CHAPTER 5

SUBALGEBRAS AND UNIONS

§a Weak and relative subalgebras

The subalgebra concept has two very natural weakenings. The first of these concerns the algebraic structure that is inherited by arbitrary subsets of the universe of an algebra, as opposed to subuniverses; one of the more compelling ways in which partial algebras arise in the study of total algebras. The second corresponds with the algebraic structure necessary for a mapping from another algebra to be a homomorphism.

a1 Definition. Let $\mathbf{A} = \langle A, I\rangle$ and $\mathbf{B} = \langle B, J\rangle$ be algebras such that $A \subseteq B$.
(a) $\mathbf{A}$ is a *relative subalgebra* of $\mathbf{B}$, notation $\mathbf{A} \subseteq \mathbf{B}$, and $\mathbf{B}$ a *outer extension* of $\mathbf{A}$, if $\mathrm{Dom}(I) = \mathrm{Dom}(J)$, and for every symbol $S \in \mathrm{Dom}(J)$, $I(S) = J(S)_A$.
(b) $\mathbf{A}$ is a *weak subalgebra* of $\mathbf{B}$, notation $\mathbf{A} \subseteq_{\mathrm{w}} \mathbf{B}$, if $\mathrm{Dom}(I) = \mathrm{Dom}(J)$, and for every symbol $S \in \mathrm{Dom}(I)$, $I(S) \subseteq J(S)$.

Observe that any subset A of the universe of an algebra $\mathbf{B}$ determines a unique relative subalgebra of $\mathbf{B}$ that has A for universe. We denote it by $A_{\mathbf{B}}$. On the other hand, there may be many weak subalgebras of $\mathbf{B}$ that have A for a universe. In particular, there may be many weak subalgebras of $\mathbf{B}$ that have B for a universe; we shall call them *weakenings* of $\mathbf{B}$. If $\mathbf{A}$ is a weakening of $\mathbf{B}$, then $\mathbf{B}$ is an *inner extension* of $\mathbf{A}$.

If $\mathbf{B}$ is a total algebra, and A is dense in $\mathbf{B}$, we say that $\mathbf{B}$ is a *completion* of $A_{\mathbf{B}}$.

Examples
(*a*) Suppose $\mathbf{A}$ is a weak subalgebra of $\mathbf{B}$. Then 1_B^A is an embedding of $\mathbf{A}$ into $\mathbf{B}$.
(*b*) Conversely, if 1_B^A is a homomorphism of $\mathbf{A}$ into $\mathbf{B}$, and $\mathrm{Nom}\mathbf{A} = \mathrm{Nom}\mathbf{B}$, then $\mathbf{A}$ is a weak subalgebra of $\mathbf{B}$.
(*c*) Let $\mathbf{K}$ be a field. Consider the weakening $\mathbf{V}$ of $\mathbf{K}^n$ in which $a \cdot b$ is defined only if $a \in \{k\}^n$ for some $k \in K$. Then $\mathbf{V}$ is, essentially, an n-dimensional vector space over $\mathbf{K}$.
(*d*) $\mathbf{I}_{\mathbb{Q}}$ als relatieve subalgebra van een algebra met oneindige intervallen.
(*e*) Concrete categories (see 4…) seldom are weak subalgebras of **Set**. For example, different algebras may have the same universe; and a fortiori different homomorphisms may reduce to the same function.

a2 Proposition. (i) Let $\mathbf{A}$ and $\mathbf{B}$ be algebras. If $\mathbf{A} \leq \mathbf{B}$, then also $\mathbf{A} \subseteq \mathbf{B}$; and if $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A} \subseteq_{\mathrm{w}} \mathbf{B}$.
(ii) If $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A} \leq \mathbf{B}$ if and only if A is a subuniverse of $\mathbf{B}$. In particular, if all the operations of $\mathbf{B}$ are predicates, $\mathbf{A} \subseteq \mathbf{B}$ if and only if $\mathbf{A} \leq \mathbf{B}$.
(iii) The [weak/relative] subalgebras of any given algebra form a set.

Proof of (iii). By (i) and Comprehension, we need only show that the *weak* subalgebras form a set. But these may be considered a subset of

$$\mathcal{P}A \times \prod_{S \in \mathrm{Dom}(J)} \mathcal{P}(J(S)) \ . \qquad \boxtimes$$

25.3 Theorem. Let **A** be a weak subalgebra of **B**, and f: **B** $\longrightarrow$ **C** a homomorphism. Then $f \upharpoonright A$ is a homomorphism from **A** to **C**.

