CHAPTER 7

CLONES AND DERIVED OPERATIONS

Basic operations are only the beginning of what makes an algebra interesting. A semigroup, for example, is not just a set with a binary operation; part of its nature is that certain different ways of composing this operation with itself lead to the same derived operation. Composite operations are one ingredient of this phenomenon; they are the subject of this chapter.

Some of the applications will require that we treat constants as constant unary functions. For the time being, we deviate from standard usage by allowing nullary operations in clones.

§A Clones

1 Definition. Let A be a set; ψ an n-ary operation on A; and $\phi_0, \ldots, \phi_{n-1}$ k-ary operations on A. Then the k-ary composite $\psi(\phi_0,\ldots, \phi_{n-1})$ is the k-ary operation χ on A defined by

$$\chi(x_0,\ldots, x_{k-1}) \simeq \psi(\phi_0(x_0,\ldots, x_{k-1}),\ldots, \phi_{n-1}(x_0,\ldots, x_{k-1})).$$

In particular, $\chi(x_0,\ldots, x_{k-1})$ exists if and only if

$$\langle x_0,\ldots, x_{k-1}\rangle \in \bigcap_{i<n} \mathrm{Dom}(\phi_i)$$

and $\langle \phi_0(x_0,\ldots, x_{k-1}),\ldots, \phi_{n-1}(x_0,\ldots, x_{k-1})\rangle \in \mathrm{Dom}(\psi)$.

If $n = 0$, $\psi(\phi_0, \ldots, \phi_{n-1})$ is determined by ψ and the supposed arity of the otherwise nonexistent ϕ_i; we may speak simply of *the k-ary constant function* ψ.

2 Lemma (associativity). For any operations $\psi, \phi_0,\ldots, \phi_{n-1}, \chi_0,\ldots, \chi_{k-1}$:

$$\psi(\phi_0,\ldots, \phi_{n-1})(\chi_0,\ldots, \chi_{k-1}) \simeq \psi(\phi_0(\chi_0,\ldots, \chi_{k-1}),\ldots, \phi_{n-1}(\chi_0,\ldots, \chi_{k-1})).$$

Proof. Exercise.

3 Definition. Let A be a set, and $0 \le i < n$. The *i-th n-ary projection operation* on A is the projection $e_i^n := \pi_i\colon A^n \longrightarrow A$, mapping $\langle x_0,\ldots, x_{n-1}\rangle$ to x_i.

4 Definition. Let A be a nonvoid set. A *clone* on A is a set Φ of operations on A that contains the projection operations $e_i^n\colon A^n \longrightarrow A$, for all $n \in \mathbb{N}$ and $i < n$, and is closed under composition:
for any $n, k \in \mathbb{N}$, $\psi\colon A^n \circ\!\!\longrightarrow A$ and $\phi_0,\ldots, \phi_{n-1}\colon A^k \circ\!\!\longrightarrow A$ such that $\psi, \phi_0,\ldots, \phi_{n-1} \in F$, $\psi(\phi_0,\ldots, \phi_{n-1}) \in \Phi$.

We denote the clone of *all* operations on A by ΩA; the subset of n-ary operations by $\Omega_n A$.

Let $\mathcal{C}lone$ be the nominator consisting of constant symbols e_i^n, for all i and n such that $0 \le i < n$, and $(m + 1)$-ary operation symbols γ_m^k for all $k, m \ge 0$. Interpret these symbols in ΩA: e_i^n as the i-th n-ary projection operation on A, and γ_m^k as the composition operation that takes an m-ary operation ϕ_0 and k-ary operations $\phi_1,\ldots, \phi_m$ to the k-ary composite $\phi_0(\phi_1,\ldots, \phi_m)$. The resulting algebra we denote by $\mathbf{\Omega}\, A$.

The clones on A are the subuniverses of $\mathbf{\Omega}\, A$. In particular, they form a closed set system.

Examples.
Ai Let O(A) be the collection of total operations of A. It is a clone.
ii. The least clone on A is be the collection Proj A of all projection operations of A.
iii. Suppose A is a singleton, say $A = \{a\}$. The operations on A are easily listed; we have
1° for every $n > 0$, the n-ary projection operation $\langle a\ldots a\rangle \mapsto a$;
2° the constant a;
3° the void operation $\emptyset$.
The clones on $\{a\}$ form a Boolean algebra:

$$\begin{tikzcd}[column sep=small]
& \Omega\{a\} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{Proj}\{a\} + \emptyset \arrow[dr, dash] & & \mathrm{Proj}\{a\} + a \arrow[dl, dash] \\
& \mathrm{Proj}\{a\} &
\end{tikzcd}$$

5 Proposition. (i) If A is finite, the number of clones on A is at most $2^{\aleph_0}$.
(ii) If A is infinite, the number of clones on A is at most $2^{2^{|A|}}$.

Proof. (i) There are countably many operations on A, and a clone is a set of operations.
(ii) Let $\mathfrak{k} = |A|$. Then $|A^n| = \mathfrak{k}$, and if $X \subseteq A^n$, $|A^X| \leq \mathfrak{k}^{\mathfrak{k}} = 2^{\mathfrak{k}}$. That is to say, for any subset X of A^n, there are at most $2^{\mathfrak{k}}$ operations with domain X. Since A^n has $2^{\mathfrak{k}}$ subsets, the number of n-ary operations on A is at most $2^{\mathfrak{k}} \cdot 2^{\mathfrak{k}} = 2^{\mathfrak{k}}$. So the total number of operations is $\aleph_0 \cdot 2^{\mathfrak{k}} = 2^{\mathfrak{k}}$. The clones on A are subsets of the set of operations, so there are at most $2^{2^{\mathfrak{k}}}$ of them. ⊠

6 Definition. Let R be an m-ary relation on A, and ϕ an n-ary operation. Then *ϕ preserves R*, or *R is invariant under ϕ*, if, for every matrix

$$(a_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n)$$

such that for every j, $\langle a_{1j},\ldots, a_{mj}\rangle \in R$, and for every i, $\langle a_{i1},\ldots, a_{in}\rangle \in \mathrm{Dom}\, \phi$, we have $\langle \phi(a_{11},\ldots, a_{1n}),\ldots, \phi(a_{m1},\ldots, a_{mn})\rangle \in R$.

In a picture:

$$\begin{pmatrix} a_{11} & \dots & \dots & a_{1n} \\ a_{21} & \dots & \dots & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} \phi(a_{11},\dots,a_{1n}) \\ \phi(a_{21},\dots,a_{2n}) \\ \vdots \\ \vdots \\ \phi(a_{m1},\dots,a_{mn}) \end{pmatrix}$$

if the columns of the matrix belong to R, then so does the column vector on the right.

Example. Assume $R \subseteq A^m$. The collection of all operations on A that preserve R is a clone.

The clone of total operations on a set A is a subclone of ΩA, on the one hand; on the other hand, there is a fairly natural injection of ΩA into the clone of total operations on a minimally enlarged set. Let $\mathbf{\Omega}'A$ be the reduct of $\mathbf{\Omega} A$ obtained by forgetting the projection operations; and $\mathbf{O}'(A)$ the corresponding reduct of $\mathbf{O}(A)$.

A7 Theorem. Let A be a nonvoid set, $\perp \notin A$. Then $\mathbf{\Omega}'A$ is isomorphic to a subalgebra of $\mathbf{O}'(A \cup \{\perp\})$.

Proof. If ϕ is an n-ary operation on A, define $F(\phi)$ on $A \cup \{\perp\}$ by

$$F(\phi)(x_1,\dots,x_n) = \phi(x_1,\dots,x_n) \text{ if } \langle x_1,\dots,x_n\rangle \in \operatorname{Dom}\phi; \\ \perp \text{ otherwise.}$$

Then F clearly is injective. We shall prove that F respects the composition operations. Consider any composite element $\psi(\phi_0,\dots,\phi_{n-1})$ of ΩA, with all ϕ_i ($i < n$) k-ary; take any $x_0,\dots,x_{k-1} \in A \cup \{\perp\}$.

1° If for some $i < n$,

$$\langle x_0,\dots,x_{k-1}\rangle \notin \operatorname{Dom}\phi_i$$

(in particular if $\perp \in \{x_0,\dots,x_{k-1}\}$), then on the one hand $\perp$ is among $F(\phi_0)(x_0\dots x_{k-1}), \dots, F(\phi_{n-1})(x_0\dots x_{k-1})$, so

$$F(\psi)(F(\phi_0)(x_0\dots x_{k-1}), \dots, F(\phi_{n-1})(x_0\dots x_{k-1})) = \perp;$$

and on the other hand $\langle x_0,\dots,x_{k-1}\rangle \notin \operatorname{Dom}\psi(\phi_0,\dots,\phi_{n-1})$, so

$$F(\psi(\phi_0,\dots,\phi_{n-1}))(x_0,\dots,x_{k-1}) = \perp.$$

2° If $\phi_i(x_0,\dots,x_{k-1}) = a_i \in A$ ($i < n$), but $\psi(a_0,\dots,a_{n-1})\uparrow$, then

$$F(\psi)(F(\phi_0),\dots,F(\phi_{n-1}))(x_0,\dots,x_{k-1}) = \perp,$$

and since $\psi(\phi_0,\dots,\phi_{n-1})(x_0,\dots,x_{k-1})\uparrow$, $F(\psi(\phi_0,\dots,\phi_{n-1}))(x_0,\dots,x_{k-1}) = \perp$ as well.

3° The remaining case, $\psi(\phi_0,\dots,\phi_{n-1})(x_0,\dots,x_{k-1})\downarrow$, is straightforward. ⊠

The embedding F does not map projection operations on A to the corresponding projection operations on $A \cup \{\perp\}$. For example, $F(e_0^2)(a, \perp) = \perp$; but $e_0^2(a, \perp) = a$.

8 Two-valued logic

Emil Post [1941] proved that the lattice of clones of total operations on a two-element set is countable. (For a modernized proof, see [Lau, 1991].) Since the standard logic recognizes precisely two truth values, *true* and *false*, this lattice holds a special interest: it orders all possible systems of truth-functional connectives.

If we allow connectives to be undefined for some possible inputs, we obtain *partial* two-valued logic. Its systems of connectives are harder to describe. In particular, their number is as great as Proposition 5 allows.

To see this, denote the truth values by 0 and 1, and define for $n > 1$ the n-ary operation ϕ_n by

$$\phi_n(x_0,\dots, x_{n-1}) = 0 \text{ if there is exactly one } i < n \text{ such that } x_i = 0,$$
$$\uparrow \text{ otherwise.}$$

For $S \subseteq \mathbb{N}$, let C_S be the clone generated by $\{\phi_{n+2} | n \in S\}$. It will suffice to show that $\phi_{n+2} \in C_S$ if and only if $n \in S$. One direction is trivial. For the other, suppose $\phi_{n+2} \in C_S$. Then $\phi_{n+2} = \phi_{k+2}(\tau_0,\dots, \tau_{k+1})$, for certain $(n + 2)$-ary operations $\tau_0,\dots, \tau_{k+1} \in C_S$ and $k \in S$. Since

$$\phi_{k+2}(\tau_0,\dots, \tau_{k+1})(0, 1,\dots, 1) = \phi_{n+2}(0, 1,\dots, 1) = 0,$$

and $1 \notin \operatorname{Ran} \phi_m$, the operations $\tau_0,\dots, \tau_{k+1}$, except at most one, must be projection operations. Let τ_i be a projection operation, say $\tau_i = e_j^{n+2}$. Since

$$\phi_{k+2}(\tau_0,\dots, \tau_{k+1})(1,\dots, 1, 0, 1,\dots, 1) = \phi_{n+2}(1,\dots, 1, 0, 1,\dots, 1) = 0$$

if the single 0 appears after $j - 1$ ones, all the other operations, $\tau_0,\dots, \tau_{i-1}$, $\tau_{i+1},\dots, \tau_{k+1}$, must be projection operations. Similar arguments establish that $\tau_0,\dots, \tau_{k+1}$ must all be distinct, and that all $n + 2$ $(n + 2)$-ary projection operations must be among them. Hence $k = n$, so $n \in S$.

History

The number of clones of total operations on the three-element set was established by Janov and Mucnik [1959] and by Hulanicki and Świerczkowski [1960]. Freivald [1966] found the number of clones on a two-element set.

