 1$. The $\nu r \times (\nu r - q)$ matrix $\bar{K}(z)$ is such that $\mathcal{N}^0(z)\bar{K}(z) \equiv 0$. In other words, the columns of $\bar{K}(z)$ form a basis for the kernel of $\mathcal{N}^0(z)$.*

Proof. Next we proceed as in the proof of Proposition 4.1. We pick the j-block column $\bar{K}^j(z)$. If $0 \leq j \leq q - 1$, then $\bar{K}_{jj}(z) = F(z)$ and $\bar{K}_{j-1,j}(z) = F'(z)$, whereas all other $\bar{K}_{ij}(z)$ are zero. The computation of $\mathcal{N}^0(z)\bar{K}^j(z)$ is then exactly as in the previous proof. Next we consider the case where $q \leq j \leq \nu - 1$, which is quite different. Again we pick the j-th block column of $\bar{K}(z)$. Recall the definition of the $\bar{K}_{ij}(z)$ for this case. We get,

$$\mathcal{N}^0(z)\bar{K}^j(z) = \sum_{i=0}^{\nu-1}\mathcal{N}_i^0(z)\bar{K}_{ij}(z)$$

$$= \frac{1}{j!}M^{(j)}(z) - \frac{1}{(j-1)!}M^{(j-1)}(z)G(z)$$

$$= 0,$$

in view of Eq. (4.10). This shows that $\bar{K}(z)$ belongs to the kernel of $\mathcal{N}^0(z)$. Since $\mathcal{N}^0(z)$ has rank q, the dimension of the kernel is equal to $\nu r - q$, which is equal to the rank of $\bar{K}(z)$. Hence the columns of $\bar{K}(z)$ span this kernel. □

Post-multiplying the matrix $\bar{K}(z)$ by

$$\begin{pmatrix} I_{r-1} & 0 & & & & & & & & \\ & I_{r-1} & 0 & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & \ddots & 0 & & & & & \\ & & & & I_{r-1} & & & & & \\ \hline & & & & & I_r & G & \cdots & \cdots & G^{\nu-q-1} \\ & & & & & & I_r & G & \cdots & G^{\nu-q-2} \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & \ddots & G \\ & & & & & & & & & I_r \end{pmatrix},$$

we obtain

$$\begin{pmatrix} F & F' & & & & & & & & \\ & F & F' & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & \ddots & F' & & & & & \\ & & & & F & -G & -G^2 & \cdots & \cdots & -G^{\nu-q} \\ \hline & & & & & I_r & & & & \\ & & & & & & I_r & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & I_r \end{pmatrix},$$

an alternative matrix whose columns span $\ker\mathcal{N}^0(z)$.

4.3. The case $\nu \geq q + r$

For this case, the kernel of $\mathcal{N}^0(z)$ is closely related to what we have obtained in the previous case.

Proposition 4.7. *Let $\nu \geq q + r$. Consider the matrix $\bar{K}_*(z)$ of (4.12) in the special case that $\nu = q + r - 1$, then $\bar{K}_*(z)$ has $(q+r)(r-1)$ columns. We have that $\ker \mathcal{N}^0(z)$, which is now $\nu r - q$-dimensional, is the product of the space spanned by the columns of $\bar{K}_*(z)$ and $\mathbb{R}^{r(\nu+1-q-r)}$.*

Proof. Since the highest power of z that appears in $\mathcal{N}^0(z)$ is $q + r - 2$, we have that the matrices $M^{(j)}(z)$ are identically zero if $\nu \geq q + r - 1$, whereas the matrix $(\frac{1}{0!}M(z), \ldots, \frac{1}{(q+r-2)!}M^{(q+r-2)}(z))$ is the same as the matrix $\mathcal{N}^0(z)$ for the case $\nu = q + r - 1$. The assertion follows by application of Proposition 4.6. $\square$

5. Characterization of the kernel of $\mathcal{N}(z)$

We have seen that we had to distinguish three different cases to describe the kernel of $\mathcal{N}^0(z)$. The same distinction has to be made in the present section.

5.1. The case $\nu \leq q$

First we introduce some more notation. For the matrix $K(z)$ of (4.1) we now write $K_0(z)$ and for $k = 1, \ldots, r - 2$, we define $K_k(z)$ in the same way as $K(z)$, but now with F replaced with $F_k(z)$, an $(r - k) \times (r - 1 - k)$ matrix having the same structure as the original $F(z)$ of Section 4.1, so $F_k(z)_{ij} = -z$, if $i = j$ and $F_k(z)_{ij} = 1$, if $i = j + 1$. In particular $F_0(z) = F(z)$. Formally, for $1 \leq j \leq r - 2$, we have

$$F_k(z) = \begin{pmatrix} I_{r-k} & 0_{(r-k)\times k} \end{pmatrix} F(z) \begin{pmatrix} I_{r-k-1} \\ 0_{k\times(k-j-1)} \end{pmatrix},$$

whereas $F_k(z)$ is the empty matrix for $k \geq r - 1$. By $\mathcal{K}^{\mu-1}(z)$ for $\mu < r$, we denote the product $K_0(z)K_1(z)\cdots K_{\mu-1}(z)$ of size $\nu r \times \nu(r - \mu)$, whereas we take $\mathcal{K}^{\mu-1}(z)$ the zero matrix for $\mu \geq r$.

For $j = 0, \ldots, r - 1$ we put

$$w_j(z) = w(z) \begin{pmatrix} I_{r-j} \\ 0_{j\times(r-j)} \end{pmatrix} = \begin{pmatrix} 1, z, \ldots, z^{r-1-j} \end{pmatrix}$$

and $M_j(z) = u(z)w_j(z)$. With this convention, we have $w_0 = w$, $M_0 = M$.

Lemma 5.1. *Let $w_1(z) = (1, z, \ldots, z^{r-2})$ and $M_1(z) = u(z)w_1(z)$. For $j \geq 1$ one has*

$$w^{(j)}(z)F(z) = -jw^{(j-1)}(z)F'(z) = jw_1^{(j-1)}(z) \tag{5.1}$$

$$M^{(j)}(z)F(z) = -jM^{(j-1)}(z)F'(z) = jM_1^{(j-1)}(z). \tag{5.2}$$

Proof. Since $w(z)F(z) = 0$, also $\frac{\mathrm{d}^j}{\mathrm{d}z^j}(w(z)F(z)) = 0$ for all $j \geq 1$. It then follows that $w^{(j)}(z)F(z) + jw^{(j-1)}(z)F'(z) = 0$, since all higher order derivatives of F vanish. Using $-w(z)F'(z) = w_1(z)$, we arrive at (5.1). Similarly, one obtains (5.2). □

Theorem 5.2. *The kernel of $\mathcal{N}(z)$ is spanned by the columns of $\mathcal{K}^{\mu-1}(z)$ and its dimension is equal to $\nu(r - \mu)^+$. Hence the rank of $\mathcal{N}(z)$ is equal to $\nu \min\{\mu, r\}$. In particular the matrix $\mathcal{N}(z)$ has full rank iff $\mu \geq r$.*

Proof. The arguments used in the proof of Proposition 4.1 can also be applied to this more general case. Let for $k \geq 0$

$$\mathcal{N}^k(z) = \frac{1}{k!}\left(\frac{1}{0!}M^{(k)}, \ldots, \frac{1}{(\nu - 1)!}M^{(k+\nu-1)}(z)\right).$$

Proposition 4.1 yields $\mathcal{N}^0(z)K_0(z) = 0$, and therefore $\mathcal{N}^0(z)\mathcal{K}^{\mu-1}(z) = 0$. Consider now $k \geq 1$. As before, $K^j(z)$ denotes the j-th block-column of $K_0(z) = K(z)$. Then, from (5.2) it follows that

$$\mathcal{N}^k(z)K^0(z) = \frac{1}{k!}M^{(k)}(z)F(z) = \frac{1}{(k-1)!}M_1^{(k-1)}(z).$$