Proof. Observe that $f \upharpoonright A = f \circ 1^A_B$; then use 23.2(ii) and example (*b*) under Definition 1. $\boxtimes$

A4 Theorem. For each $i \in I$, let $\mathbf{B}_i$ be a relative subalgebra of $\mathbf{A}_i$. Then

$$\prod_{i \in I} \mathbf{B}_i \subseteq \prod_{i \in I} \mathbf{A}_i.$$

Proof. Put $\mathbf{A} := \prod_i \mathbf{A}_i$ and $\mathbf{B} := \prod_i \mathbf{B}_i$. Suppose $b_0, \ldots, b_n \in B$. Then

$$\begin{aligned} Q^{\mathbf{A}}(b_1, \ldots, b_n) = b_0 &\Leftrightarrow \forall i \in I\ Q^{\mathbf{A}_i}(b_1(i), \ldots, b_n(i)) = b_0(i) \\ &\Leftrightarrow \forall i \in I\ Q^{\mathbf{B}_i}(b_1(i), \ldots, b_n(i)) = b_0(i), \text{ since } \mathbf{B}_i \subseteq \mathbf{A}_i \\ &\qquad\qquad \text{and } b_0(i), \ldots, b_n(i) \in B_i \\ &\Leftrightarrow Q^{\mathbf{B}}(b_1, \ldots, b_n) = b_0. \end{aligned} \qquad \boxtimes$$

§26. Union and Intersection

If $\mathbf{A} = \langle A, I \rangle$ and $\mathbf{B} = \langle B, J \rangle$ are relational structures, we can define their union $\mathbf{A} \cup \mathbf{B}$ as $\langle A \cup B, K \rangle$, where $\mathrm{Dom}(K) = \mathrm{Dom}(I) \cup \mathrm{Dom}(J)$ and for every $K \in \mathrm{Dom}(K)$, $K(R) = R^{\mathbf{A}} \cup R^{\mathbf{B}}$. The generalization to arbitrary families of relational structures is straightforward. This approach will not work for structures with operations, as long as we hold on to the restriction that operations be single-valued. We shall have to presuppose some kind of hierarchy in the structures to be combined, so that we are able to decide which has precedence in case of conflict. In principle, the order of precedence could be different for different conflicts, but we shall have no use for this ultimate subtlety.

26.1 Definition. Let $\boldsymbol{A} = \langle \mathbf{A}_\xi \,|\, \xi < \alpha \rangle$ be a sequence of algebras, $\mathbf{A}_\xi = \langle A_\xi, I_\xi \rangle$. The *union* of $\boldsymbol{A}$ is the algebra $\mathbf{B} = \langle B, J \rangle$ where

$$B = \bigcup_{\xi < \alpha} A_\xi, \ \ \mathrm{Dom}(J) = \bigcup_{\xi < \alpha} \mathrm{Dom}(I_\xi),$$

and for every $Q \in \mathrm{Dom}(J)$,

$$\mathrm{Dom}(J(Q)) = \bigcup_{\xi < \alpha} \mathrm{Dom}(I_\xi(Q)),$$

and if ξ is the first index such that $\boldsymbol{b} \in \mathrm{Dom}(I_\xi(Q))$, $J(Q)(\boldsymbol{b}) = I_\xi(Q)(\boldsymbol{b})$.

We denote the union of a sequence $\langle \mathbf{A}_\xi | \ \xi < \alpha \rangle$ of algebras by

$$\bigcup_{\xi < \alpha} \mathbf{A}_\xi.$$

Algebras **A** and **B** are *compatible* if for every operation symbol Q, for every sequence $\boldsymbol{d} \in (A \cap B)^*$, if both $Q^{\mathbf{A}}(\boldsymbol{d})\downarrow$ and $Q^{\mathbf{B}}(\boldsymbol{d})\downarrow$, then $Q^{\mathbf{A}}(\boldsymbol{d}) = Q^{\mathbf{B}}(\boldsymbol{d})$.

An *enumeration* of a set X is a mapping of an ordinal onto X. In other words, it is a listing of all the elements of X in a sequence, with repetitions allowed.

26.2 Lemma. Let $\langle \mathbf{A}_i \,|\, i \in I\rangle$ be a family of pairwise compatible algebras; suppose $\langle \mathbf{A}_\xi \,|\, \xi < \alpha\rangle$ and $\langle \mathbf{B}_\eta \,|\, \eta < \beta\rangle$ are enumerations of $\{\mathbf{A}_i \,|\, i \in I\}$. Then

$$\bigcup_{\xi<\alpha} \mathbf{A}_\xi = \bigcup_{\eta<\beta} \mathbf{B}_\eta.$$

Proof. Let

$$\mathbf{C} := \bigcup_{\xi<\alpha} \mathbf{A}_\xi \text{ and } \mathbf{D} := \bigcup_{\eta<\beta} \mathbf{B}_\eta.$$