§B Term operations and polynomial operations

Algebras have been introduced as pairs $\mathbf{A} = \langle A, I\rangle$ of a set A and a mapping I of symbols into ΩA that respects the arities of symbols. This is clearly the sort of definition required for the notion of homomorphism to work. There are, however, also contexts it which the names of operations are of small account. In particular, when we are speaking of clones, we may identify an algebra with its universe and its set of basic operations: $\mathbf{A} = \langle A, F\rangle$, with A a set and $F \subseteq \Omega\mathbf{A}$.

1 Definition. Let $\mathbf{A}$ be an algebra: $\operatorname{Clo}\mathbf{A}$, the *clone of term operations of* $\mathbf{A}$, is the subuniverse of $\mathbf{\Omega} A$ generated by the basic operations of $\mathbf{A}$.

The elements of $\operatorname{Clo}\mathbf{A}$ are the *term operations of* $\mathbf{A}$. The set of n-ary term operations of $\mathbf{A}$ will be denoted by $\operatorname{Clo}_n \mathbf{A}$.

2 Proposition. If $\mathbf{A}$ is a total algebra, $\operatorname{Clo}\mathbf{A} \subseteq \mathrm{O}(A)$.

Proof. Let **A** be a total algebra. Then the generating elements of Clo **A** are total operations. So are the projections; and if f_0 is an m-ary total operation and $f_1,\ldots,f_m$ are k-ary total operations, then $\gamma^k_m(f_0,f_1,\ldots,f_m)$ is a total operation. ☒

Let ψ be a k-ary operation. Then m-ary *composition by* ψ is the operation $F_m(\psi)$ that maps each k-tuple $\langle\phi_0,\ldots,\phi_{k-1}\rangle$ of m-ary operations to the m-ary composite $\psi(\phi_0,\ldots,\phi_{k-1})$.

B3 Theorem. Let **A** be an algebra. Then $\mathrm{Clo}_n\,\mathbf{A}$ is the least set of operations on A that contains the n-ary projection operations on A and is closed under n-ary composition by basic operations of **A**.

Proof. Let Γ_n be the least set of operations on A that
(a) contains $e^n_0,\ldots,e^n_{n-1}$;
(b) contains $\gamma^n_k(Q,\tau_0,\ldots,\tau_{k-1})$ if it contains $\tau_0,\ldots,\tau_{k-1}$ and Q is a basic k-ary operation of **A**.
Put

$$\Gamma := \bigcup_{0\le n}\Gamma_n\,.$$

It will suffice to prove that $\Gamma = \mathrm{Clo}\,\mathbf{A}$.

The inclusion $\Gamma\subseteq\mathrm{Clo}\,\mathbf{A}$ is easy. For the converse, define:

$$\Gamma' = \{\psi\in\Omega A\,|\,\text{for all } n, \Gamma_n \text{ is closed under } F_n(\psi)\}.$$

We are going to show that $\mathrm{Clo}(\mathbf{A})\subseteq\Gamma'\subseteq\Gamma$.
1° Γ' is a clone. For: the projections are in Γ', since $e^m_i(\phi_0,\ldots,\phi_{m-1}) = \phi_i$. Since composition is associative (Lemma A2), if all the Γ_n are closed under n-ary composition by ψ, $\phi_0,\ldots$, and ϕ_{k-1}, they are closed under n-ary composition by $\psi(\phi_0,\ldots,\phi_{k-1})$.
2° This implies that $\mathrm{Clo}(\mathbf{A})\subseteq\Gamma'$, for by definition every Γ_n is closed under n-ary composition by the basic operations of **A**.
3° Let ψ be an m-ary element of Γ'. Then in particular Γ_m is closed under $F_m(\psi)$, and so, since $\psi=\psi(e^m_0,\ldots,e^{m-1}_m)$, $\psi\in\Gamma_m$. ☒

Examples
i Let A be a nonvoid set, considered as a discrete algebra. Then Clo$A$ is the set Proj A of projection operations of A.
ii Let $\mathbf{A} = \langle\{0, 1\}, +\rangle$ be the two-element group. Let us denote e^2_0 by x and e^2_1 by y, and the single element of $\{0\}^{A\times A}$ by 0. Then from Theorem 3, we find the elements of $\mathrm{Clo}_2\,\mathbf{A}$ are

$$x, y, x+y \text{ and } 0,$$

where 0 can be construed as $x + x$.

Thus equations holding in **A**, such as $x + y = y + x$ or $x + x = y + y$, may be taken as statements that various ways of combining operations of **A** lead to the same result. In the sequel we shall meet yet another, more formal, approach to equations.
iii Let $\mathbf{Q} = \langle\mathbb{Q}, +, \cdot, -, (.)^{-1}, 0, 1\rangle$ be the meadow of rational numbers. Among the n-ary term operations of **Q** are the rational functions

$$\frac{p(x_0,\ldots,x_{n-1})}{q(x_0,\ldots,x_{n-1})}$$

where p and q are polynomials in $x_0, \ldots, x_{n-1}$ with coefficients in $\mathbb{Q}$. (Here the variable x_i represents the projection e_i^n.)
iv The n-ary term operations of the meadow $\mathbf{R} = \langle \mathbb{R}, +, \cdot, -, (.)^{-1}, 0, 1\rangle$ of real numbers correspond one-to-one with the n-ary term operations of $\mathbf{Q}$.

B4 Definition. Let $\mathbf{A}$ be an algebra. Then $\mathbf{Clo}_n\,\mathbf{A}$ is the algebra $\mathbf{B}$ with universe $B = \mathrm{Clo}_n\,\mathbf{A}$ and the same nominator as $\mathbf{A}$, the interpretation being determined by

$$Q^{\mathbf{B}}(\phi_0, \ldots, \phi_{k-1}) = Q^{\mathbf{A}}(\phi_0, \ldots, \phi_{k-1}),$$

where on the left-hand side a k-ary operation is *applied to* $\phi_0, \ldots, \phi_{k-1}$, and on the right the defining operation is *composed with* them.

5 Proposition. Let $\mathbf{A}$ be an algebra. Then
(a) $\mathbf{Clo}_n\,\mathbf{A}$ is a total algebra;
(b) if $\mathbf{A}$ is total, $\mathrm{Clo}_n\,\mathbf{A}$ consists of total operations.

Proof. (a) Immediate by the definition.
(b) Immediate by Proposition 2. ⊠

6 Definition. Let $\mathbf{A}$ be an algebra, and $B \subseteq A$: B-Pol$\,\mathbf{A}$, the *clone of B-polynomial operations of* $\mathbf{A}$, is the subuniverse of $\mathbf{\Omega}\,A$ generated by the basic operations of $\mathbf{A}$ and the nullary operations with values in B.

The elements of B-Pol$\,\mathbf{A}$ are the *B-polynomial operations of* $\mathbf{A}$. The set of n-ary B-polynomial operations of $\mathbf{A}$ will be denoted by B-Pol$_n\,\mathbf{A}$. If $B = A$, we omit the præfix: Pol$\,\mathbf{A}$ = A-Pol$\,\mathbf{A}$, the clone of polynomial operations of $\mathbf{A}$; Pol$_n\,\mathbf{A}$ = B-Pol$_n\,\mathbf{A}$, the n-ary polynomial operations of $\mathbf{A}$.

Example **v**. Let $\mathbf{R}$ be the meadow of real numbers. Among the n-ary polynomial operations of $\mathbf{R}$ are the functions

$$\frac{p(x_0, \ldots, x_{n-1})}{q(x_0, \ldots, x_{n-1})}$$

where p and q are polynomials in $x_0, \ldots, x_{n-1}$ with real coefficients. The $\mathbb{Q}$-polynomial operations of $\mathbf{R}$ are the same as the term operations.

There is a close connection between term operations and polynomial operations. We formulate it in terms of expansions by *special constants*.

7 Definition (special constants). Let $\mathbf{A} = \langle A, I\rangle$ be an algebra, and $B \subseteq A$. Let C be a set of special constant symbols (in particular, $C \cap \mathrm{Nom}\,\mathbf{A} = \emptyset$), in one-to-one correspondence with B; let c_b be the constant symbol corresponding to $b \in B$. Then $\mathbf{A}_B$ =

$$\langle A, I \cup \{\langle b, c_b\rangle \mid b \in B\}\rangle.$$

One easily sees that B-Pol$\,\mathbf{A} = \mathrm{Clo}\,\mathbf{A}_B$ and B-Pol$_n\,\mathbf{A} = \mathrm{Clo}_n\,\mathbf{A}_B$.

8 Corollary. Let $\mathbf{A}$ be an algebra.
(a) If $\mathbf{A}$ is total, Pol$\,\mathbf{A} \subseteq \mathrm{O}(A)$.
(b) B-Pol$_n\,\mathbf{A}$ is the least set of operations on A that contains the n-ary projection operations and the constant functions from A^n into B, and that is closed under composition by basic operations of $\mathbf{A}$.

Proof. (a) $\mathrm{Pol}\,\mathbf{A} = \mathrm{Clo}\,\mathbf{A}_A \subseteq \mathrm{O}(A)$ by Proposition 2.
(b) B-$\mathrm{Pol}_n\,\mathbf{A} = \mathrm{Clo}_n\,\mathbf{A}_B$; now apply Theorem 3, taking into account that the nullary functions into B are basic operations of $\mathbf{A}_B$. (The constant n-ary operation $\{b\} \times A^n$ is the n-ary composite of the nullary b and the void sequence of n-ary operations.) ⊠

B9 Definition. Let $\mathbf{A}$ be an algebra. Then B-$\mathbf{Pol}_n\,\mathbf{A}$ is the algebra $\mathbf{P}$ with universe $P = B$-$\mathrm{Pol}_n\,\mathbf{A}$ and the same nominator as $\mathbf{A}$, the interpretation being determined by

$$Q^{\mathbf{P}}(\phi_0,\dots,\phi_{k-1}) = Q^{\mathbf{A}}(\phi_0,\dots,\phi_{k-1}).$$

10 Corollary. Let $\mathbf{A}$ be an algebra. Then
(a) B-$\mathbf{Pol}_n\,\mathbf{A}$ is a total algebra;
(b) if $\mathbf{A}$ is total, B-$\mathrm{Pol}_n\,\mathbf{A}$ consists of total operations.

11 Theorem. Let $\mathbf{A}$ be an algebra, and $B \subseteq A$. If $t \in \mathrm{Clo}_{m+n}\,\mathbf{A}$ and $b_0,\dots, b_{m-1} \in B$, and p is the operation on A defined by

$$(*) \qquad p(u_0,\dots, u_{n-1}) \simeq t(a_0,\dots, a_{m-1}, u_0,\dots, u_{n-1}),$$

then $p \in B$-$\mathrm{Pol}_n\,\mathbf{A}$. Conversely, if $p \in B$-$\mathrm{Pol}_n\,\mathbf{A}$, then there exist $m \in \mathbb{N}$ and $t \in \mathrm{Clo}_{m+n}\,\mathbf{A}$, $b_0,\dots, b_{m-1} \in B$, such that (*) holds.

Proof. We use Theorem 3.

Suppose $t = e_i^{m+n}$. If $i < m$, let p be the n-ary constant operation with value a_i. If $i \geq m$, take $p = e_{i-m}^n$. If t is an $(m + n)$-ary constant operation, let p be the corresponding n-ary constant operation.

Suppose $t = Q(t_0,\dots, t_{k-1})$, $k \geq 1$, and for all $j < k$, we have $p_j \in \mathrm{Pol}_n\,\mathbf{A}$ such that

$$p_j(u_0,\dots, u_{n-1}) \simeq t_j(a_0,\dots, a_{m-1}, u_0,\dots, u_{n-1}).$$

Then take $p = Q(p_0,\dots, p_{k-1})$.