For $j \geq 1$ we get

$$\mathcal{N}^k(z)K^j(z) = \frac{1}{k!}\left(\frac{1}{(j-1)!}M^{(k+j-1)}(z)F'(z) + \frac{1}{j!}M^{(k+j)}(z)F(z)\right)$$

$$= \frac{1}{(k-1)!j!}M_1^{(k+j-1)}(z),$$

where we used (5.2) again. It follows that for $k \geq 1$

$$\mathcal{N}^k(z)K_0(z) = \mathcal{N}_1^{k-1}(z), \tag{5.3}$$

where

$$\mathcal{N}_1^{k-1}(z) = \frac{1}{(k-1)!}\left(\frac{1}{0!}M_1^{(k-1)}, \ldots, \frac{1}{(\nu-1)!}M_1^{(k+\nu-2)}(z)\right),$$

which is a matrix of size $q \times \nu(r-1)$. Invoking Proposition 4.1 again, we obtain for $k = 1$

$$\mathcal{N}^1(z)K_0(z)K_1(z) = \mathcal{N}_1^0(z)K_1(z) = 0,$$

and hence $\mathcal{N}^1(z)\mathcal{K}^{\mu-1}(z) = 0$. Assume that $\mu < r$. To have $\mathcal{N}(z)\mathcal{K}^{\mu-1}(z) = 0$, we need $\mathcal{N}^k(z)\mathcal{K}^{\mu-1}(z) = 0$ for $k = 0, \ldots, \mu - 1$. This now follows by iteration of (5.3). In fact, by induction, one can show

$$\mathcal{N}^k(z)K_0(z)\cdots K_{j-1}(z) = \mathcal{N}_j^{k-j}(z) \quad \text{for } j \leq k, \tag{5.4}$$

where

$$\mathcal{N}_j^{k-j}(z) = \frac{1}{(k-j)!}\left(\frac{1}{0!}M_j^{(k-j)}, \ldots, \frac{1}{(\nu-1)!}M_j^{(k+\nu-j-1)}(z)\right).$$

For $j \geq k + 1$ one has $\mathcal{N}^k(z)K_0(z)\cdots K_{j-1}(z) = 0$.

Since each of the matrices $K_k(z)$ for $k < r - 1$ has full rank, which is equal to $\nu(r - k - 1)$, we get that $\mathcal{K}^{\mu-1}(z)$ has rank equal to $\nu(r - \mu)$. All assertions for $\mu < r$ now follow. On the other hand, for $\mu \geq r$, the matrix $\mathcal{N}(z)$ has rank equal to νr, and therefore has a zero kernel. $\quad\square$

Remark 5.3. The matrix $K^{\mu-1}(z)$ for $\mu \leq r-1$, which is of size $\nu r \times \nu(r-\mu)$ turns out to be upper block-triangular. Consider for this case the product $\mathcal{F}_{\mu-1}(z) = F_0(z)\cdots F_{\mu-1}(z)$. Then one has for $j \geq i$ the ij-block

$$\mathcal{K}^{\mu-1}(z)_{ij} = \frac{1}{(j-i)!}\mathcal{F}_{\mu-1}^{(j-i)}(z),$$

which can easily be shown by induction.

5.2. The case $q + 1 \leq \nu < q + r$

Next we extend the result of Proposition 4.6 to obtain the kernel of the matrix $\mathcal{N}(z)$ for the present case. The approach that we follow is the same as the one leading to Theorem 5.2.

To obtain our results, we need to introduce additional notation. Let $G_{kj}(z)$ denote the upper left block of $G(z)$ having size $(r - k) \times (r - j)$ for $0 \leq k, j \leq r - 1$. So

$$G_{kj}(z) = \begin{pmatrix} I_{r-k} & 0_{(r-k)\times k} \end{pmatrix} G(z) \begin{pmatrix} I_{r-j} \\ 0_{j\times(r-j)} \end{pmatrix}.$$

We also need the matrices $\mathcal{G}_0(z) = I_r$, and $\mathcal{G}_i(z) = G_{i,i-1}(z)\cdots G_{10}(z) \in \mathbb{R}^{(r-i)\times r}$, for $0 < i \leq r - 1$.

Lemma 5.4. *It holds that* $G_{jj}(z)G_{j,j-1}(z) = G_{j,j-1}(z)G_{j-1,j-1}(z)$ *for* $j \geq 1$ *and for* $j > i > k \geq 0$ *one has* $G_{ji}(z)G_{ik}(z) = G_{j,i-1}(z)G_{i-1,k}(z) = G_{jk}(z)G_{kk}(z)$.

Proof. The first assertion follows from the decomposition

$$G_{j-1,j-1} = \begin{pmatrix} G_{j,j-1} \\ 0 \end{pmatrix} = \begin{pmatrix} G_{jj} & g \\ 0 & 0 \end{pmatrix},$$

where g is the last column of $G_{j,j-1}$. For the proof of the second assertion we need the following property of the shift matrix $S \in \mathbb{R}^k$ (k according to the context): $\widetilde{I}S = S$, where

$$\widetilde{I} = \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since any $G_{ii}(z)$ is of the form $S(I-zS)^{-1}$ (with S of size $(r-i) \times (r-i)$, see Lemma 4.3), we have $I_0 G_{ii}(z) = G_{ii}(z)$. Then we compute

$$G_{ji}G_{ik} = (\,I \quad 0\,)\,G_{i-1,i-1}\begin{pmatrix} I \\ 0 \end{pmatrix}(\,I \quad 0\,)\,G_{i-1,i-1}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$= (\,I \quad 0\,)\,G_{i-1,i-1}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}G_{i-1,i-1}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$= (\,I \quad 0\,)\,G_{i-1,i-1}\widetilde{I}G_{i-1,i-1}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$= G_{j,i-1}G_{i-1,k}. \quad \square$$

Lemma 5.5. *For* $i \geq 1$, *it holds that*

$$\frac{1}{i!}G_{i,0}^{(i)}(z) = G_{i0}(z)G(z)^i$$

$$\mathcal{G}_i(z) = G_{i0}(z)G(z)^{i-1} \tag{5.5}$$

$$G_{ii}(z)\mathcal{G}_i(z) = \mathcal{G}_i(z)G(z). \tag{5.6}$$

Proof. Using the definition of G_{i0}, the equality $\frac{1}{i!}G_{i,0}^{(i)} = G_{i0}G^i$ immediately follows from (4.4). The second equality (5.5) is obviously true for $i = 0$. We use induction. Let $i \geq 1$ and assume that $\mathcal{G}_i = G_{i0}G^{i-1}$. Then, using Lemma 5.4, $\mathcal{G}_{i+1}G = G_{i+1,i}\mathcal{G}_iG = G_{i+1,i}G_{i0}G^i = G_{i+1,0}G^{i+1}$. To prove (5.6), we use (5.5) and Lemma 5.4 to write $G_{ii}\mathcal{G}_i = G_{ii}G_{i0}G^i = G_{i0}GG^i = \mathcal{G}_iG$. $\quad \square$

Lemma 5.6. *Let* $w_j(z)$ *and* $M_j(z)$ *be as in Section 5.1. It holds that*

$$w_j^{(k)}(z)\mathcal{G}_j(z) = w^{(k)}G^j(z) \tag{5.7}$$

$$\frac{1}{k!}w_j^{(k)}(z)\mathcal{G}_j(z) = \frac{1}{(k+j)!}w^{(k+j)}(z) \tag{5.8}$$

$$M_j^{(k)}(z)\mathcal{G}_j(z) = M^{(k)}(z)G(z)^j. \tag{5.9}$$

Moreover, for $k \geq 0$ it holds that

$$\frac{M_j^{(q-1+k)}(z)}{(q-1+k)!}\mathcal{G}_j(z) = \frac{1}{(q-1)!}M^{(q-1)}(z)G^{k+j+1}. \tag{5.10}$$

Proof. We need the following observation. For a row vector x of appropriate length and a scalar y, one has

$$(x,y)G_{jj} = xG_{j,j-1}, \tag{5.11}$$

because

$$G_{jj} = \begin{pmatrix} G_{j,j-1} \\ 0 \end{pmatrix}.$$

We now show (5.7). It is obviously true for $j = 0$. Assume it holds for some $j \geq 0$. We get, using (5.6) and (5.11)

$$w_{j+1}^{(k)}\mathcal{G}_{j+1} = w_{j+1}^{(k)}G_{j+1,j}\mathcal{G}_j$$
$$= w_j^{(k)}G_{jj}\mathcal{G}_j$$
$$= w_j^{(k)}\mathcal{G}_j G$$
$$= w^{(k)}G^{j+1}.$$

