Threshold behavior of a laser with nonorthogonal polarization modes

A. M. van der Lee,* M. P. van Exter, H. A. Assadian, N. J. van Druten,[†] and J. P. Woerdman

Huygens Laboratory, Leiden University, P.O. Box 9504, 2300 RA Leiden, The Netherlands

We investigate experimentally and theoretically the influence of the excess quantum noise in a laser on the laser's input–output curve near threshold. As an experimental system we use a He–Xe gas laser with nonorthogonal polarization modes. We observe that the excess quantum noise is absent far below threshold and steadily builds up as threshold is approached. The excess noise is fully developed when the mode that (above threshold) becomes the lasing mode dominates in power the other, nonlasing, modes. This situation may already occur considerably below threshold, namely, when the hot-cavity photon lifetime of the dominant mode exceeds the coloring time of the excess noise. © 2002 Optical Society of America

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1. INTRODUCTION

Nonorthogonality of the modes of a laser resonator leads to excess quantum noise, which is quantified by an excess-noise factor that is usually denoted K.[1,2] The existence of excess quantum noise is well established nowadays. It has been demonstrated in unstable-resonator lasers,[3,4] stable-resonator lasers with small apertures,[5,6] and lasers with nonorthogonal polarization modes.[7,8] The usual way to study excess quantum noise has been by measurement of the noise's effect on the linewidth of the laser.

In this paper we investigate the effect of excess quantum noise on the time-averaged output power of a laser as a function of pump strength P. It is well known that the input–output curve of a laser when it is plotted on a logarithmic scale is an S curve, as shown in Fig. 2 below. This curve is given by[9]

$$s = \frac{M - 1 + [(M - 1)^2 + 4\beta M]^{1/2}}{2\beta}, \qquad (1)$$

where s is the number of photons in the cavity of a single mode and $M = P/P_{\mathrm{thr}}$ is the normalized pump strength. The shape of the S curve is fully determined by parameter β, i.e., the fraction of spontaneous emission emitted into the lasing mode. The question now is whether Eq. (1) is the same for a laser with nonorthogonal modes and, if not, in what way should it be modified. We show both theoretically and experimentally how the S curve is modified as a result of the nonorthogonality of the modes of the laser cavity.

As we shall see, this is a nontrivial modification of Eq. (1), i.e., not simply a change of β to $K\beta$, as was postulated previously.[10] This is so because the strength of the excess noise (i.e., the K factor) is determined not only by the nonorthogonality of the cavity modes (i.e., by the geometry) but also by the time constants involved in gain and loss of these modes (i.e., by the dynamics). The latter property leads to coloring of excess quantum noise.[7]

Experimentally we have control over the dynamics of the excess noise, which includes its buildup time.[11] In this way we demonstrate in the input–output curve the transition below threshold from normal spontaneous emission into the lasing mode toward an enhanced spontaneous emission rate.

The paper is organized as follows: In Section 2 we derive theoretically a modified expression of the input–output curve for a laser with nonorthogonal modes. In Section 3 we introduce the experimental setup, and in Section 4 we give the experimental results for the S curve. We compare experiment and theory and end in Section 5 with a conclusion.

2. INPUT–OUTPUT CURVE FOR A LASER WITH NONORTHOGONAL MODES

The standard input–output curve for a single-mode laser is obtained from the steady-state solution of the coupled rate equations for the photon number in the lasing mode and the inversion number, including saturation of the inversion.[9] This curve leads to the well-known result of Eq. (1).

In the case of a laser with nonorthogonal modes the situation is more complicated. Above threshold the S curve will not change appreciably, as the output is determined by the saturation that is due to stimulated emission, which is not affected by excess noise.[1,2] In contrast, below threshold the output power is mainly amplified spontaneous emission noise. A first-order correction was proposed in Ref. 10: it was assumed that the spontaneous emission is always amplified by K, but this assumption is in fact too simplistic and fails far below threshold,[12] where the excess-noise factor can disappear completely because of its spectral coloring. To be more specific, it was shown in Ref. 11 that the excess spontaneous emission that ends up in the lasing mode can be divided into two types: normal spontaneous emission, which is emitted directly into the lasing mode, and extra-spontaneous emission, which is emitted in orthogonal di-

rections to the laser mode and evolves also into the lasing mode. Thus the excess spontaneous emission takes time (the so-called coloring time) to build up. Above threshold, the photon decay time of the hot cavity is long enough that the excess noise can build up completely. Below threshold, however, the photon decay time of the hot cavity will at a certain point get shorter than the coloring time of the excess quantum noise; hence the excess quantum noise will not build up properly, resulting in less enhancement of the spontaneous emission in the lasing mode.