In the other direction, if $p = e_i^n$, we can take $m = 0$ and $t = p$. If p is the n-ary constant function with value a, take $m = 1$ and $t = e_0^{n+1}$. If $p = Q(p_0,\dots, p_{k-1})$, $k \geq 1$, and for all $j < k$ we have $m_j \in \mathbb{N}$, $t_j \in \mathrm{Clo}_{m_j+n}\,\mathbf{A}$ and $a_{m_{j-1}},\dots, a_{m_j-1} \in A$ (where $m_{-1} = 0$) such that

$$p_j(u_0,\dots, u_{n-1}) \simeq t_j(a_{m_{j-1}},\dots, a_{m_j-1}, u_0,\dots, u_{n-1}),$$

put $m := m_{k-1}$ and $s_j := t_j(e_{m_{j-1}}^{m+n},\dots, e_{m_j-1}^{m+n}, e_m^{m+n},\dots, e_{m+n-1}^{m+n})$. Then

$$s_j(a_0,\dots, a_{m-1}, u_0,\dots, u_{n-1}) \simeq p_j(u_0,\dots, u_{n-1});$$

so we may put $t = Q(s_0,\dots, s_{k-1})$. ⊠

31.12 Definition. Let f be an n-ary operation on a set A, $i < n$. Then f is *i-dependent*, or *depends on the index i*, if there exist $a_0,\dots, a_{i-1}, a, b, a_{i+1},\dots, a_{n-1}$ such that

$$f(a_0,\dots, a_{i-1}, a, a_{i+1},\dots, a_{n-1}) \not\simeq f(a_0,\dots, a_{i-1}, b, a_{i+1},\dots, a_{n-1}).$$

We call f *i-independent* if it is not i-dependent.

Now suppose $i_0,\dots, i_{m-1}$ are the distinct indices on which f depends. If $m = k$, we say f is *essentially k-ary*. If $m \leq k$, then f is *at most k-ary*.

31.13 Theorem. Suppose $0 \le k < n$, and f is an n-ary operation on some set A. Then f is essentially at most k-ary if and only if there exist a k-ary operation g on A and n-ary projection operations $g_0,\dots, g_{k-1}$ such that

$$f = \gamma_k^n(g, g_0,\dots, g_{k-1}).$$

Moreover, if f is essentially at most k-ary and $k \ge 1$, we can take

$$g = f(f_0,\dots,f_{n-1})$$

for certain k-ary projection operations $f_0,\dots,f_{n-1}$.

Proof. ($\Leftarrow$) Suppose $f = \gamma_k^n(g, g_0,\dots, g_{k-1})$. There are at most k indices $i < n$ for which the projection operation π_i^n is among $g_0,\dots, g_{k-1}$. If j is not among them, then f is j-independent; which leaves f essentially at most k-ary.
($\Rightarrow$) Assume f is essentially at most k-ary. If $k = 0$, f is constant, hence $f = \gamma_0^n(g)$ for some (possibly undefined) nullary operation g. Otherwise let $i_0 < \dots < i_{k-1}$ be a list containing all the indices on which f depends. Define

$$f_{i_0} = \pi_0^k, \dots, f_{i_{k-1}} = \pi_{k-1}^k,$$
$$\text{and } f_i = \pi_0^k \text{ if } i \notin \{i_0,\dots, i_{k-1}\}.$$

Now put $g = f(f_0,\dots,f_{n-1})$. Then $f_i(a_{i_0},\dots, a_{i_{k-1}}) = a_i$ if $i \in \{i_0,\dots, i_{k-1}\}$. Since f does not depend on the other indices,

$$f(a_0,\dots, a_{n-1}) \simeq f(f_0(a_{i_0},\dots, a_{i_{k-1}}),\dots,f_{n-1}(a_{i_0},\dots, a_{i_{k-1}})).$$

Now define for $j < k$,

$$g_j = \pi_{i_j}^n.$$

Then $g_j(a_0,\dots, a_{n-1}) = a_{i_j}$. Hence with $g = f(f_0,\dots,f_{n-1})$, $f = g(g_0,\dots, g_{k-1})$. ⊠

b14 Theorem. Let **A** be an algebra.
(i) A subset of A is a subuniverse of **A** if and only if it is closed under the term operations of **A**.
(ii) Suppose $X \subseteq A$. Then $\mathrm{Sg}^{\mathbf{A}}X = \{t(\boldsymbol{x}) \mid \boldsymbol{x} \in X^* \text{ and } t \in \mathrm{Clo}\,\mathbf{A}\}$.
(iii) If $X = \{x_0,\dots, x_{n-1}\}$, $n > 0$, then $\mathrm{Sg}^{\mathbf{A}}X = \{t(x_0,\dots, x_{n-1}) \mid t \in \mathrm{Clo}_n\mathbf{A}\}$.

Proof. (i) ($\Leftarrow$) Trivial.
($\Rightarrow$) Suppose S is a subuniverse of **A**; let F be the set of operations on A that S is closed under. We must prove that $\mathrm{Clo}\,\mathbf{A} \subseteq F$. Clearly F contains the projection operations and the basic operations of **A**. It remains to show that F is closed under composition. Suppose S is closed under $f, g_0,\dots, g_{k-1}$; we may assume $k > 0$. If $f(g_0,\dots, g_{k-1})$ exists and $\langle s_0,\dots, s_{n-1}\rangle$ is in its domain, then, since $g_j(s_0,\dots, s_{n-1}) \in S$, for $j < k$, and $f \in F$, we have

$$f(g_0(s_0,\dots, s_{n-1}),\dots, g_{k-1}(s_0,\dots, s_{n-1})) \in S.$$

So indeed $f(g_0,\dots, g_{k-1}) \in F$.
(ii) ($\supseteq$) Immediate by (i), since $\mathrm{Sg}^{\mathbf{A}}X \in \mathrm{Sub}\,\mathbf{A}$.
($\subseteq$) Abbreviate: $\{t(\boldsymbol{x}) \mid \boldsymbol{x} \in X^* \text{ and } t \in \mathrm{Clo}\,\mathbf{A}\} =: U$. Since

$$X = \{\pi_0^1(x) \mid x \in X\},$$

$X \subseteq U$. Let Q be an n-ary basic operation of **A**, and $u_1,\dots, u_n \in U$. Then there are $\boldsymbol{x}_1,\dots, \boldsymbol{x}_n \in X^*$ and $t_1,\dots, t_n \in \mathrm{Clo}\,\mathbf{A}$ such that $u_i = t_i(\boldsymbol{x}_i)$, for $1 \le i \le n$. Let $\boldsymbol{x}$ be the concatenation $\boldsymbol{x}_1\dots\boldsymbol{x}_n$. Then for each i, there are projection operations $f_1^i,\dots,f_{k_i}^i$ such that $t_i(f_1^i,\dots,f_{k_i}^i)(\boldsymbol{x}) = t_i(\boldsymbol{x}_i)$. Then

$$Q(u_1,\dots, u_n) = Q(t_1(f_1^1,\dots,f_{k_1}^1), \dots, t_n(f_1^n,\dots,f_{k_n}^n))(\boldsymbol{x}) \in U.$$

(iii) ($\supseteq$) Trivial.
($\subseteq$) Suppose $y \in \mathrm{Sg}^{\mathbf{A}}X$. By (ii), there are a term operation s and $i_0,\dots, i_{k-1} < n$ such that $y = s(x_{i_0},\dots, x_{i_{k-1}})$. Take $t = \gamma_k^n(s, \pi_{i_0}^n,\dots, \pi_{i_{k-1}}^n) \in \mathrm{Clo}_n\mathbf{A}$. Then $y = t(x_0,\dots, x_{n-1})$. ⊠

31.15 Theorem. Let **A** be an algebra. Then there exists a semigroup **S** such that $S \supseteq A$ and the basic operations of **A** are limitations to A of polynomial operations of **S**.

Proof. Assume that the elements of A and those of Nom**A** are primitive, and that A and Nom**A** are disjoint. Put $S = (A \cup \mathrm{Nom}\mathbf{A})^+$, identifying each sequence of length 1 with its single element. For $s, t \in S$, let $s{\cdot}t$ be the concatenation (written st) of s and t, except when $st = Qa_0\dots a_{n-1}$ with $Q \in \mathrm{Nom}\mathbf{A}$ and $\langle a_0,\dots, a_{n-1}\rangle \in \mathrm{Dom}\,Q^{\mathbf{A}}$, in which case $s{\cdot}t = Q^{\mathbf{A}}(a_0,\dots, a_{n-1})$. ⊠

§. Integral subalgebras/ outward extensions

A subalgebra of **B**, $B - A$ gesloten onder **B**-polynomen. De eenpuntscompletering.

§32. Terms and polynomials

The comparison of different algebras hinges on the circumstance that basic operations are denoted by operation symbols. We want to extend this comparability to derived operations. To this purpose we define terms and polynomials, as a kind of derived operation symbols. They are organized in *term algebras*, that embody the linguistics of our subject.

32.1 Definition. Let $\mathcal{N}$ be a nominator, and X any set disjoint with $\mathcal{N}$. Then $T_{\mathcal{N}}(X)$, the set of *X-ary terms* of type $\mathcal{N}$, is the least set $T \subseteq (\mathcal{N} \cup X)^+$ such that
(i) $X \subseteq T$, and
(ii) if $Q \in \mathcal{N}$ is n-ary, and $\mathbf{t}_0,\dots, \mathbf{t}_{n-1} \in T$, then $Q\mathbf{t}_0\dots\mathbf{t}_{n-1} \in T$.

N.B. In clause (i), we identify sequences of length 1 with their elements. Nonetheless we insist that the length of any term is uniquely determined. In other words, if for some odd reason some $a \in \mathcal{N} \cup X$ is a sequence $a_1a_2a_3$ of three other elements of $\mathcal{N} \cup X$, we assume we know that by a we mean $\langle\langle a_1, a_2, a_3\rangle\rangle$ and not $\langle a_1, a_2, a_3\rangle$.

In the context of $T_{\mathcal{N}}(X)$, the elements of X are called *variables*. Terms in which no variables occur are *ground terms*.

We use bold lower case letters, in particular **s**, **t**, to refer to terms, with sub- and superscripts as needed. We sometimes insert brackets and commas into terms for readibility, as in $Q(\mathbf{t}_0,\dots, \mathbf{t}_{n-1})$.

The definition of $T_{\mathcal{N}}(X)$ as the least set containing the variables that satisfies a certain closure condition implies that any class that contains the variables and is suitably closed is a superclass of $T_{\mathcal{N}}(X)$. This justifies an induction principle, *induction on terms*: if a certain property P belongs to all the variables, and whenever $Q \in \mathcal{N}$ is n-ary, and $\mathbf{t}_0,\dots, \mathbf{t}_{n-1}$ have property P, $Q\mathbf{t}_0\dots\mathbf{t}_{n-1}$ has property P as well, then all X-ary terms over $\mathcal{N}$ have property P.

Term induction may even be used without the limitations imposed by a fixed nominator. If a class C
(1) contains the variables, and
(2) for any n-ary operation symbol Q,

$$\mathbf{t}_0, \ldots, \mathbf{t}_{n-1} \in C \text{ implies } Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1} \in C,$$

then all terms belong to C. A justification of this more general principle may be taken from complete induction: if all terms of length less than m belong to C, then the terms of length m belong to C, so the terms of *any* length belong to C. Indeed, there are no terms of length 0; if $m = 1$, terms of length m must be variables, which are in C by (1), or constant symbols, which are in C by (2) with $n = 0$; if $m > 1$, terms of lengt m have the form $Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1}$ for some $n \geq 1$, and then since the sum of the lengths of $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}$ is $m - 1$, $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}$ must be in C, hence by (2), $Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1} \in C$.

32.2 Definition. Let $\mathcal{N}$ be a nominator, and X any set disjoint with $\mathcal{N}$. Then $\mathbf{T}_{\mathcal{N}}(X)$, the *term algebra* of type $\mathcal{N}$ *over* X is the total algebra with universe $T_{\mathcal{N}}(X)$ and operations

$$\langle \mathbf{t}_0, \ldots, \mathbf{t}_{n-1} \rangle \longmapsto Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1},$$

for all $n \in \mathbb{N}$ and n-ary $Q \in \mathcal{N}$.

32.3 Examples.
(a) $\langle \mathbb{N}, 0, S \rangle$, the algebra of the natural numbers with zero and the successor operation, is isomorphic to the term algebra of type $\{0, S\}$ over the empty set — the *algebra of ground terms over* $\{0, S\}$.
(b) $\langle \mathbb{N}, S \rangle$, the algebra of the natural numbers with the successor operation, is isomorphic to the term algebra of type $\{S\}$ over $\{0\}$.