Eq. (5.8) follows by combining (5.7) and (4.7), whereas (5.9) is an immediate consequence of (5.7). Next we compute, using (5.9) and (4.11),

$$\frac{M_j^{(q-1+k)}(z)}{(q-1+k)!}\mathcal{G}_j(z) = \frac{M^{(q-1+k)}(z)}{(q-1+k)!}G(z)^j$$
$$= \frac{M^{(q-1)}(z)}{(q-1)!}G(z)^{k+j+1},$$

which yields (5.10). $\square$

Let for $j \leq r - 2$ the matrix $\bar{K}_j(z)$ be given by

$$
\bar{K}_j = \left(\begin{array}{cccc:cccc}
F_j & F_j' & & & & & & \\
 & F_j & F_j' & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & \ddots & F_j' & & & \\
 & & & & F_j & -\frac{1}{j!}G_{j0}^{(j)} & & \\
\hdashline
 & & & & I & -G & & \\
 & & & & & I & -G & \\
 & & & & & & \ddots & \ddots \\
 & & & & & & \ddots & -G \\
 & & & & & & & I
\end{array}\right). \tag{5.12}
$$

Here $\bar{K}_j$ has q diagonal entries F_j and $\nu - q$ diagonal entries $I = I_r$. Hence $\bar{K}^j$ has dimensions $(\nu r - qj) \times (\nu r - q(j+1))$. A compact expression of $\bar{K}^j$ is as follows. Let ℓ_q be the last standard basis vector of $\mathbb{R}^q$, $f_{\nu-q}$ the first basis vector of $\mathbb{R}^{\nu-q}$, and S_q the shift matrix of size $q \times q$. Then, similar to (4.13),

$$
\bar{K}_j = \begin{pmatrix} I_q \otimes F_j + S_q \otimes F_j' & -\ell_q f_{\nu-q}^\top \otimes \frac{1}{j!}G_{j0}^{(j)} \\ 0 & I_{\nu-q} \otimes I_r - S_{\nu-q} \otimes G \end{pmatrix}.
$$

Note that the matrices F_j are empty for $j \geq r - 1$, $\frac{1}{(r-1)!}G_{r-1,0}^{(r-1)}(z) = (0, \dots, 0)$ and that $G_{j0}^{(j)}$ is empty for $j \geq r$. Hence we define

$$
\bar{K}_{r-1} = \left(\begin{array}{c:ccccc}
0_{q \times r} & & & & & \\
\hdashline
I & -G & & & & \\
 & I & -G & & & \\
 & & \ddots & \ddots & & \\
 & & & \ddots & -G & \\
 & & & & I &
\end{array}\right), \tag{5.13}
$$

a matrix of size $(\nu r - q(r-1)) \times (\nu - q)r$, whereas for $j \geq r$ we define

$$
\bar{K}_j = \begin{pmatrix}
I & -G & & & \\
 & I & -G & & \\
 & & \ddots & \ddots & \\
 & & & \ddots & -G \\
 & & & & I
\end{pmatrix}, \tag{5.14}
$$

a matrix of size $(\nu - q)r \times (\nu - q)r$.

In what follows, we need the matrices $\bar{\mathcal{K}}^i(z) = \bar{K}_0(z) \cdots \bar{K}_i(z)$, where the matrices $\bar{K}_i(z)$ have been introduced in (5.12), (5.13), (5.14). Then $\bar{\mathcal{K}}^i(z)$ is of size $\nu r \times (\nu r - q(i+1))$ for $i \leq r - 2$ and of size $\nu r \times (\nu - q)r$ for $i \geq r - 1$. Note that $\bar{\mathcal{K}}^i$ is always of full column rank. The next lemma extends Eq. (5.4), obtained for the case $\nu \leq q$.

Lemma 5.7. *Let $0 \leq i \leq r - 1$. For $0 \leq i < k$ one has*

$$\mathcal{N}^k(z)\bar{\mathcal{K}}^i(z) = \mathcal{N}_{i+1}^{k-i-1} \begin{pmatrix} I_q \otimes I_r & 0 \\ 0 & I_{\nu-q} \otimes \mathcal{G}_{i+1} \end{pmatrix}$$

$$= \left(\mathcal{R}_{ik}^1(z), \mathcal{R}_{ik}^2(z) \right) \tag{5.15}$$

where $\mathcal{R}_{ik}^1(z) \in \mathbb{R}^{q \times (r-i-1)q}$ and $\mathcal{R}_{ik}^2(z) \in \mathbb{R}^{q \times (r-i-1)(\nu-q)}$ are explicitly given by

$$\mathcal{R}_{ik}^1(z) = \frac{1}{(k-i-1)!} \left(\frac{M_{i+1}^{(k-i-1)}(z)}{0!}, \cdots, \frac{M_{i+1}^{(k+q-i-2)}(z)}{(q-1)!} \right),$$

$$\mathcal{R}_{ik}^2(z) = \frac{1}{(k-i-1)!} \left(\frac{M^{(k+q-i-1)}(z)G(z)^{i+1}}{q!}, \cdots, \frac{M^{(k+\nu-i-2)}(z)G(z)^{i+1}}{(\nu-1)!} \right).$$

For $i \geq k$ it holds that $\mathcal{N}^k(z)\bar{\mathcal{K}}^i(z) = 0$.

Proof. The case $k = 0$ has been verified in the proof of Proposition 4.6. Let therefore $k \geq 1$. To prove that the assertion holds true for $i < k$, we assume the right hand side of formula (5.15) to be valid for $\mathcal{N}^k(z)\bar{\mathcal{K}}^{i-1}(z)$ and proceed by induction. To that end we multiply it by $\bar{K}_i$ and verify the answer. As before, we denote the j-th block column of $\bar{K}_i$ by $\bar{K}_i^j$, for $j = 0, \ldots, \nu - 1$. We will discern the four cases $j = 0$, $j = 1, \ldots, q - 1$, $j = q$ and $j = q + 1, \ldots, \nu - 1$.

Let $j = 0$. Then the product $\mathcal{N}^k\bar{\mathcal{K}}^{i-1}\bar{K}_i^0$ becomes $\frac{1}{(k-i)!}M_i^{(k-i)}F_i$. The analogue of (5.2), with M_i and F_i substituted for M and F, yields that this equals $\frac{1}{(k-i-1)!}M_{i+1}^{(k-i-1)}$, as should be the case.

Let $1 \leq j \leq q - 1$. One obtains

$$\mathcal{N}^k\bar{\mathcal{K}}^{i-1}\bar{K}_i^j = \frac{1}{(k-i)!} \left(\frac{M_i^{(k+j-i-1)}}{(j-1)!}F_i' + \frac{M_i^{(k+j-i)}}{j!}F_i \right). \tag{5.16}$$

The analogue of Eq. (5.2) yields $M_i^{(k+j-i-1)}F_i' = -\frac{M_i^{(k+j-i)}}{k+j-i}$. Hence the right hand side of (5.16) becomes

$$\frac{1}{(k-i)!} \frac{1}{j!} \left(-\frac{j}{k+j-i}M_i^{(k+j-i)}F_i + M_i^{(k+j-i)}F_i \right) = \frac{1}{(k-i-1)!} \frac{1}{j!} \frac{M_i^{(k+j-i)}F_i}{k+j-i}.$$

Invoking the analog of (5.2) again, we can rewrite this as $\frac{1}{(k-i-1)!}\frac{M_{i+1}^{(k+j-i-1)}}{j!}$, a typical block of $\mathcal{R}_{ik}^1$, as required.