The universes C and D are obviously the same. Let Q be an operation symbol, and suppose $\boldsymbol{c}$ is a sequence of elements of C such that $Q^{\mathbf{C}}(\boldsymbol{c})\downarrow$. By definition, $Q^{\mathbf{C}}(\boldsymbol{c}) = Q^{\mathbf{A}_\gamma}(\boldsymbol{c})$, where γ is the least ξ such that $Q^{\mathbf{A}_\xi}(\boldsymbol{c})\downarrow$. Now if δ is the least η such that $Q^{\mathbf{B}_\eta}(\boldsymbol{c})\downarrow$ — such η certainly exist, since $\mathbf{A}_\gamma$ occurs in the enumeration $\langle \mathbf{B}_\eta \,|\, \eta < \beta\rangle$ — then since $\mathbf{A}_\gamma$ and $\mathbf{B}_\delta$ are compatible, $Q^{\mathbf{B}_\delta}(\boldsymbol{c}) = Q^{\mathbf{A}_\gamma}(\boldsymbol{c})$. Hence $Q^{\mathbf{D}}(\boldsymbol{c}) = Q^{\mathbf{C}}(\boldsymbol{c})$. ⊠

By this lemma, we may define the union $\bigcup_i \mathbf{A}_i$, for a family $\langle \mathbf{A}_i \,|\, i \in I\rangle$ of pairwise compatible algebras, as $\bigcup(\mathbf{A}_\xi \,|\, \xi < \alpha)$ for *any* enumeration $\langle \mathbf{A}_\xi\rangle_{\xi<\alpha}$ of $\{\mathbf{A}_i \,|\, i \in I\}$.

26.3 Corollary. Let $\langle \mathbf{A}_j \,|\, j \in J\rangle$ be a family of weak subalgebras of an algebra $\mathbf{B}$. Then $\bigcup\limits_{j\in J} \mathbf{A}_j$ is a weak subalgebra of $\mathbf{B}$.

Proof. The union exists: since the operations of the $\mathbf{A}_j$ are contained in operations of $\mathbf{B}$, the $\mathbf{A}_j$ are pairwise compatible. ⊠

26.4 Proposition. If $\langle \mathbf{A}_i \,|\, i \in I\rangle$ is a family of pairwise compatible structures, then the canonical injection $a \mapsto a$ of any A_i into $\bigcup_i A_i$ is an embedding of $\mathbf{A}_i$ into $\bigcup_i \mathbf{A}_i$.

26.5 Definition. The *intersection* $\bigcap_i \mathbf{A}_i$ of a family $\langle \mathbf{A}_i \,|\, i \in I\rangle$ of algebras is the algebra $\mathbf{B} = \langle B, K\rangle$ where $B = \bigcap_i A_i$, and K is the function on $\bigcap_i \mathrm{Dom} I_{\mathbf{A}_i}$ defined, for every symbol S in its domain, by $K(S) = \bigcap_i I_{\mathbf{A}_i}(S)$.

The canonical injection of $\bigcap_i A_i$ into any A_i is an embedding of $\bigcap_i \mathbf{A}_i$ into $\mathbf{A}_i$.

Exercises

§25

1. Verify that the relative subalgebra relation and the weak subalgebra relation are orderings of the class of all algebras.

2. Verify Example (*b*).

2. Prove (i) and (ii) of Proposition 2.

§26

1. Is the union of two orders necessarily an order?

2. Prove Proposition 4.

A *unary algebra* is an algebra with only unary basic operations. The algebra $\langle \mathbb{N}, S \rangle$ of the natural numbers with the successor operation is a prime example.

3. Let **B** be a unary algebra, and $\langle \mathbf{A}_j \mid j \in J \rangle$ a family of subalgebras of **B**. Prove that $\bigcup_j \mathbf{A}_j \leq \mathbf{B}$.

4. (a) Show by example that in general the union of a family of subalgebras of an algebra **B** need not even be a relative subalgebra of **B**.

(b) Show by example that the union of a family of relative subalgebras of a unary algebra **B** need not be a relative subalgebra of **B**.

5. Let $\mathcal{A} = \{A_j \mid j \in J\}$ be a collection of subsets of the universe of an algebra **B**. Assume that $\mathcal{A}$ is *directed by inclusion*, that is, $\langle \mathcal{A}, \subseteq \rangle$ is a directed order. Put $\mathbf{A}_j := (A_j)_{\mathbf{B}}$.

(a) Prove that $\bigcup_j \mathbf{A}_j \subseteq \mathbf{B}$.

(b) Prove that $\bigcup_j \mathbf{A}_j \leq \mathbf{B}$ if all the $\mathbf{A}_j$ are subalgebras of **B**.