32.4 Unique Readability Theorem. Let $\mathbf{T} = \mathbf{T}_{\mathcal{N}}(X)$, the term algebra of type $\mathcal{N}$ over X.
(i) Suppose $Q \in \mathcal{N}$, $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1} \in T$. Then $Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}) \notin X$.
(ii) Suppose $P, Q \in \mathcal{N}$, $\mathbf{s}_0, \ldots, \mathbf{s}_{m-1}, \mathbf{t}_0, \ldots, \mathbf{t}_{n-1} \in T$. If

$$P^{\mathbf{T}}(\mathbf{s}_0, \ldots, \mathbf{s}_{m-1}) = Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}),$$

then $P = Q$, $m = n$, and $\mathbf{s}_i = \mathbf{t}_i$ for all $i < n$.

Proof. Put $A := \mathcal{N} \cup X$.
(i) By definition, $Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1})$ is a word over A of a certain positive length k. If $k = 1$, then $Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}) = Q \in \mathcal{N}$, and by assumption $\mathcal{N} \cap X = \emptyset$. If $k > 1$, then $Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}) \notin A$ since, as explained under Definition 1, elements of A have length 1.
(ii) Assign weights to the elements of A:
— $g(Q) = 1 - n$ if $Q \in \mathcal{N}$ is n-ary;
— $g(x) = 1$ for all $x \in X$.
Now extend g over A^+ by

$$g(a_1 \ldots a_k) = \sum_{i=1}^{k} g(a_i).$$

We claim that $w \in A^+$ is a term (that is, belongs to T) if and only if:

(1) $g(w) = 1$ and for every proper initial segment v of w, $g(v) \leq 0$.

The proof from left to right is a straightforward structural induction. The other direction is by induction on the length $|w|$ of w. Assume (1).
— $|w| = 1$. Then w must be a variable or a nullary operation symbol, so w is a term.
— $|w| > 1$. Then the weight of the first letter of w cannot be positive, so w begins with an operation symbol Q of arity $n > 0$. Let $w = Qw_0$, and for $k \geq 0$ define v_k to be the least initial segment of w_k that has positive weight; let w_{k+1} be the remainder, so

$$w_k = v_k w_{k+1}.$$

Say that the process stops after m steps, with v_m undefined: the reason must be that $g(w_m) \leq 0$. Now $g(w_0) = g(w) - g(Q) = 1 - (1 - n) = n$; and since the weight of a single letter is at most 1, v_k, if it exists, has weight 1; so $g(w_k) = n - k$. Hence $m = n$. Observe that

$$g(Qv_0 \ldots v_{n-1}) = (1 - n) + n = 1;$$

so $Qv_0 \ldots v_{n-1}$ is not a proper initial segment of w, that is, w_m is the empty word ε. By induction hypothesis, $v_0, \ldots, v_{n-1}$ are terms; so $w = Qv_0 \ldots v_{n-1}$ is a term.

Now suppose $P^{\mathbf{T}}(\mathbf{s}_0, \ldots, \mathbf{s}_{m-1}) = Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1})$, for terms $\mathbf{s}_0, \ldots, \mathbf{s}_{m-1}$, $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}$. Then the words $P\mathbf{s}_0 \ldots \mathbf{s}_{m-1}$ and $Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1}$ must be the same, so $P = Q$, and since arities are fixed, $m = n$. Moreover $\mathbf{s}_0$ is an initial segment of $\mathbf{t}_0 \ldots \mathbf{t}_{n-1}$, so either $\mathbf{s}_0$ is an initial segment of $\mathbf{t}_0$ or $\mathbf{t}_0$ is an initial segment of $\mathbf{s}_0$. In the first case $\mathbf{s}_0$ cannot be a *proper* initial segment, since $\mathbf{t}_0$ is a term and $g(\mathbf{s}_0) = 1$; so $\mathbf{s}_0 = \mathbf{t}_0$. The same conclusion follows in the other case. We can repeat the argument for $\mathbf{s}_1$ and $\mathbf{t}_1 \ldots \mathbf{t}_{n-1}$, and so on, finding $\mathbf{s}_i = \mathbf{t}_i$ for all $i < n$. ⊠

OP dit punt kan de interpretatie van termen worden gedefinieerd. Daarvoor gebruik je de recursiestelling of een daarvan afgeleide vorm van recursie op termen. R is 'onmiddellijke subterm' (argument). De operatie $\mathbf{q}$:
$\mathbf{q}(x, \psi) = a_x$; $\mathbf{q}(Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1}, \psi) \simeq Q^{\mathbf{A}}(\psi(\mathbf{t}_0), \ldots, \psi(\mathbf{t}_{n-1}))$. ($\mathbf{q}$ is een operatie wegens unieke leesbaarheid.) De interpretatie is per definitie een gegrond quomorfisme (growmorphism).

32.5 Lemma. Let $\mathbf{T} = \mathbf{T}_{\mathcal{N}}(X)$ be a term algebra. Then
(a) $\mathbf{T}$ is X-generated;
(b) if $\mathbf{S} \subseteq \mathbf{T}$ is X-generated, then

$$Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}) \in S \text{ implies } \mathbf{t}_0, \ldots, \mathbf{t}_{n-1} \in S.$$

Proof. (a) If $T \neq \mathrm{Sg}^{\mathbf{T}}X$, then $T - \mathrm{Sg}^{\mathbf{T}}X$ contains a term $\mathbf{t}$ of minimal length: $\mathbf{t} \notin \mathrm{Sg}^{\mathbf{T}}X$, but any term $\mathbf{s}$ that, viewed as a word over $\mathcal{N} \cup X$, is shorter than $\mathbf{t}$, belongs to $\mathrm{Sg}^{\mathbf{T}}X$. Since $X \subseteq \mathrm{Sg}^{\mathbf{T}}X$, $\mathbf{t}$ must be of the form $Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1}$, with $Q \in \mathcal{N}$ and $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}$ terms shorter than $\mathbf{t}$. So $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1} \in \mathrm{Sg}^{\mathbf{T}}X$; hence

$$\mathbf{t} = Q\mathbf{t}_0 \ldots \mathbf{t}_{n-1} = Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}) \in \mathrm{Sg}^{\mathbf{T}}X,$$

a contradiction.
(b) Assume

$$\mathbf{t} := Q^{\mathbf{T}}(\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}) \in S. \tag{2}$$

Then there must be $P \in \mathcal{N}$ and terms $\mathbf{s}_0, \ldots, \mathbf{s}_{m-1} \in S$ such that

$$\mathbf{t} = P^{\mathbf{S}}(\mathbf{s}_0, \ldots, \mathbf{s}_{m-1}), \tag{3}$$

for otherwise, since by Unique Readability $\mathbf{t} \notin X$, $S - \{\mathbf{t}\}$ would be a proper subuniverse of $\mathbf{S}$ containing X. However, (3) implies

$$\mathbf{t} = P^{\mathbf{T}}(\mathbf{s}_0,\dots, \mathbf{s}_{m-1}). \tag{4}$$

By (2) and (4) and Unique Readability, $P = Q$, $m = n$, and $\mathbf{s}_i = \mathbf{t}_i$ for all $i < n$. So $\mathbf{t}_0,\dots, \mathbf{t}_{n-1} \in S$. ☒

32.6 Lemma. Let $\mathbf{T} = \mathbf{T}_{\mathcal{N}}(X)$ be a term algebra, and $\langle \mathbf{S}_j \mid j \in J\rangle$ a nonempty family of X-generated relative subalgebras of $\mathbf{T}$. Then $\bigcup_j \mathbf{S}_j$ is an X-generated relative subalgebra of $\mathbf{T}$.

Proof. Put $\mathbf{S} := \bigcup_j \mathbf{S}_j$. To prove that $\mathbf{S} \subseteq \mathbf{T}$, we must show that if $\mathbf{t}_0,\dots, \mathbf{t}_{n-1} \in S$, and (again)

$$\mathbf{t} := Q^{\mathbf{T}}(\mathbf{t}_0,\dots, \mathbf{t}_{n-1}) \in S, \tag{2}$$

then $\mathbf{t} = Q^{\mathbf{S}}(\mathbf{t}_0,\dots, \mathbf{t}_{n-1})$. Now there must be some j such that $\mathbf{t} \in S_j$. Then by Lemma 5, $\mathbf{t}_0,\dots, \mathbf{t}_{n-1} \in S_j$. But $\mathbf{S}_j \subseteq \mathbf{T}$, so

$$\mathbf{t} = Q^{\mathbf{S}_j}(\mathbf{t}_0,\dots, \mathbf{t}_{n-1}) = Q^{\mathbf{S}}(\mathbf{t}_0,\dots, \mathbf{t}_{n-1}).$$

It remains to be shown that $\mathbf{S}$ is X-generated. Let U be a subuniverse of $\mathbf{S}$ that contains X. By the definition of $\mathbf{T}$, an element $\mathbf{s}$ of S is a word, and as such it has a length $|\mathbf{s}|$. We prove $\mathbf{s} \in U$ by induction on $|\mathbf{s}|$.

If $|\mathbf{s}| = 1$, then either $\mathbf{s} \in X \subseteq U$, or $\mathbf{s}$ is a constant symbol, in which case it belongs to U since $U \in \mathrm{Sub}(\mathbf{S})$. If $|\mathbf{s}| > 1$, then for some $P \in \mathcal{N}$ and $\mathbf{s}_0,\dots, \mathbf{s}_{m-1} \in T$,

$$\mathbf{s} = P\mathbf{s}_0\dots\mathbf{s}_{m-1} = P^{\mathbf{T}}(\mathbf{s}_0,\dots, \mathbf{s}_{m-1}).$$

By lemma 5, $\mathbf{s}_0,\dots, \mathbf{s}_{m-1} \in S$, so by induction hypothesis, $\mathbf{s}_0,\dots, \mathbf{s}_{m-1} \in U$; and $\mathbf{s} = P^{\mathbf{S}}(\mathbf{s}_0,\dots, \mathbf{s}_{m-1})$, hence belongs to U since U is a subuniverse. ☒

32.7 Theorem. Let $\mathcal{N}$ be a nominator, X a set, $\mathbf{A}$ an arbitrary algebra, and

$$\alpha : X \longrightarrow A$$

a mapping. Then the class of homomorphisms from X-generated relative subalgebras of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{A}$ that extend α contains a largest element (under inclusion).

Proof. Put $\mathbf{T} = \mathbf{T}_{\mathcal{N}}(X)$. Let $\mathcal{B}$ be the said class,

$$\mathcal{B} = \{\beta\colon \mathbf{S}_\beta \longrightarrow \mathbf{A} \mid \mathbf{S}_\beta \subseteq \mathbf{T} \text{ is } X\text{-generated and } \beta \supseteq \alpha\}.$$

Then $\mathcal{B} \neq \emptyset$, since $\langle X, \emptyset\rangle \subseteq \mathbf{T}$ by Unique Readability, and $\alpha\colon X \longrightarrow \mathbf{A}$. By Lemma 6,

$$\mathbf{S} := \bigcup_{\beta\in\mathcal{B}} \mathbf{S}_\beta$$

is an X-generated relative subalgebra of $\mathbf{T}$. So if we define a homomorphism by

$$\alpha^* = \bigcup\mathcal{B},$$

we shall be done.