Next we consider the more involved case $j = q$. In this case the block column $\bar{K}_i^q$ has entry $-\mathcal{G}_i G$ on the $(q-1)$st row (see Lemma 5.5) and I on the q-th row. Hence we get

$$\mathcal{N}^k \bar{\mathcal{K}}^{i-1} \bar{K}_i^j = \frac{1}{(k-i)!} \left(-\frac{M_i^{(k-i+q-1)}}{(q-1)!} \mathcal{G}_i G + \frac{M^{(k-i+q)}}{q!} G^i \right). \tag{5.17}$$

Using (5.9) we obtain $M_i^{(k-i+q-1)} \mathcal{G}_i G = M^{(k-i+q-1)} G^{i+1}$. In view of (4.11), it holds that $M^{(k-i+q)} = (q+k-i) M^{(k-i+q-1)} G$. Hence we van rewrite the right hand side of (5.17) as

$$\frac{1}{(k-i)!} \left(-\frac{M^{(k-i+q-1)}}{(q-1)!} + (q+k-i) \frac{M^{(k-i+q-1)}}{q!} \right) G^{i+1}$$

which is equal to

$$\frac{1}{(k-i-1)!} \frac{M^{(k-i+q-1)}}{q!} G^{i+1},$$

the first block of $\mathcal{R}_{ik}^2$, as was to be shown.

Finally we treat the case $q + 1 \le j \le \nu - 1$. The block columns $\bar{K}_i^j$ have $-G$ at the $(j-1)$st row and I at the j-th row. Hence, we obtain

$$\mathcal{N}^k \bar{\mathcal{K}}^{i-1} \bar{K}_i^j = \frac{1}{(k-i)!} \left(-\frac{M^{(k-i+j-1)} G^i}{(j-1)!} G + \frac{M^{(k-i+j)} G^i}{j!} \right). \tag{5.18}$$

Since $k - i + j - 1 > q$, we apply (4.11) to get $M^{(k-i+j)} = (k-i+j) M^{(k-i+j-1)} G$, and the right hand side of (5.18) reduces to

$$\frac{1}{(k-i-1)!} \frac{M^{(k-i+j-1)} G^{i+1}}{j!},$$

a typical block of $\mathcal{R}_{ik}^2$, as desired. This settles the proof of the validity of Eq. (5.15). $\quad \square$

Theorem 5.8. *It holds that $\mathcal{N}^k(z) \bar{\mathcal{K}}^i(z) = 0$, for $i \ge k$. For $\mu \le r-1$, the matrix $\bar{\mathcal{K}}^{\mu-1}(z)$ is of size $\nu r \times (\nu r - q\mu)$ and has full rank, equal to $\nu r - q\mu$. If $\mu \ge r$, $\bar{\mathcal{K}}^{\mu-1}(z)$ is of size $\nu r \times (\nu - q)r$ and has full rank, equal to $(\nu - q)r$. Summarizing, the kernel of $\mathcal{N}(z)$ is $(\nu r - q \min\{\mu, r\})$-dimensional and spanned by the columns of $\bar{\mathcal{K}}^{\mu-1}(z)$. The rank of $\mathcal{N}(z)$ is equal to $q \min\{\mu, r\} < \nu r$ and therefore $\mathcal{N}(z)$ never has full column rank.*

Proof. We show that $\mathcal{N}^k(z) \bar{\mathcal{K}}^i(z) = 0$ for $i \ge k$, for which it is clearly sufficient to show that $\mathcal{N}^k(z) \bar{\mathcal{K}}^k(z) = 0$. Starting point is Eq. (5.15) for $i = k - 1$. We have

$$\mathcal{N}^k(z) \bar{\mathcal{K}}^{k-1}(z) = \left(\frac{M_k^{(0)}(z)}{0!}, \cdots, \frac{M_k^{(q-1)}(z)}{(q-1)!}, \frac{M^{(q)}(z) G(z)^k}{q!}, \cdots, \frac{M^{(\nu-1)}(z) G(z)^k}{(\nu-1)!} \right).$$

We multiply this equation with the block columns $\bar{K}_k^j$ and, as above, we discern the case $j = 0$, $1 \leq j \leq q - 1$, $j = q$ and $j = q + 1, \ldots, \nu - 1$.

For $j = 0$ we get $\mathcal{N}^k(z)\bar{\mathcal{K}}^{k-1}\bar{K}_k^0 = M_k^{(0)}F_k = uw_k F_k = 0$, whereas for $1 \leq j \leq q - 1$ one computes

$$\mathcal{N}^k\bar{\mathcal{K}}^{k-1}\bar{K}_k^j = \frac{M_k^{(j-1)}}{(j-1)!}F_k' + \frac{M_k^{(j)}}{j!}F_k = 0,$$

in view of an analogue of (5.2).

For $j = q$, we obtain

$$\mathcal{N}^k\bar{\mathcal{K}}^{k-1}\bar{K}_k^q = -\frac{M_k^{(q-1)}\mathcal{G}_k G}{(q-1)!} + \frac{M^{(q)}G^k}{q!}. \tag{5.19}$$

We can now use Eqs. (5.10) and (4.10) to get

$$\frac{M_k^{(q-1)}\mathcal{G}_k G}{(q-1)!} = \frac{M^{(q-1)}G^{k+1}}{(q-1)!} = \frac{M^{(q)}G^k}{q!}.$$

Hence, the right hand side of (5.19) is zero.

Next we consider the case $q + 1 \leq j \leq \nu + 1$. We then get, parallel to (5.18),

$$\mathcal{N}^k\bar{\mathcal{K}}^{i-1}\bar{K}_k^j = \left(-\frac{M^{(j-1)}G}{(j-1)!} + \frac{M^{(j)}}{j!}\right)G^k,$$

which is zero, in view of Eq. (5.10).

To show that $\mathcal{N}(z)\bar{\mathcal{K}}^{\mu-1}(z) = 0$, one has to show that $\mathcal{N}^k(z)\bar{\mathcal{K}}^{\mu-1}(z) = 0$, for all $k \leq \mu - 1$, but this has implicitly been shown above. The other statements in the theorem have already been addressed before. The theorem is proved. □

5.3. The case $\nu \geq q + r$

We follow the approach leading to Proposition 4.7. We observe that the matrix $\mathcal{N}(z)$ for $\nu \geq q + r$ can be decomposed as

$$\mathcal{N}(z) = \left(\mathcal{N}_*(z) \quad 0_{\mu q \times r(\nu-q-r-1)}\right),$$

where $\mathcal{N}_*(z)$ is the "$\mathcal{N}(z)$ matrix" for the case $\nu = q + r - 1$, since all derivatives of $M(z)$ of order higher than $q + r - 2$ vanish. Let $\bar{\mathcal{K}}_*^{\mu-1}(z)$ be the $\bar{\mathcal{K}}^{\mu-1}(z)$ matrix for the case $\nu = q + r - 1$. Put

$$\bar{\bar{\mathcal{K}}}^{\mu-1} = \begin{pmatrix} \bar{\mathcal{K}}_*^{\mu-1} & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity matrix of order $r(\nu - q - r - 1)$. If $\mu < r$, then $\bar{\mathcal{K}}_*^{\mu-1}(z)$ is of size $(q + r - 1)r \times (r(r - 1) + (r - \mu)q)$, and if $\mu \geq r$, then it has size $(q + r - 1)r \times r(r - 1)$. Then $\bar{\bar{\mathcal{K}}}^{\mu-1}(z)$ has size $\nu r \times (\nu r - \mu q)$ for $\mu < r$ and has size $\nu r \times (\nu - q)r$ for $\mu \geq r$. In short, $\bar{\bar{\mathcal{K}}}^{\mu-1}(z)$ has dimensions $\nu r \times (\nu r - q \min\{\mu, r\})$.

Theorem 5.9. *Let $\nu \geq q + r$. The kernel of the matrix $\mathcal{N}(z)$ is spanned by the columns of $\bar{\bar{\mathcal{K}}}^{\mu-1}(z)$, has dimension $\nu r - \mu q$ if $\mu < r$ and dimension $(\nu - q)r$ if $\mu \geq r$. So $\dim \ker(\mathcal{N}(z)) = \nu r - q \min\{\mu, r\}$.*

Proof. Similar to the proof of Proposition 4.7, using the results of Theorem 5.8 for the case $\nu = q + r - 1$. □

6. Intermezzo, properties of $\mathcal{A}^0(z)$

The results of this section will be used in Section 7, where we want to find (special) right inverses of the matrix $\mathcal{M}^0(z)$.

We focus on the matrix $\mathcal{A}^0 = (A^0, \ldots, A^{\nu-1}) \in \mathbb{R}^{q \times \nu r}$, the first block row of $\mathcal{A}$, the matrix defined in Section 2. One directly sees that the rank of $\mathcal{A}^0$ is equal to $\min\{q, \nu\}$, although it also follows from Theorem 2.3 with $\mu = 1$. Hence $\mathcal{A}^0$ has full rank iff $\nu \geq q$. We introduce the matrix $\mathcal{B}^0 \in \mathbb{R}^{\nu r \times q}$ consisting of the $r \times q$ blocks B^k as follows.

$$\mathcal{B}^0 = \begin{pmatrix} B^0 \\ \vdots \\ B^{\nu-1} \end{pmatrix} \tag{6.1}$$

where each B^k has elements

$$B_{ij}^k = \begin{cases} (-1)^i \binom{q}{k+1} & \text{if } i + j = k \\ 0 & \text{else,} \end{cases}$$

for $i = 0, \ldots, r - 1$, $j = 0, \ldots, q - 1$.

Lemma 6.1. *Let $\nu \geq q$. Then $\mathcal{A}^0$ has full row rank and $\mathcal{A}^0 \mathcal{B}^0 = I$. In other words, $\mathcal{B}^0$ is a right inverse of $\mathcal{A}^0$.*

Proof. We have to compute the ij-elements of $T := \sum_{k=0}^{\nu-1} A^k B^k$. Using the definitions of the matrices A^k and B^k that only have nonzero entries on corresponding anti-diagonals, we see that $A^k B^k$ is a diagonal matrix. Hence we only have to consider the ii-entries of T. Note that $B^k = 0$ for $k \geq q$. One obtains

$$T_{ii} = \sum_{k=0}^{q-1} \left(A^k B^k\right)_{ii}$$

$$= \sum_{k=0}^{q-1} \binom{k}{i} (-1)^{k-i} \binom{q}{k+1}$$

$$= \frac{q!}{i!(q-1-i)!} \sum_{k=i}^{q-1} \binom{q-1-i}{k-i} \frac{(-1)^{k-i}}{k+1}.$$

To compute the latter summation, we write it as

$$\int_0^1 \sum_{k=i}^{q-1} \binom{q-1-i}{k-i} (-1)^{k-i} x^k \, \mathrm{d}x = \int_0^1 \sum_{j=0}^{q-1-i} \binom{q-1-i}{j} (-x)^j x^i \, \mathrm{d}x$$

$$= \int_0^1 (1-x)^{q-1-i} x^i \, \mathrm{d}x$$

$$= B(q-i, i+1),$$

by definition of the β-function $B(\cdot, \cdot)$. Using the well-known fact that this can be computed in terms of Γ-functions $(B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta))$ we obtain

$$B(q-i, i+1) = \frac{(q-1-i)!i!}{q!}.$$

It follows that $T_{ii} = 1$. $\square$

We need some additional properties.

Lemma 6.2. *It holds that* $X^k := B^k S_q^\top + S_r B^k = 0$, *if* $k \geq q$ *or* $k \leq r-1$. *For the case* $r \leq k \leq q-1$ *(which requires* $q > r$) *only the last row of this matrix is nonzero. In fact, this row is equal to* $(-1)^{r-1}\binom{q}{k+1}e_{k-r}^\top$, *with the convention that* e_i *denotes the* i*-th basis vector of* $\mathbb{R}^q$ $(i = 0, \ldots, q-1)$.

Proof. We compute the ij-element of $X^k = B^k S_q^\top + S_r B^k$. For $i = 0, \ldots, r-1$ and $j = 0, \ldots, q-1$ it is equal to

$$X_{ij}^k = \sum_{l=0}^{q-1} B_{il}^k 1_{\{l=j+1\}} + \sum_{l=0}^{r-1} 1_{\{i+1=l\}} B_{lj}^k$$

$$= \sum_{l=0}^{q-1} B_{i,j+1}^k 1_{\{l=j+1\}} + \sum_{l=0}^{r-1} 1_{\{i+1=l\}} B_{i+1,j}^k$$

$$= B_{i,j+1}^k 1_{\{0 \leq j+1 \leq q-1\}} + 1_{\{0 \leq i+1 \leq r-1\}} B_{i+1,j}^k$$

$$= B^k_{i,j+1} 1_{\{0 \le j \le q-2\}} + 1_{\{0 \le i \le r-2\}} B^k_{i+1,j}$$

$$= \binom{q}{k+1} 1_{\{i+j+1=k\}} \left((-1)^i 1_{\{0 \le j \le q-2\}} + (-1)^{i+1} 1_{\{0 \le i \le r-2\}} \right)$$

$$= \binom{q}{k+1} 1_{\{i+j+1=k\}} (-1)^i (1_{\{0 \le j \le q-2\}} - 1_{\{0 \le i \le r-2\}}).$$

Clearly, for $i = 0, \ldots, r-2$ and $j = 0, \ldots, q-2$, the last expression in the display equals zero, as is the case for $i = r-1$ and $j - q - 1$. We next consider the two remaining cases, the first being $i \le r-2$ and $j = q-1$. Since we only have to consider $i = k - j - 1$, we get $i = k - q$, which has to be nonnegative, so $k \ge q$. But then the binomial coefficient is equal to zero. The remaining case is $i = r-1$. Then we only have to consider $j = k - r$, the other values of j again give zero. Note that this implies that $k \ge r$ is necessary to get a nonzero outcome, whereas we already know that also $k \le q-1$ is necessary. Hence nonzero elements in the last row of X^k can only occur if $r \le q-1$. Under this last condition we find $X^k_{r-1,j} = \binom{q}{k+1}(-1)^{r-1} 1_{\{j=k-r\}}$. Hence the bottom row of X^k equals $\binom{q}{k+1}(-1)^{r-1}(1_{\{k=r\}}, \ldots, 1_{\{k=r+q-1\}})$, which is equal to $\binom{q}{k+1}(-1)^{r-1} e^\top_{k-r}$, for $k = r, \ldots, q-1$. $\square$

Remark 6.3. Here is an example where X^k as defined in Lemma 6.2 is not equal to zero. Take $k = r = 1$ and $q = 2$. Then $B^1 = (0 \quad 1)$ and $X^1 = (1 \quad 0)$.

Proposition 6.4. *Define $H_k : \mathbb{R} \to \mathbb{R}^{r \times q}$ by $H_k(z) = \widetilde{W}_r(z) D^{-1}_r B^k D^{-1}_q \widetilde{U}_q(z)$. Then H_k is a constant mapping, $H_k(z) \equiv D^{-1}_r B^k D^{-1}_q$, under the condition $k \ge q$ or $k \le r-1$.*

Proof. First we prove the following auxiliary results. One has

$$\widetilde{U}'_q(z) = \widetilde{U}_q(z) D_q S^\top_q D^{-1}_q \tag{6.2}$$

$$\widetilde{W}'_r(z) = D^{-1}_r S_r D_r \widetilde{W}_r(z). \tag{6.3}$$

Eq. (6.2) follows from the definition of $\widetilde{U}_q(z)$ and the elementary identity $U'_q(z) = U_q(z) S^\top_q$. Eq. (6.3) can be proved similarly.