1° To prove that α^* is single-valued, we show, by induction on terms $\mathbf{t}$, that $\beta(\mathbf{t})$ is the same for all $\beta \in \mathcal{B}$ for which it is defined. Suppose

$$\mathbf{t} \in S_\beta \cap S_\gamma.$$

Then either $\mathbf{t} \in X$, and $\beta(\mathbf{t}) = \alpha(\mathbf{t}) = \gamma(\mathbf{t})$; or there are $Q \in \mathcal{N}$ and terms $\mathbf{t}_0, \ldots, \mathbf{t}_{n-1}$ such that $\mathbf{t} = Q^{\mathbf{T}}(\mathbf{t}_0,\ldots, \mathbf{t}_{n-1})$. By Lemma 5, $\mathbf{t}_0,\ldots, \mathbf{t}_{n-1} \in S_\beta \cap S_\gamma$. So

$$\begin{aligned}\beta(\mathbf{t}) = \beta(Q^{\mathbf{S}_\beta}(\mathbf{t}_0,\ldots, \mathbf{t}_{n-1})) &= Q^{\mathbf{A}}(\beta(\mathbf{t}_0),\ldots, \beta(\mathbf{t}_{n-1})) \text{ since } \beta \text{ is a homomorphism,}\\ &= Q^{\mathbf{A}}(\gamma(\mathbf{t}_0),\ldots, \gamma(\mathbf{t}_{n-1})) \text{ by induction hypothesis,}\\ &= \gamma(Q^{\mathbf{S}_\gamma}(\mathbf{t}_0,\ldots, \mathbf{t}_{n-1})) = \gamma(\mathbf{t}) \text{ since } \gamma \text{ is a homomorphism.}\end{aligned}$$

2° α^* is a homomorphism: suppose that $\mathbf{t} = Q^{\mathbf{S}}(\mathbf{t}_0,\ldots, \mathbf{t}_{n-1})$. Then for some $\beta \in \mathcal{B}$, $\mathbf{t} \in S_\beta$, and by Lemma 5, $\mathbf{t}_0,\ldots, \mathbf{t}_{n-1} \in S_\beta$ as well. Since $\alpha^*\lceil S_\beta = \beta$,

$$\begin{aligned}\alpha^*(\mathbf{t}) = \beta(Q^{\mathbf{S}_\beta}(\mathbf{t}_0,\ldots, \mathbf{t}_{n-1})) &= Q^{\mathbf{A}}(\beta(\mathbf{t}_0),\ldots, \beta(\mathbf{t}_{n-1})) \text{ since } \beta \text{ is a homomorphism,}\\ &= Q^{\mathbf{A}}(\alpha^*(\mathbf{t}_0),\ldots, \alpha^*(\mathbf{t}_{n-1})).\end{aligned}$$ ⊠

32.8 Corollary. Let $\mathcal{N}$, X, $\mathbf{A}$ and α be as in the statement of the theorem, and α^* the largest homomorphism from an X-generated relative subalgebra of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{A}$. Then α^* is $\mathcal{N}$-closed, and it is the only $\mathcal{N}$-closed homomorphism from an X-generated relative subalgebra of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{A}$ that extends α.

Proof. Take $\mathcal{B}$ as in the proof of the theorem. It will suffice to show that $\beta \in \mathcal{B}$ is not $\mathcal{N}$-closed if and only if $\mathcal{B}$ contains a proper extension of β. Abbreviate $\mathbf{T}_{\mathcal{N}}(X)$ to $\mathbf{T}$.
($\Rightarrow$) Say β: $\mathbf{U} \longrightarrow \mathbf{A}$, $Q \in \mathcal{N}$, $\mathbf{s}_0, \ldots, \mathbf{s}_{n-1} \in U$, $Q^{\mathbf{A}}(\beta(\mathbf{s}_0),\ldots, \beta(\mathbf{s}_{n-1})) = a \in A$, but $Q^{\mathbf{U}}(\mathbf{s}_0,\ldots, \mathbf{s}_{n-1})\uparrow$. Since $\mathbf{U} \subseteq \mathbf{T}$, we must have

$$\mathbf{t} := Q^{\mathbf{T}}(\mathbf{s}_0,\ldots, \mathbf{s}_{n-1}) \notin U.$$

Let $U' = U \cup \{\mathbf{t}\}$, and $\mathbf{U}' = U'_{\mathbf{T}}$. Define $\gamma := \beta \cup \{\langle a, \mathbf{t}\rangle\}$. By definition $\mathbf{U}' \subseteq \mathbf{T}$; and $\mathbf{U}'$ is obviously X-generated. Moreover, γ is a homomorphism. In particular, if

$$\mathbf{t} = P^{\mathbf{U}'}(\mathbf{t}_0,\ldots, \mathbf{t}_{m-1}),$$

then $\mathbf{t} = P^{\mathbf{T}}(\mathbf{t}_0,\ldots, \mathbf{t}_{m-1})$, so by Unique Readability $P = Q$, $m = n$, and $\mathbf{s}_i = \mathbf{t}_i$ for all $i < n$. Hence

$$\gamma(Q^{\mathbf{U}'}(\mathbf{s}_0,\ldots, \mathbf{s}_{n-1})) = \gamma(\mathbf{t}) = a = Q^{\mathbf{A}}(\gamma(\mathbf{s}_0),\ldots, \gamma(\mathbf{s}_{n-1}))$$

is all there is to it.
($\Leftarrow$) Suppose $\beta, \gamma \in \mathcal{B}$, $\beta \subset \gamma$. Let $\mathbf{U} := (\mathrm{Dom}(\beta))_{\mathbf{T}}$. Take $\mathbf{t} \in \mathrm{Dom}(\gamma) - U$ of minimal word-length. Then $\mathbf{t} = Q^{\mathbf{T}}(\mathbf{s}_0,\ldots, \mathbf{s}_{n-1})$ for some $Q \in \mathcal{N}$, with $\mathbf{s}_0,\ldots, \mathbf{s}_{n-1} \in U$. Then

$$\gamma(\mathbf{t}) = Q^{\mathbf{A}}(\gamma(\mathbf{s}_0),\ldots, \gamma(\mathbf{s}_{n-1})) = Q^{\mathbf{A}}(\beta(\mathbf{s}_0),\ldots, \beta(\mathbf{s}_{n-1})),$$

but $Q^{\mathbf{U}}(\mathbf{s}_0,\ldots, \mathbf{s}_{n-1})\uparrow$. ⊠

32.9 Corollary. Let X be a set, $\mathbf{A}$ a total algebra of type $\mathcal{N}$, and

$$\alpha: X \longrightarrow A$$

a mapping. Then there exists exactly one homorphism of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{A}$ that extends α.

Proof. Let $\mathbf{T} = \mathbf{T}_{\mathcal{N}}(X)$. By Example [between total], Lemma 5(a) and the previous corollary, it is enough to show that $\mathrm{Dom}(\alpha^*) = T$. We prove this, or rather $T \subseteq \mathrm{Dom}(\alpha^*)$, by structural induction (22.1).

By definition, $X \subseteq \mathrm{Dom}(\alpha^*)$. Now suppose $\mathbf{t}_0,\ldots, \mathbf{t}_{n-1} \in \mathrm{Dom}(\alpha^*)$, and $Q \in \mathcal{N}$ is an n-ary operation symbol. Since $Q^{\mathbf{A}}(\alpha^*(\mathbf{t}_0),\ldots, \alpha^*(\mathbf{t}_{n-1}))\downarrow$, by the previous corollary $Q^{\mathbf{T}}(\mathbf{t}_0,\ldots, \mathbf{t}_{n-1})$ belongs to $\mathrm{Dom}(\alpha^*)$. ☒

32.10 Examples

(a) In propositional logic, we deal with atomic statements and abstractions from them, denoted by *proposition letters*; and *connectives*, logical conjunctions that combine statements in various ways. *Formulas* are the expressions denoting statements. Let Ω be a set of connectives, say $\Omega = \{\wedge, \vee, \rightarrow, \neg, \bot\}$, and P a set of proposition letters. Then $T_\Omega(P)$ is the set of formulas.

The format specified by our definition of terms, as words over $\Omega \cup P$, is called *Polish notation* in logic. This notation may be considered difficult to read, even though, according to the Unique Readability Theorem, it is perfectly unambiguous. Fortunately they is no reason why we should insist on trying to denote words by themselves; we are free to use, say, $\neg(p \rightarrow \neg q)$ as a name of $\neg{\rightarrow}p\neg q$.

The truth tables for the connectives (Cayley tables for $\vee$, $\wedge$ and $\neg$ are on p.23, $\bot$ stands for 0, and the Cayley table for $\rightarrow$ has been drawn below) determine a total algebra $\mathbf{2}$ of type Ω on the universe $\{0, 1\}$. Then by Corollary 8, a valuation V: $P \longrightarrow \{0, 1\}$ has a unique extension V^*: $\mathbf{T}_\Omega(P) \longrightarrow \mathbf{2}$ that assigns a truth value to every formula.

$\rightarrow$	1	0
1	1	0
0	1	1

Cayley table of $\rightarrow$ (if…then).

(b) First order logic is considerably more complex, and its algebraization is accordingly harder. An atomic statement is built from a *predicate letter* and a fixed number of *terms*, or the equality symbol and two terms (so it is not really atomic, just like the atom of modern physics). Terms are built from *function letters* and *individual variables*, as in Definition 1, except that the roles of variables and constants has been reversed (as will be seen below). From atomic statements, composite statements are constructed as in propositional logic; moreover, a new statement may be constructed from an individual variable and a statement by applying a quantifier. Let Ω consist of:

an $(n + 1)$-ary operation symbol R_n for every $n \in \mathbb{N}$;
an $(n + 1)$-ary operation symbol Q_n for every $n \in \mathbb{N}$;
a binary operation symbol $\approx$;
constant symbols $v_0, v_1, v_2, v_3, \ldots, v_n, \ldots$: the individual variables;
connectives, say $\wedge, \vee, \rightarrow, \neg, \bot$, with arities as in (a);
and binary operation symbols $\forall$ and $\exists$, the quantifiers.

Let P be a set. The formulas of the first order languages over P are included in $T_\Omega(P)$.

Let A be any nonvoid set. We shall turn A into an algebra of type Ω. The universe is the union of two sets $\mathcal{R}A$ and $\mathcal{Q}A$ constructed from A. The first of these is the set of continuations of finitary relations on A:

$$\mathcal{R}A = \{R \subseteq A^\omega \mid \exists n\ \forall \boldsymbol{a}, \boldsymbol{b} \in A^\omega\ (\forall x < n.\ \boldsymbol{a}(x) = \boldsymbol{b}(x) \Rightarrow (\boldsymbol{a} \in R \Leftrightarrow \boldsymbol{b} \in R))\}.$$

The second is the set of continuations of finitary operations on A,

$$\mathcal{Q}A = \{Q\colon A^{\omega} \longrightarrow A \mid \exists n\ \forall \boldsymbol{a}, \boldsymbol{b} \in A^{\omega}(\forall x < n.\ \boldsymbol{a}(x) = \boldsymbol{b}(x) \Rightarrow Q(\boldsymbol{a}) = Q(\boldsymbol{b}))\}.$$

In words, the universe $B = \mathcal{R}A \cup \mathcal{Q}A$ consists of the sets of and operations on infinite sequences of elements of A that depend only on initial segments of uniform length.

For $\boldsymbol{a} \in A^{\omega}$ and any sequence $f_0,\ldots,f_{n-1}$ of maps from A^{ω} into A, we define $\boldsymbol{a}[f_0,\ldots,f_{n-1}]$ to be the sequence $\boldsymbol{b} \in A^{\omega}$ such that

$$\boldsymbol{b}(i) = \begin{cases} f_i(\boldsymbol{a}) \text{ if } i < n, \\ \boldsymbol{a}(i) \text{ otherwise.} \end{cases}$$

The interpretation J on Ω is as follows.

1° $\mathrm{Dom}(J(R_n)) = \mathcal{R}A \times (\mathcal{Q}A)^n$, and

$$J(R_n)(R, Q_0,\ldots, Q_{n-1}) \simeq \{\boldsymbol{a} \in A^{\omega} \mid \boldsymbol{a}[Q_0,\ldots, Q_{n-1}] \in R\}.$$

2° $\mathrm{Dom}(J(Q_n)) = (\mathcal{Q}A)^{n+1}$; if $F, G_0,\ldots, G_{n-1} \in \mathcal{Q}A$, then

$$J(Q_n)(F, G_0,\ldots, G_{n-1}) = \lambda \boldsymbol{a} \in A^{\omega}.\ F(\boldsymbol{a}[G_0,\ldots, G_{n-1}]).$$

3° $J(\approx)$ maps $(\mathcal{Q}A)^2$ into $\mathcal{R}A$; $J(\approx)(F, G) = \{\boldsymbol{a} \in A^{\omega} \mid F(\boldsymbol{a}) = G(\boldsymbol{a})\}$.

4° $J(v_n) \in A^{A^{\omega}}$; it maps $\boldsymbol{a} \in A^{\omega}$ to $\boldsymbol{a}(n)$.