We now compute

$$H'_k(z) = \widetilde{W}'_r(z) D^{-1}_r B^k D^{-1}_q \widetilde{U}_q(z) + \widetilde{W}_r(z) D^{-1}_r B^k D^{-1}_q \widetilde{U}'_q(z)$$

$$= D^{-1}_r S_r D_r \widetilde{W}_r(z) D^{-1}_r B^k D^{-1}_q \widetilde{U}_q(z) + \widetilde{W}_r(z) D^{-1}_r B^k D^{-1}_q \widetilde{U}_q(z) D_q S^\top_q D^{-1}_q,$$

according to Eqs. (6.2) and (6.3). Putting $\hat{S}_r = D^{-1}_r S_r D_r$ and $\hat{S}_q = D_q S_q D^{-1}_q$, we see that H_k satisfies the linear differential equation

$$H'_k(z) = \hat{S}_r H_k(z) + H_k(z) \hat{S}^\top_q. \tag{6.4}$$

This equation has a unique solution and we claim that it is given by the constant function as asserted. To that end we check

$$\hat{S}_r D_r^{-1} B^k D_q^{-1} + D_r^{-1} B^k D_q^{-1} \hat{S}_q^\top = D_r^{-1} \big(S_r B^k + B^k S_q^\top \big) D_q^{-1}$$
$$= 0,$$

by Lemma 6.2, since $k \geq q$ or $k \leq r - 1$. Furthermore, we have $H_k(0) = D_r^{-1} B^k D_q^{-1}$, since $\widetilde{U}_q(0) = I_q$. □

Remark 6.5. Eq. (6.4) has as the general solution

$$H_k(z) = \exp(\hat{S}_r z) H_k(0) \exp\big(\hat{S}_q^\top z\big), \tag{6.5}$$

where the exponentials can be computed as finite sums, since $S_q^\top$ and S_r are nilpotent. Elementary computations yield for instance that the ij-element of $\exp(\hat{S}_q^\top z)$ is given by $\binom{i}{j} z^{i-j}$ for $i \geq j$ and zero otherwise. Hence we obtain $\exp(\hat{S}_q z) = \widetilde{U}_q(z)$, which is in agreement with the definition of $H_k(z)$.

An example of a solution that is not constant is obtained for $r = 1$ and $q = 2$. For the case $k = 1$ one finds directly from the definition of $H_k(z)$ that $H_1(0) = B^1 = (0 \quad 1)$ and $H_k(z) = (z \quad 1)$ in view of Remark 6.3. This is in agreement with Eq. (6.5), whose right hand side is equal to

$$(0 \quad 1) \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

7. The equation $\mathcal{M}^0(z)C = I$

We return to one of our original aims, finding a right inverse of the $q \times \nu r$ matrix $\mathcal{M}^0(z) = (M(z), \ldots, M^{(\nu-1)}(z)) \in \mathbb{R}^{q \times r\nu}$. Recall from Theorem 2.3 and Proposition 3.1 that $\mathcal{M}^0(z)$ has rank equal to $\min\{\nu, q\}$. Hence the matrix is of full rank iff $\nu \geq q$. Equations like $\mathcal{M}^0(z)X = b$ will in general not have a solution X for a given $b \in \mathbb{R}^{\nu r \times 1}$, if $\nu < q$. In fact, we are interested in solutions X that are independent of z. It is easy to see that such solutions only exist if $b = 0$ and then $X = 0$. The uninteresting case $\nu < q$ will therefore be ignored and the standing assumption in the remainder of this section is $\nu \geq q$. Under this assumption, there are two subcases to discern, $r \geq q$ and $r < q$.

Proposition 7.1. *Assume that $r \geq q$ and $\nu \geq q$. Let I_q be the q-dimensional unit matrix. There exists a constant (not depending on z) matrix $C \in \mathbb{R}^{\nu r \times q}$ such that $\mathcal{M}^0(z)C = I_q$ for all z. The equation $\mathcal{M}^0(z)X = b$ for $b \in \mathbb{R}^q$ then has the constant solution $X = Cb$. The constant matrix C is unique iff $\nu = q$. In all cases one can take $C = (I_\nu \otimes D_r^{-1})\mathcal{B}^0 D_q^{-1}$, with $\mathcal{B}^0$ as in (6.1).*

Proof. In this proof we simply write I for I_q. Suppose that we have found a constant matrix C with the desired property

$$\mathcal{M}^0(z)C = I. \tag{7.1}$$

By differentiation of (7.1) k times, with $k = 0, \dots, r-1$, we obtain, recall the definition of $\mathcal{M}(z)$ with $\mu = r$, that

$$\mathcal{M}(z)C = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7.2}$$

We note that now $\mathcal{M}(z)$ is of size $rq \times r\nu$ and has rank equal to rq. Hence $\mathcal{M}(z)$ has a right inverse, $\mathcal{M}(z)^+$ say, and a true inverse in the case that $\nu = q$, see e.g. Corollary 3.6. It follows that C should be the first block-column of $\mathcal{M}(z)^+$. Next we use the factorization (3.1) and note that also $\mathcal{A}$ has a right inverse, $\mathcal{A}^+$ say. Then

$$\mathcal{M}(z)^+ = \big(I_\nu \otimes W_r(z)^{-1}\big)\mathcal{A}^+\big(I_r \otimes U_q(z)^{-1}\big).$$

Hence, we can choose

$$C = \big(I_\nu \otimes W_r(z)^{-1}\big)\mathcal{A}^+\big(I_r \otimes U_q(z)^{-1}\big) \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which means that C is the first block-column of $\mathcal{M}(z)^+$, so

$$C = \big(I_\nu \otimes W_r(z)^{-1}\big)\mathcal{A}^+ \begin{pmatrix} U_q(z)^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \big(I_\nu \otimes W_r(z)^{-1}\big)\big(\mathcal{A}^0\big)^+ U_q(z)^{-1}, \tag{7.3}$$

where $(\mathcal{A}^0)^+$ is a right inverse of the matrix $\mathcal{A}^0$, since $\mathcal{A}^0$ is the first block-row of $\mathcal{A}$. But, a right inverse of $\mathcal{A}^0$ is in Proposition 6.1 shown to be $\mathcal{B}^0$. Therefore, we can now explicitly pose our candidate for C,

$$C = \big(I \otimes W_r(z)^{-1}\big)\mathcal{B}^0 U_q(z)^{-1}, \tag{7.4}$$

where $\mathcal{B}^0$ as defined in (6.1). Hence we have to show that
(1) the matrix C in (7.4), in fact doesn't depend on z,
(2) $\mathcal{M}^0(z)C = I$. Using matrices introduced in Section 3, we write

$$C = \big(I \otimes \big(\widetilde{W}_r(z)^{-1} D_r^{-1}\big)\big)\mathcal{B}^0 D_q^{-1} \widetilde{U}_q(z)^{-1}$$
$$= \big(I \otimes \widetilde{W}_r(-z) D_r^{-1}\big)\mathcal{B}^0 D_q^{-1} \widetilde{U}_q(-z).$$

Decomposing C as

$$C = \begin{pmatrix} C^0 \\ \vdots \\ C^{\nu-1} \end{pmatrix},$$

where each block C^k $(k = 0, \ldots, \nu - 1)$ is of size $r \times q$, we get

$$C^k = \widetilde{W}_r(-z)^\top D_r^{-1} B^k D_q^{-1} \widetilde{U}_q(-z).$$

Hence we see that $C^k = H_k(-z)$, which was in Proposition 6.4 shown to be constant and equal to $D_r^{-1} B^k D_q^{-1}$, if we have $k \le r - 1$ or $k \ge q$. Obviously, this is true of $k = 0, \ldots, r - 1$, but for $k = r, \ldots, \nu - 1$, we have $k \ge r \ge q$ by assumption. This proves claim (1). Since C is constant in z, we can take $z = 0$ in (7.4).

For the second one we have

$$\mathcal{M}^0(z)C = U_q(z)\mathcal{A}^0\big(I \otimes W_r(z)\big)C$$
$$= U_q(z)\mathcal{A}^0\big(I \otimes W_r(z)\big)\big(I \otimes W_r(z)^{-1}\big)\mathcal{B}U_q(z)^{-1}$$
$$= U_q(z)\mathcal{A}^0\mathcal{B}U_q(z)^{-1}$$
$$= I,$$

in view of Lemma 6.1. Finally, if $\nu = q$, then $\mathcal{M}(z)$ is invertible, which implies that C is the unique constant matrix solving $\mathcal{M}^0(z)C = I$, since in this case Eq. (7.2) has a unique solution. $\quad\square$

Remark 7.2. The special choice $(\bar{\mathcal{A}}^0)^+ = \mathcal{B}^0$ in the proof of Proposition 7.1 is rather crucial in finding a right inverse of $\mathcal{M}^0(z)$ that doesn't depend on z. We illustrate this with the following example. Our point of departure is Eq. (7.3) with $\mu = 1$.

Recalling (2.10), we can take $(\mathcal{A}^0)^+ = \mathcal{L}_{\nu,r}(0)^{-1} \bar{\mathcal{A}}^+ \mathcal{L}_{1,q}(0)^{-1}$, with $\bar{\mathcal{A}}^+$ any right inverse of $\bar{\mathcal{A}}$. We choose $\bar{\mathcal{A}}^+ = \bar{\mathcal{A}}^\top$ and compute

$$\bar{\mathcal{B}} := \mathcal{L}_{\nu,r}(0)^{-1} \bar{\mathcal{A}}^\top \mathcal{L}_{1,q}(0)^{-1} = \mathcal{L}_{\nu,r}(0)^{-1} \bar{\mathcal{A}}^\top \in \mathbb{R}^{\nu r \times q},$$

because $\mathcal{L}_{1,q}(0) = I_q$. Since $\bar{\mathcal{A}}_j^\top = f_0 e_j^\top$ (Theorem 2.3) and $\mathcal{L}_{\nu,r}(0)_{ij}^{-1} = \binom{i}{j}(-S_r)^{i-j}$, for the k-th block $\bar{B}^k$ of $\bar{\mathcal{B}}$ we get $\bar{B}^k = \sum_{l=0}^{\nu-1}\binom{l}{k}(-1)^{l-k} f_{k-l} e_l^\top = f_0 e_k^\top$ $(k = 0, \ldots, \nu - 1)$. We conclude that $\bar{\mathcal{B}} = \bar{\mathcal{A}}^\top$.

In order to see that this may result in a right inverse of $\mathcal{M}^0(z)$ that depends on z, we choose $q = r = \nu = 2$. The conditions of Proposition 7.1 are then satisfied. We have

$$\mathcal{M}^0(z) = \begin{pmatrix} 1 & z & 0 & 1 \\ z & z^2 & 1 & 2z \end{pmatrix}$$

and it follows from the above that

$$\mathcal{M}^0(z)^+ = \left(I_2 \otimes W_2(z)^{-1}\right)\bar{\mathcal{B}}U_2(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -z & 1 \\ 0 & 0 \end{pmatrix}.$$

We close this remark by noting that there also other right inverses of $\mathcal{M}^0(z)$, depending on z, but still having a simple structure. An example (essentially taken from [8], where it was only given for $\nu = q = r$ in a slightly different situation) is

$$\mathcal{M}^0(z)^+ = \begin{pmatrix} U_q(z)^{-1} \\ 0_{(\nu-q)\times q} \end{pmatrix} \otimes f_0,$$

where f_0 is the first basis vector of $\mathbb{R}^r$. This follows from the easy to verify identity

$$\mathcal{M}^0(z)(I_\nu \otimes f_0) = \begin{pmatrix} U_q(z) & 0_{q\times(\nu-q)} \end{pmatrix}.$$

The assertion of Proposition 7.1 is not true if $r < q$ (the second subcase). Indeed, in the proof of this proposition we used the fact that all C^k are indeed constant matrices, under the condition $r \geq q$. If this is not the case, $r < q$, the matrices C^k for $k = r, \ldots, q-1$ are not constant, in view of Proposition 6.4. Let us give an example to illustrate this. Consider the case $q = \nu = 2$ and $r = 1$. Then

$$\mathcal{M}^0(z) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

and the equation $\mathcal{M}^0(z)C = I_2$ has the *unique* on z depending solution $C = C(z) = \mathcal{M}^0(z)^{-1}$.

We now treat the case $\nu > q$ in more detail. To that end we need the following auxiliary result.

Lemma 7.3. *The subspace of the kernel of $\mathcal{M}^0(z)$ that consists of vectors that are constant in z, i.e. the intersection $\bigcap_z \ker(\mathcal{M}^0(z))$, is $(\nu - q)^+r$-dimensional. This subspace is equal to the kernel of $\mathcal{M}(z)$ with $\mu = r$, which is the same for all z and hence can be parametrized free of z.*

Proof. The first observation is that a vector x in $\ker \mathcal{M}^0(z)$ that doesn't depend on z also satisfies $\mathcal{M}(z)x = 0$ for arbitrary μ, in particular for $\mu = r$. The case $\nu \leq q$ follows from Theorem 5.2, since in this case the kernel of $\mathcal{M}(z)$ is the null space for all $\mu \geq r$.

Let then $\nu > q$. Let x be a column vector consisting of r-dimensional sub-vectors $x_0, \ldots, x_{\nu-1}$ that don't depend on z. Recalling that $\mathcal{M}^0$ consists of a row of blocks $(uw)^{(n)}$, we have

$$
\mathcal{M}^0 x = \sum_{n=0}^{\nu-1} (uw)^{(n)} x_n
$$

$$
= \sum_{n=0}^{\nu-1} \sum_{k=0}^{n} \binom{n}{k} u^{(k)} w^{(n-k)}
$$

$$
= \sum_{k=0}^{\nu-1} u^{(k)} \sum_{n=k}^{\nu-1} \binom{n}{k} w^{(n-k)} x_n.
$$

The vectors $u^{(k)}$ are zero for $k \geq q$ and otherwise linear independent. Hence, to have the above sum equal to zero is equivalent to

$$
\sum_{n=k}^{\nu-1} \binom{n}{k} w^{(n-k)} x_n = 0 \quad \text{for } k = 0, \ldots, q-1.
$$

The equation for arbitrary $1 \leq k \leq q-1$ can be differentiated to get

$$
\sum_{n=k}^{\nu-1} \binom{n}{k} w^{(n+1-k)} x_n = 0,
$$

which, subtracting from the equation for $k-1$ yields

$$
\binom{n}{k-1} w^{(0)} x_{k-1} = 0,
$$

valid for $k = 1, \ldots, q$. The only constant solutions to these equations are the zero vectors, so $x_0 = \cdots = x_{q-2} = 0$.

On the other hand, for $k = q-1$ we keep the equation

$$
\sum_{m=0}^{\nu-q} \binom{m+q-1}{q-1} w^{(m)} x_{m+q-1} = 0.
$$

Now we relabel the unknowns by setting $y_m = \binom{m+q-1}{q-1} x_{m+q-1}$ to get

$$
\sum_{m=0}^{\nu-q} w^{(m)} y_m = 0.
$$

We differentiate this equation j times, with $j = 0, \ldots, r-1$ to get

$$
\begin{pmatrix} w^{(0)} & \cdots & w^{(\nu-q)} \\ \vdots & & \vdots \\ w^{(r-1)} & \cdots & w^{(\nu-q+r-1)} \end{pmatrix} y = 0, \tag{7.5}
$$

where y is obtained by stacking the y_m. The first block-column in the above matrix is W, the second can be written as $S_r W$, up to the last one equal to $S_r^{\nu-q} W$. Hence the above system of equations can be compactly written as

$$(W \quad S_r W \quad \cdots \quad S_r^{\nu-q} W) \, y = 0.$$

Let Δ be the diagonal matrix with elements $\Delta_{ii} = i$. A simple computation shows that $S_r W = W S_r \Delta$, and therefore $S_r^k W = W(S_r \Delta)^k$. Writing $S_r \Delta = T$, we can rewrite the last equation in y as

$$(W \quad WT \quad \cdots \quad WT^{\nu-q}) \, y = 0.$$

Since $W = W_r(z)$ is invertible for any z, this reduces to

$$(I \quad T \quad \cdots \quad T^{\nu-q}) \, y = 0.$$

Since the coefficient matrix has full row rank equal to r, its kernel has dimension $(\nu - q + 1)r - r = (\nu - q)r$. Actually, this kernel is spanned by the columns of the $(\nu - q + 1)r \times (\nu - q)r$ matrix

$$\begin{pmatrix} -T & & & \\ I_r & -T & & \\ & I_r & & \\ & & \ddots & -T \\ & & & I_r \end{pmatrix}.$$

This proves the claim. □

Remark 7.4. The result of Lemma 7.3 is also valid for $\mu > r$. This can be seen from Eq. (7.5). Indeed, if $\mu > r$ one has to extend the coefficient matrix with additional rows, all involving derivatives $w^{(k)}$, with $k \geq r$. But these are all equal to zero.