5° $J(\wedge) = \lambda R, S \in \mathcal{R}A.\, R \cap S$; $J(\vee) = \lambda R, S \in \mathcal{R}A.\, R \cup S$; $J(\rightarrow) = \lambda R, S \in \mathcal{R}A.\, (A^{\omega} - R) \cup S$; $J(\neg) = \lambda R \in \mathcal{R}A.\, A^{\omega} - R$; $J(\perp) = \emptyset$.

$$J(R_n)(R, Q_0,\ldots, Q_{n-1}) = \{\boldsymbol{a} \in A^{\omega} \mid \boldsymbol{a}[Q_0,\ldots, Q_{n-1}] \in R\}.$$

6° $J(\forall)$ and $J(\exists)$ are binary operations. They are defined only if their first argument is the interpretation of an individual variable, and the second argument belongs to $\mathcal{R}A$. Denote by $\boldsymbol{a};[d/n]$ the function that maps n to d and any $n' \in \omega - \{n\}$ to $\boldsymbol{a}(n')$. Then if $f = J(v_n)$, $R \in \mathcal{R}A$,

$$J(\forall)(f, R) = \bigcap_{d \in A} \{\boldsymbol{a} \in A^{\omega} \mid \boldsymbol{a};[d/n] \in R\};$$
$$J(\exists)(f, R) = \bigcup_{d \in A} \{\boldsymbol{a} \in A^{\omega} \mid \boldsymbol{a};[d/n] \in R\}.$$

Now consider a term algebra $\mathbf{T}_{\Omega}(P)$. A map $\alpha\colon P \longrightarrow B$ assigns to a variable p either an essentially finitary operation (in $\mathcal{Q}A$) or an essentially finitary relation (in $\mathcal{R}A$). So α forces a partition of P into predicate letters and function letters. The *formulas*, under this partition, are the composite elements of $\mathrm{Dom}(\alpha^*)$ that are mapped to $\mathcal{R}A$. In general, a P-ary term φ of type Ω is a formula if there exists a map $\alpha\colon P \longrightarrow B$ such that $\alpha^*(\varphi) \in \mathcal{R}A$.

*This description of first order logic is not entirely standard. Normally, we would insist that a symbol be combined with exactly the number of terms specified by its arity. In the present description, no arities are given. Every letter is assigned a maximal number of arguments that can make a difference, and this would be a natural choice of arity, but these arities are not enforced. If p is assigned a relation of arity n in this sense, then still $R_k p \mathbf{t}_1 \ldots \mathbf{t}_k$ will be meaningful, as long as $\mathbf{t}_1,\ldots, \mathbf{t}_k$ are meaningful as terms, whatever k is. If k is too small, $R_k p \mathbf{t}_1 \ldots \mathbf{t}_k$ is treated as $p \mathbf{t}_1 \ldots \mathbf{t}_k v_{k+1} \ldots v_n$ in the standard notation; if k is too large, as $p \mathbf{t}_1 \ldots \mathbf{t}_n$.

*Stepping over these anomalies, if a P-ary term φ is a formula relative to α, then we have

$$\langle A, \alpha \rangle \models \varphi(\boldsymbol{a}) \Leftrightarrow \boldsymbol{a} \in \alpha^*(\varphi).$$

(c) Let $\mathcal{R}$ consist of + and · (binary), – (unary), and the constant symbols 0 and 1. Then $\mathbf{T}_{\mathcal{R}}(\{x_1,\dots, x_n\})$, or, as we shall prefer, $\mathbf{T}_{\mathcal{R}}(x_1,\dots, x_n)$, is related to the ring of polynomials with integer coefficients. It is only related, for it is definitely not a ring; for example, $x_1 + x_2 \neq x_2 + x_1$, if, as convention requires, $x_1 \neq x_2$. There will be more on this later.

32.11 Notation and terminology. Let $\mathbf{T}_{\mathcal{N}}(X)$ be a term algebra, and

$$\boldsymbol{a}\colon X \longrightarrow A$$

a map into the universe of an algebra $\mathbf{A}$. Such a map is called an *assignment*, of values to the variables. We shall usually write a_x instead of $\boldsymbol{a}(x)$, and a_i instead of $\boldsymbol{a}(x_i)$. Let $\boldsymbol{a}^*$ be the greatest homomorphism extending $\boldsymbol{a}$ — it exists by Theorem 7. For $\mathbf{t} \in T_{\mathcal{N}}(X)$ we usually write $\mathbf{t}^{\mathbf{A}}(\boldsymbol{a})$ instead of $\boldsymbol{a}^*(\mathbf{t})$. By this notation we stress that terms have the character of composite operation symbols. The interpretation of $\mathbf{t}$ acts on an X-indexed family $\boldsymbol{a}$, just as the interpretation of an k-ary operation symbol Q acts on a k-indexed family $\langle a_0,\dots, a_{k-1}\rangle$. Indeed, denoting $i < k$ by x_i, we have

$$(Qx_0\dots x_{k-1})^{\mathbf{A}} = Q^{\mathbf{A}}.$$

This may be generalized:

32.12 Theorem. Let $\mathbf{A}$ be an algebra, $\mathcal{N} = \operatorname{Nom}\mathbf{A}$, and $k \in \mathbb{N}$. Then

$$\phi\colon \mathbf{t} \longmapsto \mathbf{t}^{\mathbf{A}}$$

is a homomorphism of $\mathbf{T}_{\mathcal{N}}(k)$ onto $\mathbf{Clo}_k\,\mathbf{A}$.

Proof. Take any $a_0,\dots, a_{k-1} \in A$, and let $\alpha = \langle a_0,\dots, a_{k-1}\rangle$, that is, α maps the variable i (which we denote by x_i) to a_i. Suppose $\mathbf{t} = Q\mathbf{t}_0\dots\mathbf{t}_{n-1}$. Then

(5) $\qquad \phi(\mathbf{t})(a_0,\dots, a_{k-1}) = \mathbf{t}^{\mathbf{A}}(\alpha) = \alpha^*(\mathbf{t}) = Q^{\mathbf{A}}(\alpha^*(\mathbf{t}_0),\dots, \alpha^*(\mathbf{t}_{n-1}))$

since α^* is a homomorphism; hence

(6) $\phi(\mathbf{t})(a_0,\dots, a_{k-1}) = Q^{\mathbf{A}}(\mathbf{t}_0^{\mathbf{A}}(\alpha),\dots, \mathbf{t}_{n-1}^{\mathbf{A}}(\alpha)) = Q^{\mathbf{A}}(\mathbf{t}_0^{\mathbf{A}},\dots, \mathbf{t}_{n-1}^{\mathbf{A}})(a_0,\dots, a_{k-1})$,

where in the last step, the notation $Q^{\mathbf{A}}(\dots)$ changes meaning from 'application of $Q^{\mathbf{A}}$' to 'k-ary composition with $Q^{\mathbf{A}}$'. So (6) may be rendered as

(7) $\qquad \phi(\mathbf{t})(a_0,\dots, a_{k-1}) = Q^{\mathbf{Clo}_k\mathbf{A}}(\mathbf{t}_0^{\mathbf{A}},\dots, \mathbf{t}_{n-1}^{\mathbf{A}})(a_0,\dots, a_{k-1})$.

Since (7) holds for any $a_0,\dots, a_{k-1} \in A$, we conclude

(8) $\qquad \phi(Q\mathbf{t}_0\dots\mathbf{t}_{n-1}) = Q^{\mathbf{Clo}_k\mathbf{A}}(\phi(\mathbf{t}_0),\dots, \phi(\mathbf{t}_{n-1}))$,

showing ϕ is a homomorphism from $\mathbf{T}_{\mathcal{N}}(k)$ into $\mathbf{Clo}_k\,\mathbf{A}$.

To prove surjectivity, it will suffice, by Theorem 31.3, to show that $\operatorname{Ran}\phi$ (1°) contains the k-ary projection operations and (2°) is closed under k-ary composition by basic operations.

1° With $a_0,\dots, a_{k-1}$ and α as above, for all $i < k$,

$$\phi(x_i)(a_0,\dots, a_{k-1}) = \alpha(x_i) = a_i = \pi_i^k(a_0,\dots, a_{k-1}),$$

so $\pi_i^k \in \operatorname{Ran}\phi$.

2° Let $Q \in \mathcal{N}$ be n-ary. Suppose $f_0,\dots,f_{n-1} \in \operatorname{Ran}\phi$; say

$$f_0 = \phi(\mathbf{t}_0),\ \dots,\ f_{n-1} = \phi(\mathbf{t}_{n-1}).$$

Then by (8),

$$\phi(Q\mathbf{t}_0\ldots\mathbf{t}_{n-1}) = Q^{\mathbf{Clo}_k\mathbf{A}}(\phi(\mathbf{t}_0),\ldots, \phi(\mathbf{t}_{n-1})) = Q^{\mathbf{A}}(f_0,\ldots,f_{n-1}),$$

so Ran ϕ is closed under k-ary composition by $Q^{\mathbf{A}}$ as required. ⊠

We defined terms (Definition 1) with reference to a nominator $\mathcal{N}$ and a set X of variables. By itself, however, a term is just a word; and there are many $\mathcal{N}$ and X over which (for example) the three-letter word Qxy is a term. The nominator $\mathcal{N}$ could be $\{Q\}$, $\{P, Q\}$, $\{Q, R\}$ and so on; $\{x, y\}$ could be the set of variables, or $\{x, y, z\}$, etcetera. In principle it is even thinkable that x and/or y belongs to the nominator; but this last freedom we shall rule out: a term is not just a word, but also the knowledge which of its letters are operation symbols — including their arities — and which are variables.

32.13 Definition. Let $\mathbf{t}$ be a term; then Nom($\mathbf{t}$) is the set of all operation symbols that occur in $\mathbf{t}$, and Var($\mathbf{t}$) the set of variables that occur in $\mathbf{t}$.

If Nom($\mathbf{t}$) $\subseteq \mathcal{N}$ and Var($\mathbf{t}$) $\subseteq X$, then $\mathbf{t} \in T_{\mathcal{N}}(X)$. Now let $\mathbf{T}_{\mathcal{M}}(Y)$ be a term algebra with $\mathcal{M} \supseteq \mathcal{N}$ and $Y \supseteq X$. Then $\mathbf{t} \in T_{\mathcal{M}}(Y)$ as well; indeed, it is easily seen that $\mathbf{T}_{\mathcal{N}}(X) \leq \mathbf{T}_{\mathcal{M}}(Y)\lceil\mathcal{N}$. Let $\mathbf{A}$ be an algebra, and β: $Y \longrightarrow A$ an assignment. Then by Corollary 8, we have a unique $\mathcal{M}$-closed extension β^*: $\mathbf{T}_{\mathcal{M}}(Y) \longrightarrow \mathbf{A}$. Since $\beta\lceil X$ is an assignment for X, by the same token we have a unique $\mathcal{N}$-closed extension $(\beta\lceil X)^*$: $\mathbf{T}_{\mathcal{N}}(X) \longrightarrow \mathbf{A}$. Now following Definition 11 we may write $\mathbf{t}^{\mathbf{A}}(\beta) \simeq \beta^*(\mathbf{t})$ and $\mathbf{t}^{\mathbf{A}}(\beta\lceil X) \simeq (\beta\lceil X)^*(\mathbf{t})$, suggesting the existence of an object $\mathbf{t}^{\mathbf{A}}$ working on assignments from different sets. A felicitous suggestion, or confusing? Does it matter whether we think of $\mathbf{t}$ as an element of $\mathbf{T}_{\mathcal{M}}(Y)$ or an element of $\mathbf{T}_{\mathcal{N}}(X)$? The following theorem implies that it does not.

32.14 Proposition. Let f: $\mathbf{T}_{\mathcal{N}}(X) \longrightarrow \mathbf{T}_{\mathcal{M}}(Y)$ be a homomorphism, $\mathbf{t} \in T_{\mathcal{N}}(X)$, $\mathbf{A}$ an algebra, and β: $Y \longrightarrow A$ an assignment with $\mathcal{M}$-closed extension β^*. Then $f(\mathbf{t})^{\mathbf{A}}(\beta) \simeq \mathbf{t}^{\mathbf{A}}(\beta^*\circ f\lceil X)$.