One might think that the assertion of the lemma can alternatively be proven by explicitly computing the matrix $\bar{\mathcal{K}}^{\mu-1}(z)$ for $\mu = r$ (noting that $(D_\nu^{-1} \otimes I_r)\bar{\mathcal{K}}^{\mu-1}(z)$ represents the kernel of $\mathcal{M}(z)$ in view of (1.2)) and showing that it is not depending on z. It turns out that this idea is false, as shown by the following simple example.

Let $q = 1$, $r = \mu = \nu = 3$. We compute $\bar{\mathcal{K}}^{\mu-1}(z)$ and show that it is not free of z. According to the results of Section 5.2 we find for $\bar{K}_0$, $\bar{K}_1$ and $\bar{K}_2$ the following.

$$
\bar{K}_0(z) = \begin{pmatrix}
-z & 0 & 0 & -1 & -z & 0 & 0 & 0 \\
1 & -z & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & -z \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
\bar{K}_1(z) = \begin{pmatrix}
-z & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -z \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \qquad
\bar{K}_2(z) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & -z \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

and the product $\bar{\mathcal{K}}^2(z) = \bar{K}_0(z)\bar{K}_1(z)\bar{K}_2(z)$ yields the kernel of $\mathcal{M}(z)$ spanned by the columns of

$$
\left(D_3^{-1} \otimes I_3\right)\bar{\mathcal{K}}^2(z) = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 2 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -3 & -3z \\
0 & 1 & 0 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}.
$$

We see that the last column of $\bar{\mathcal{K}}^2(z)$ has a term $-3z$ in the fourth row, hence this parametrization of $\ker(\mathcal{M}(z))$ is not the one we are looking for. The reparametrization of $\ker(\mathcal{M}(z))$ given by $(D_3^{-1} \otimes I_3)\bar{\mathcal{K}}^2(z)R(z)$ with

$$
R(z) = \begin{pmatrix}
1 & 0 & 0 & 0 & 5 & 6z \\
0 & 1 & 0 & 0 & 0 & 4 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

yields $(D_3^{-1} \otimes I_3)\bar{\mathcal{K}}^2(z)R(z) =: \hat{\mathcal{K}}$, with

$$\hat{\mathcal{K}} = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -2 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

which is free of z. The procedure in the proof of Lemma 7.3 (in this case $y_m = x_m$, since $q = 1$) yields a kernel of $\mathcal{M}(z)$ spanned by column vectors that are not depending on z. The result of that procedure is the matrix $\hat{\mathcal{K}}$ above.

Proposition 7.5. *Let $q < \nu \leq r$. Then the equation $\mathcal{M}^0(z)C = I_q$ admits a constant solution $C \in \mathbb{R}^{\nu r \times q}$. The dimension of the affine space of constant solutions is equal to $(\nu - q)rq$.*

Proof. According to Proposition 7.1 a constant solution C exists. Any other constant right inverse C' is such that the q columns of $C' - C$ belongs to the kernel of $\mathcal{M}^0$ and hence to the kernel of $\mathcal{M}$ for $\mu = r$. In view of Lemma 7.3, a basis of this kernel can be obtained can be obtained by choosing $(\nu - q)r$ linearly independent vectors. Applying this result to each the columns of $C' - C$, we obtain the result. □

To illustrate the fact that the constant solution C of Proposition 7.5 is in general not unique, we consider the case $q = 1$, $r = \nu = 2$. Then $\mathcal{M}^0(z) = \begin{pmatrix} 1 & z & 0 & 1 \end{pmatrix}$ and all constant solutions are given by $C = C_{a,b} = \begin{pmatrix} a & 0 & b & 1-a \end{pmatrix}^\top$ with $a, b \in \mathbb{R}$, which form an affine space of dimension $(\nu - q)qr = 2$.

Of course all right inverses of $\mathcal{N}^0(z)$, also those that depend on z, are given by a much larger affine subspace. Assume $\nu \geq q$ and let $C_0(z)$ be any right inverse of $\mathcal{N}^0(z)$. Then any matrix $C(z) = C_0(z) + X(z)$, with $X(z) \in \mathbb{R}^{\nu r \times q}$ a matrix whose columns belong to $\ker \mathcal{N}^0(z)$ is a right inverse. Since, $\dim \ker \mathcal{N}^0(z) = \nu r - q$, the affine subspace of these right inverses has dimension $(\nu r - q)q$.

The natural extension of the equation $\mathcal{M}^0(z)C = I_q$ is $\mathcal{M}(z)C = I_{\mu q}$, with $\mathcal{M}(z)$ of order $\mu q \times \nu r$ and $I_{\mu q}$ the identity matrix of order μq. The matrix $\mathcal{M}(z)$ has rank equal to $\min\{\mu, r\} \times \min\{\nu, q\}$ and therefore has full row rank if and only if $\nu \geq q$ and $\mu \leq r$. Hence, under the latter condition, and only then, a right inverse exists, and the equation $\mathcal{M}(z)C = I_{\mu q}$ has a solution. This equation can be decomposed as

$$\begin{pmatrix} \mathcal{M}^0(z) \\ \vdots \\ \mathcal{M}^{\mu-1}(z) \end{pmatrix} \begin{pmatrix} C_0 & \cdots & C_{\mu-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 & \cdots & 0 \\ 0 & I_q & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & I_q \end{pmatrix},$$

where every C_j has size $\nu r \times q$. Parallelling our previous aim, also here one could be interested in finding solutions C that are constant in z. For C_0 we are in the previous situation, since a constant C_0 satisfying $\mathcal{M}^0(z)C_0 = I_q$, also satisfies $\mathcal{M}^k(z)C_0 = 0$ for all $k \geq 1$. The situation for the other C_k is different. Consider for example C_1. It should satisfy $\mathcal{M}^0(z)C_1 = 0$ and $\mathcal{M}^1(z)C_1 = I_q$. However, this is impossible for a C_1 that is constant in z, since differentiating $\mathcal{M}^0(z)C_1 = 0$ yields $\mathcal{M}^1(z)C_1 = 0$. We conclude that the equation $\mathcal{M}(z)C = I_{\mu q}$ for $\mu \geq 2$ has no constant solutions.

Nonconstant solutions are for instance Moore–Penrose inverses. These can be obtained by using the Moore–Penrose inverse of the matrix $\bar{A}$. It follows from the proof of Theorem 2.3 that for $\mu \leq r$ and $\nu \geq q$, the matrix $\bar{A}(\bar{A})^\top$ is the identity matrix. Hence $(\bar{A})^\top$ is a right inverse of $\bar{A}$. Using then Theorem 2.2 and Proposition 3.1 we obtain that

$$\mathcal{M}(z)^+ = \left(I_\nu \otimes W_r(z)^{-1}\right)\mathcal{L}_{\nu,r}(z)^{-1}\bar{A}^\top \mathcal{L}_{\mu,q}(z)^{-1}\left(I_\mu \otimes U_q(z)^{-1}\right)$$

is a right inverse of $\mathcal{M}(z)$. The inverses $U_q(z)^{-1}$ and $W_r(z)^{-1}$ can be computed easily, since one has for instance $\widetilde{U}_q(z) = U_q(z)D_\mu$ and $\widetilde{U}_q(z)^{-1} = \widetilde{U}_q(-z)$. The inverses $\mathcal{L}_{\nu,r}^{-1}$ and $\mathcal{L}_{\mu,q}^{-1}$ can be computed in view of the formulas just above Theorem 2.2. Since $\mathcal{L}_{\nu,r}^{-1} = \mathcal{L}_{\nu,r}(0)^{-1}$, one obtains that its ij-block ($i \geq j$) is given by $\binom{i}{j}(S_q^\top)^{i-j}(-1)^{i-j}$. Summarizing, we have

Proposition 7.6. *The matrix $\mathcal{M}(z)$ has a right inverse iff $\nu \geq q$ and $\mu \leq r$, in which case a right inverse is*

$$\mathcal{M}(z)^+ = \left(I_\nu \otimes W_r(z)^{-1}\right)\mathcal{L}_{\nu,r}(0)^{-1}\bar{A}^\top \mathcal{L}_{\mu,q}(0)^{-1}\left(I_\mu \otimes U_q(z)^{-1}\right).$$

All right inverses form an affine space of dimension $(\nu r - \mu q)\mu q$.

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