Proof. Let $(\beta^*\circ f\lceil X)^*$ be the $\mathcal{N}$-closed extension of $\beta^*\circ f\lceil X$. Note that the existence of f and $\mathbf{t}$ implies that $\mathcal{M} \supseteq \mathcal{N}$.

$$\begin{tikzcd}
\mathbf{T}_{\mathcal{N}}(X) \arrow[rr, "f"] \arrow[drr, "(\beta^*\circ f\lceil X)^*"'] & & \mathbf{T}_{\mathcal{M}}(Y) \arrow[d, "\beta^*"] \\
& & \mathbf{A}
\end{tikzcd}$$

Since $\mathbf{T}_{\mathcal{N}}(X)$ is total, f is $\mathcal{N}$-closed; and β^* is $\mathcal{M}$-closed, hence $\mathcal{N}$-closed; so β^*f is $\mathcal{N}$-closed. Moreover β^*f is an extension of $\beta^*f\lceil X$; therefore, $(\beta^*f\lceil X)^*$ being the unique $\mathcal{N}$-closed extension of $\beta^*f\lceil X$, $\beta^*f = (\beta^*f\lceil X)^*$. So $f(\mathbf{t})^{\mathbf{A}}(\beta) \simeq \beta^*(f(\mathbf{t})) \simeq (\beta^*f\lceil X)^*(\mathbf{t}) \simeq \mathbf{t}^{\mathbf{A}}(\beta^*f\lceil X)$. ⊠

32.15 Corollary. Let $\mathbf{T}_{\mathcal{M}}(Y)$ and $\mathbf{T}_{\mathcal{N}}(X)$ be term algebras, with $\mathcal{M} \supseteq \mathcal{N}$ and $Y \supseteq X$, $\mathbf{t} \in T_{\mathcal{N}}(X)$, $\mathbf{A}$ an algebra, and β: $Y \longrightarrow A$ an assignment. Then

$$\mathbf{t}^{\mathbf{A}}(\beta) \simeq \mathbf{t}^{\mathbf{A}}(\beta\lceil X).$$

Proof. Let β^* be the $\mathcal{M}$-closed extension of β, and ι be the identical embedding of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{T}_{\mathcal{M}}(Y)$. Applying the Proposition, we have

$$\mathbf{t}^{\mathbf{A}}(\beta) \simeq \iota(\mathbf{t})^{\mathbf{A}}(\beta) \simeq \mathbf{t}^{\mathbf{A}}(\beta^{*} \circ \iota\lceil X) \simeq \mathbf{t}^{\mathbf{A}}(\beta\lceil X). \qquad \boxtimes$$

32.16 Corollary. Let $\mathbf{t}$ be a term, $\mathbf{A}$ an algebra, and α and β assignments in A such that $\alpha\lceil\mathrm{Var}(\mathbf{t}) = \beta\lceil\mathrm{Var}(\mathbf{t})$. Then $\mathbf{t}^{\mathbf{A}}(\alpha) \simeq \mathbf{t}^{\mathbf{A}}(\beta)$.

Proof. By the previous corollary,

$$\mathbf{t}^{\mathbf{A}}(\alpha) \simeq \mathbf{t}^{\mathbf{A}}(\alpha\lceil\mathrm{Var}(\mathbf{t})) \simeq \mathbf{t}^{\mathbf{A}}(\beta\lceil\mathrm{Var}(\mathbf{t})) \simeq \mathbf{t}^{\mathbf{A}}(\beta). \qquad \boxtimes$$

Let $\mathbf{t} \in T_{\mathcal{N}}(X)$ be a term, $\mathbf{A}$ an algebra. We may consider $\mathbf{t}^{\mathbf{A}}$ as a function of X-indexed families, just as n-ary operations are functions of n-indexed families. Definition 31.12 generalizes to this case as follows.

32.17 Definition. Let f be a function of X-indexed families in a set A, $i \in X$. Then f is *i-dependent*, or *depends on the index i*, if there exist α, β: $X \longrightarrow A$ such that for all $j \in X - \{i\}$, $\alpha(j) = \beta(j)$, and $f(\alpha) \not\simeq f(\beta)$. We call f *i-independent* if it is not i-dependent.

Now Corollary 16 implies, for $\mathbf{t} \in T_{\mathcal{N}}(X)$, $i \in X$, and an algebra $\mathbf{A}$, that $\mathbf{t}^{\mathbf{A}}$ depends on i only if $i \in \mathrm{Var}(\mathbf{t})$.

Another interesting special case of Proposition 14 arises when $\mathbf{A} = \mathbf{T}_{\mathcal{M}}(Y)$ and β is the identical embedding of Y. It is customary to identify, more or less, terms with the term operations they induce in term algebras. So when $\mathbf{t}$ is a k-ary term, $\mathbf{t} = \mathbf{t}(x_0,\ldots, x_{k-1})$ (instead of $\mathbf{t}^{\mathbf{T}\cdots}(x_0,\ldots, x_{k-1})$), and so on. Even when $\mathbf{t}$ is a infinitary term, we may write $\mathbf{t} = \mathbf{t}(x_0,\ldots, x_{k-1})$ if all the variables that actually occur in $\mathbf{t}$ are among $x_0,\ldots, x_{k-1}$, thus presenting $\mathbf{t}$ as an interpretation of a k-ary term. We have:

32.18 Corollary. Let f: $\mathbf{T}_{\mathcal{N}}(X) \longrightarrow \mathbf{T}_{\mathcal{M}}(Y)$ be a homomorphism, $\mathbf{t} \in T_{\mathcal{N}}(X)$. Then $f(\mathbf{t}) = \mathbf{t}(f\lceil X)$.

In other words, the action of a homomorphism of term algebras consists in uniformly replacing variables by terms: such homomorphisms are *substitutions*.

32.19 Corollary. Let X and Y be equipollent sets, disjoint with the nominator $\mathcal{N}$. Then $\mathbf{T}_{\mathcal{N}}(X) \cong \mathbf{T}_{\mathcal{N}}(Y)$.

The subuniverse generated by a set $X \subseteq A$ in an algebra $\mathbf{A}$ is the closure of X under the basic operations of $\mathbf{A}$ (§22). By Theorem 31.14, $\mathrm{Sg}^{\mathbf{A}}(X)$ is also the closure under the term operations of $\mathbf{A}$; and more, every element of $\mathrm{Sg}^{\mathbf{A}}(X)$ may be reached from X by just one application of a term operation. There are several ways of phrasing this with terms.

32.20 Corollary. Let $\mathbf{A}$ be an algebra, $\mathcal{N} = \mathrm{Nom}\,\mathbf{A}$, and $X \subseteq A$. Then

$$\begin{aligned} \mathrm{Sg}^{\mathbf{A}}X &= \{\mathbf{t}^{\mathbf{A}}[X^{*}] \mid \mathbf{t} \in \bigcup_{k\in\mathbb{N}} T_{\mathcal{N}}(k)\} && \text{(a)}\\ &= \{\mathbf{t}^{\mathbf{A}}[X^{\omega}] \mid \mathbf{t} \in T_{\mathcal{N}}(\omega)\} && \text{(b)}\\ &= \{\mathbf{t}^{\mathbf{A}}(1_A^X) \mid \mathbf{t} \in T_{\mathcal{N}}(X)\}. && \text{(c)} \end{aligned}$$

Proof. (a) By 31.14 and Theorem 12.
(b) For any $\mathbf{t} \in T_{\mathcal{N}}(\omega)$ and $x_0,\ldots, x_n,\ldots$ $(n \in \omega)$, there exists k such that $\mathbf{t}^{\mathbf{A}}(x_0,\ldots, x_n,\ldots) \simeq \mathbf{t}^{\mathbf{A}}(x_0,\ldots, x_{k-1})$; and conversely $\mathbf{T}_{\mathcal{N}}(k) \leq \mathbf{T}_{\mathcal{N}}(\omega)$.

(c) By 14.3.6 and 31.14(iii),

$$\mathrm{Sg}^{\mathbf{A}}X = \bigcup_{X_0 \in p(X)} \mathrm{Sg}^{\mathbf{A}}X_0 = \{s(x_0 \ldots, x_{n-1}) \mid n \in \mathbb{N}, x_0, \ldots, x_{n-1} \in X \text{ and } s \in \mathrm{Clo}_n\mathbf{A}\}.$$

By Theorem 12, the term operations $s \in \mathrm{Clo}_n\mathbf{A}$ are the term operations $\mathbf{s}^{\mathbf{A}}$ for $\mathbf{s} \in T_{\mathcal{N}}(n)$. Now consider $\mathbf{t} = \mathbf{s}(x_0, \ldots, x_{n-1}) \in T_{\mathcal{N}}(X_0)$, where

$$X_0 = \{x_0, \ldots, x_{n-1}\}.$$

Let α be the mapping $x_i \mapsto i$. By Proposition 14,

$$\mathbf{s}^{\mathbf{A}}(x_0, \ldots, x_{n-1}) \simeq \mathbf{t}^{\mathbf{A}}(\langle x_0, \ldots, x_{n-1}\rangle \circ \alpha) \simeq \mathbf{t}^{\mathbf{A}}(1_A^{X_0}).$$

And finally, if X_0 is the set of variables actually occurring in $\mathbf{t} \in T_{\mathcal{N}}(X)$, then by Corollary 15, $\mathbf{t}^{\mathbf{A}}(1_A^X) \simeq \mathbf{t}^{\mathbf{A}}(1_A^{X_0})$. ☒

Example. There is no general upper bound on the arity of terms required to obtain the generated subuniverse. To see this, take for each $m \in \mathbb{N}$ an m-ary operation symbol Q_m; let I be the interpretation of these symbols in the universe $\mathbb{N}$ defined by

$$I(Q_m)(n_0, \ldots, n_{m-1}) = \begin{cases} 2m+1 & \text{if } n_0, \ldots, n_{m-1} \text{ are distinct even numbers,} \\ n_0 & \text{otherwise.} \end{cases}$$

Put $\mathbf{N} = \langle \mathbb{N}, I\rangle$, and let $2\mathbb{N}$ be the set of even numbers. It is easy to see, by induction on terms, that if $\mathbf{t}$ is a k-ary term and $n_0, \ldots, n_{k-1}$ are even,

$$\mathbf{t}^{\mathbf{A}}(n_0, \ldots, n_{k-1}) \in \{n_0, \ldots, n_{k-1}\} \cup \{1, \ldots, 2k+1\}.$$

So $\bigcup\{\mathbf{t}^{\mathbf{A}}[(2\mathbb{N})^k] \mid \mathbf{t} \in T_{\mathrm{Nom}(\mathbf{N})}(k)\} \neq \mathrm{Sg}^{\mathbf{N}}2\mathbb{N}$.

Homomorphisms preserve term operations:

32.21 Theorem. Let f: $\mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism, α: $X \longrightarrow A$ an assignment; let $\mathbf{t} \in T_{\mathcal{N}}(X)$ have interpretation $\mathbf{t}^{\mathbf{A}}(\alpha)$ in $\mathbf{A}$. Then

$$f(\mathbf{t}^{\mathbf{A}}(\alpha)) = \mathbf{t}^{\mathbf{B}}(f \circ \alpha).$$

Proof. Let α^* be the unique $\mathcal{N}$-closed homomorphism from an X-generated relative subalgebra of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{A}$ that extends α, and $(f \circ \alpha)^*$ the unique $\mathcal{N}$-closed homomorphism from an X-generated relative subalgebra of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{B}$ that extends $f \circ \alpha$. Then $f \circ \alpha^*$ is a homomorphism from an X-generated relative subalgebra of $\mathbf{T}_{\mathcal{N}}(X)$ into $\mathbf{B}$ that extends $f \circ \alpha$; and since by Theorem 7.4 $(f \circ \alpha)^*$ is the *largest* such homomorphism, we have

$$f \circ \alpha^* \subseteq (f \circ \alpha)^*.$$

So $f(\mathbf{t}^{\mathbf{A}}(\alpha)) = f(\alpha^*(\mathbf{t})) = (f \circ \alpha^*)(\mathbf{t}) = (f \circ \alpha)^*(\mathbf{t}) = \mathbf{t}^{\mathbf{B}}(f \circ \alpha)$. ☒

32.22 Corollary. Let f: $\mathbf{A} \longrightarrow \mathbf{B}$ be a closed homomorphism, α: $X \longrightarrow A$ an assignment, and $\mathbf{t} \in T_{\mathcal{N}}(X)$. Then $f(\mathbf{t}^{\mathbf{A}}(\alpha)) \simeq \mathbf{t}^{\mathbf{B}}(f \circ \alpha)$.

Proof. Following the proof of the theorem, we now get that $f \circ \alpha^*$ is an $\mathcal{N}$-closed extension of $f \circ \alpha$; since by Corollary 8 $(f \circ \alpha)^*$ is the *only* such extension, we have $f \circ \alpha^* = (f \circ \alpha)^*$. So

$$f(\mathbf{t}^{\mathbf{A}}(\alpha)) \simeq f(\alpha^*(\mathbf{t})) \simeq (f \circ \alpha^*)(\mathbf{t}) \simeq (f \circ \alpha)^*(\mathbf{t}) \simeq \mathbf{t}^{\mathbf{B}}(f \circ \alpha).$$ ☒

32.23 Corollary. Let $\mathbf{B}$ be an algebra and $\mathbf{t}$ a term. If $\mathbf{A} \leq \mathbf{B}$, then $\mathbf{t}^{\mathbf{A}} = (\mathbf{t}^{\mathbf{B}})_A$.

Proof. Let f in the previous corollary be the identical embedding. The domain of $\mathbf{t}^{\mathbf{A}}$ is included in A^X, and for $\alpha \in A^X$, $\mathbf{t}^{\mathbf{A}}(\alpha) \simeq \mathbf{t}^{\mathbf{B}}(\alpha)$. $\boxtimes$

Example. Theorem 4 does not hold for *relative* subalgebras. Let $A = \{0,1,2\}$, $\mathbf{A} = \langle A, S\rangle$ with $S0 = 1$, $S1 = 2$, $S2 = 2$, and $\mathbf{B}$ the relative subalgebra with universe $\{0, 2\}$. Take $\mathbf{t} = SSv_0$; then $\mathbf{t}^{\mathbf{A}}(0) = 2$, but $\mathbf{t}^{\mathbf{B}}(0)\uparrow$, since $1 \notin B$.

In a natural way, the terms over a nominator $\mathcal{N}$ form an algebra $\mathbf{T} = \mathbf{T}(\mathrm{Var})_{\mathcal{N}}$ of type $\mathcal{N}$: its universe is the set $T(\mathrm{Var})_{\mathcal{N}}$, and the operations are defined by

$$Q^{\mathbf{T}}(\mathbf{t}_0,\dots, \mathbf{t}_{n-1}) = Q\mathbf{t}_0\dots\mathbf{t}_{n-1}.$$

Terms may be interpreted in $\mathbf{T}$ as in any algebra. Usually, instead of $\mathbf{t}^{\mathbf{T}}(\mathbf{s}_0,\dots, \mathbf{s}_{k-1})$ we write $\mathbf{t}(\mathbf{s}_0,\dots, \mathbf{s}_{k-1})$; we call this term an *instance* of $\mathbf{t}$, obtained by *substitution* of $\mathbf{s}_0$ for $v_0,\dots, \mathbf{s}_{k-1}$ for v_{k-1}. If the substituted terms $\mathbf{s}_0, \dots, \mathbf{s}_{k-1}$ are variables, we call $\mathbf{t}(\mathbf{s}_0,\dots, \mathbf{s}_{k-1})$ a *variant* of $\mathbf{t}$.

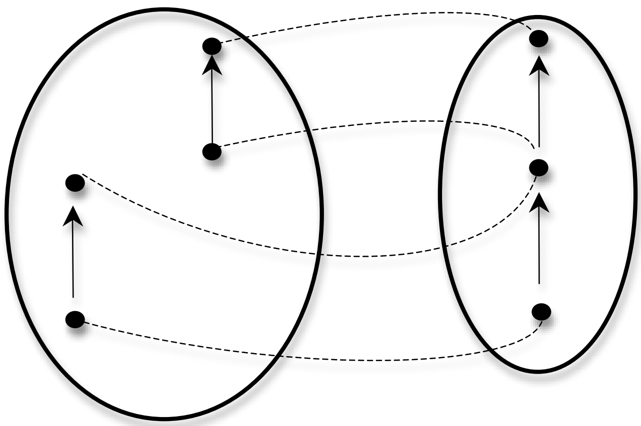
29.6 Theorem. Let f: $\mathbf{A} \longrightarrow \mathbf{B}$ be a homomorphism, $\mathbf{t}$ a k-ary term, and $a_0, \dots, a_{k-1} \in A$.
(i) If f is reflective, then

$$\mathbf{t}^{\mathbf{B}}(f(a_0),\dots,f(a_{k-1})) \in \mathrm{Ran} f \text{ implies } \mathbf{t}^{\mathbf{A}}(a_0,\dots, a_{k-1})\downarrow.$$

(ii) If f is closed, then $\mathbf{t}^{\mathbf{B}}(f(a_0),\dots,f(a_{k-1}))\downarrow$ implies $\mathbf{t}^{\mathbf{A}}(a_0,\dots, a_{k-1})\downarrow$.

Proof. By induction on terms. If $\mathbf{t} = v_i$, the succedent is true unconditionally. Now assume $\mathbf{t} = Q\mathbf{t}_0\dots\mathbf{t}_{n-1}$. By induction hypothesis for each $j < n$ $\mathbf{t}_j^{\mathbf{A}}(a_0,\dots, a_{k-1})\downarrow$. Let $c_j = \mathbf{t}_j^{\mathbf{A}}(a_0,\dots, a_{k-1})$.
(i) For some $c_n, f(c_n) = \mathbf{t}^{\mathbf{B}}(f(a_0),\dots, f(a_{k-1})) = Q^{\mathbf{B}}(f(c_0),\dots, f(c_{n-1}))$. Since f is reflective, $c_n = Q^{\mathbf{A}}(c_0,\dots, c_{n-1})$. Hence $\mathbf{t}^{\mathbf{A}}(a_0,\dots, a_{k-1})\downarrow$.
(ii) Since $\mathbf{t}^{\mathbf{B}}(f(a_0),\dots, f(a_{k-1})) = Q^{\mathbf{B}}(f(c_0),\dots, f(c_{n-1}))$, $\langle f(c_0),\dots, f(c_{n-1})\rangle \in \mathrm{Dom}\, Q^{\mathbf{B}}$. Since f is closed, it follows that $\langle c_0,\dots, c_{n-1}\rangle \in \mathrm{Dom}\, Q^{\mathbf{A}}$. So $\mathbf{t}^{\mathbf{A}}(a_0,\dots, a_{k-1})\downarrow$. $\boxtimes$

Example. Theorem 6 does not hold for *weakly* reflective or closed homomorphisms. The diagram below suggests a counterexample.



$\mathbf{A}_A$, en algemener, voor $B \subseteq A$, $\mathbf{A}_B$. Polynomen als termen over $\mathrm{Nom}\,\mathbf{A}_A$, en algemener, voor $B \subseteq A$, B-Polynomen als termen over $\mathrm{Nom}\,\mathbf{A}_B$.

§ Nominal Clones

Exercises

§A

1. Prove Lemma 2.

2. Let A be a finite nonvoid set. Prove that $\mathbf{Op}(A)$ is generated by a single operation.

3. Let A be a nonvoid set. For all $k > 0$ and every nullary operation a on A, let $\gamma^k(a)$ be the constant k-ary operation with the same value as a if a is defined, and the void operation otherwise. For $m > 0$, let γ_m be the operation that takes an m-ary operation f_0 and m operations $f_1,\ldots,f_m$ of the same arity to the composite $f_0(f_1,\ldots,f_m)$. Let $\mathbf{O}(A)$ be the algebra with universe $\mathrm{Op}(A)$, the projection operators for constants, unary partial operations γ^k for all $k > 0$, and $(m+1)$-ary operations γ_m for all $m > 0$. Prove that $\mathbf{O}(A)$ has the same subuniverses as $\mathbf{Op}(A)$.

3. Let $\mathfrak{k}$ be an infinite cardinal. Prove that the number of clones on a set of cardinality $\mathfrak{k}$ is precisely $2^{2^{\mathfrak{k}}}$.

4. Let A and B be nonvoid sets, and f a mapping from A to B.
(a) Show that there exists a homomorphism $\phi = \mathrm{Op}(f)$: $\mathbf{Op}(A) \to \mathbf{Op}(B)$ that maps, for every $a \in A$, the nullary operation with value a to the nullary operation with value $f(a)$, if and only if $|B| = 1$ or f is injective.
(b) Show that if such a homomorphism ϕ exists, there exists one that maps total operations to total operations.

5. Suppose $\emptyset \neq A \subseteq B$; define ϕ: $\mathrm{Op}(B) \to \mathrm{Op}(A)$ by $\phi(g) = g_A$. Show that ϕ is a homomorphism if and only if $A = B$.

6. The *Mal'cev operations* are defined as follows. For an n-ary operation f, $n \geq 2$,
$(\xi f)(x_1,\ldots,x_n) \simeq f(x_2,\ldots,x_n,x_1)$;
$(\tau f)(x_1,\ldots,x_n) \simeq f(x_2,x_1,x_3,\ldots,x_n)$;
$(\Delta f)(x_1,\ldots,x_{n-1}) \simeq f(x_1,x_1,x_2,\ldots,x_{n-1})$;
for unary or nullary f, $\xi f = \tau f = \Delta f = f$. For any m and n, n-ary operation f, and m-ary operation g, $(\nabla f)(x_1,\ldots,x_{n+1}) \simeq f(x_2,\ldots,x_{n+1})$;
$(f * g)(x_1,\ldots,x_{m+n-1}) \simeq f(g(x_1,\ldots,x_m),x_{m+1},\ldots,x_{m+n-1})$.
Let $\mathbf{M}(A) = \langle \mathrm{Op}(A), \xi, \tau, \Delta, \nabla, *, 1_A \rangle$ be the algebra of the Mal'cev operations on $\mathrm{Op}(A)$. Prove that the subuniverses of $\mathbf{M}(A)$ are precisely the clones on A.

§31

1. Let $\mathbf{A}$ be an algebra. Show that $\mathbf{A}$ is minimal if and only if $\mathbf{A} \cong \mathbf{Clo}_0\mathbf{A}$.

2. Let $\mathbf{R}$ be a commutative ring with identity element.
(a) Verify that the term operations of $\mathbf{R}$ are the operations determined by polynomials (in the sense of classical algebra) with integer coefficients.
(b) The same for polynomial operations and polynomials with coefficients from $\mathbf{R}$.

§32

1. Let Ω be a type, and X a set of variables; $A := \Omega \cup X$. Define the weight function g on A^+ as in the proof of the Unique Readability Theorem. Suppose $w = a_1\ldots a_k \in A^+$ has weight $g(w) = 1$. Prove that there is a unique cyclic variant $w' = a_i\ldots a_k a_1\ldots a_{i-1}$ of w that is a term.

2. Let **A** be a nonvoid algebra, and X a nonvoid set. Show that there exists a homomorphism of $\mathbf{T}_\Omega(X)$ *onto* **A** (that is, with surjective underlying map) only if $\mathbf{A}\lceil\Omega$ is a total algebra.

3. Show: if $Y \subseteq X$ and $\Omega \subseteq \Omega'$, then $\mathbf{T}_\Omega(Y) \leq \mathbf{T}_{\Omega'}(X)\lceil\Omega$.

4. Prove Corollary 18.

1. (a) Let **A** be an algebra. Show that if the arity of operation symbols in Nom(**A**) is at most 1, then for any $X \subseteq A$, $\mathrm{Sg}^{\mathbf{A}}X = \bigcup\{\mathbf{t}^{\mathbf{A}}[X] \mid \mathbf{t} \in T(\mathrm{Var})_{\mathrm{Nom}(\mathbf{A})}\}$.
(b) Construct an example to show that it is not true that, if the arity of operation symbols in Nom(**A**) is at most 2, then for any $X \subseteq A$, $\mathrm{Sg}^{\mathbf{A}}X = \bigcup\{\mathbf{t}^{\mathbf{A}}[X^2] \mid \mathbf{t} \in T(\mathrm{Var})_{\mathrm{Nom}(\mathbf{A})}\}$.

n. standaard algebraïseringen van de predikaatlogica: cylindrische algebra's, polyadische algebra's.