For a laser with nonorthogonal modes, the output power below threshold is generally difficult to calculate, as a large number of modes will contribute to that power. For a laser with nonorthogonal polarization modes considered in this paper we can perform this calculation, as only two modes are involved. We use the following procedure: Starting with the evolution equation of the optical field (instead of the power), we calculate the field–field time-correlation matrix. From the trace of this correlation matrix at zero time difference we obtain the output intensity. Finally, this output power is again related to the saturated inversion such that we obtain the input–output curve.

The evolution equations for the x- and y-polarized modes, E_x and E_y, of our laser with nonorthogonal polarization modes are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}E_x\\E_y\end{pmatrix} = \begin{bmatrix}-\xi - A & -\Omega \\ \Omega & -\xi + A\end{bmatrix}\begin{pmatrix}E_x\\E_y\end{pmatrix} + \begin{pmatrix}f_x\\f_y\end{pmatrix}, \quad (2)$$

where ξ (>0) is the net polarization isotropic loss (polarization isotropic loss minus gain). A is the linear dichroism, i.e., the difference in loss between x- and y-polarized modes; this difference leads to a polarization anisotropic loss. Ω is the circular birefringence, i.e., the difference in refractive index between σ_+- and σ_--polarized modes, which leads to coupling of the two linearly polarized modes E_x and E_y. In Appendix A we briefly discuss the eigenvalues and the polarization eigenmodes of the evolution matrix. (Note that one can also create nonorthogonal polarization modes by using linear birefringence in combination with linear dichroism.[8]) The spontaneous emission noise is given by the Langevin noise sources $f_{x,y}$, with diffusion constants given by $\langle f_i^\dagger(t_1)f_j(t_2)\rangle = R_{sp}\delta_{ij}\delta(t_1 - t_2)$. R_{sp} is the spontaneous emission rate into the lasing mode, with R_{sp} approximately constant near threshold. To incorporate the polarization isotropic saturation of the gain we need an equation to relate the net (amplitude) loss ξ to the pump rate. We model the behavior of the laser near threshold to the following standard rate equation for inversion N:

$$\frac{\mathrm{d}}{\mathrm{d}t}N = P - \gamma_n N(1 + \beta s), \quad (3)$$

where P is the pump rate and γ_n is the inversion decay rate, β is the fraction of spontaneous emission in the lasing mode, and s is the number of photons in the lasing mode. Because the value of K and the (intensity) cavity loss rate (Γ_c) depend on the values of A and Ω, it is practical to define a threshold inversion value and a normalized pump rate for a laser with isotropic polarization losses (i.e., $A = 0$).[13] The value for which the unsatur-

ated gain that is due to the inversion is equal to the losses yields the threshold inversion value $N_{th,iso} = \Gamma_{c,iso}/(\beta\gamma_n)$, and by normalizing the pump rate to its threshold pump rate we obtain $M_{iso} = P\beta/\Gamma_{c,iso}$. Using these definitions, we find the steady-state net-loss rate for a polarization isotropic laser:

$$\frac{2\xi}{\Gamma_{c,iso}} \equiv \left(1 - \frac{N}{N_{iso,th}}\right) = 1 - \frac{M_{iso}}{1 + \beta s}. \quad (4)$$

In typical experimental situations the polarization anisotropies are relatively small, so in general the polarization anisotropic saturation barely influences the output power.[14]

For a fixed ξ Eq. (2) is easily formally solved, leading to the following correlation matrix ($\tau > 0$):

$$C(\tau) = \exp(L\tau)\int_0^\infty \mathrm{d}\tau' \exp(L\tau')R_{sp}\exp(L^\dagger\tau'), \quad (5)$$

where L is the evolution matrix of Eq. (2). The correlation function can be calculated from the eigenvalues, eigenmodes, and adjoint modes of matrix L.[16] The linearly polarized eigenmodes and adjoint modes are depicted graphically in Fig. 1. The vectors represent the directions of linear polarization.

When we use mode-nonorthogonality theory[1,2] we obtain for the excess-noise factor K

$$K = \frac{1}{\cos^2(2\theta)} = \frac{1}{1 - (\Omega/A)^2}. \quad (6)$$

Note that Eq. (6) shows the direct relation between the nonorthogonality of the eigenmodes (2θ) and the K factor. When $|\Omega| < A$, the eigenmodes have a difference in loss, as the eigenvalues are $\lambda_{1,2} = -\xi \pm \Delta$, with $\Delta = A/\sqrt{K}$. In the case of this two-mode laser the coloring time of the excess noise corresponds to the difference in loss rate, 2Δ. The threshold of the mode involved is reached when the mode's eigenvalue approaches zero; so for the laser the
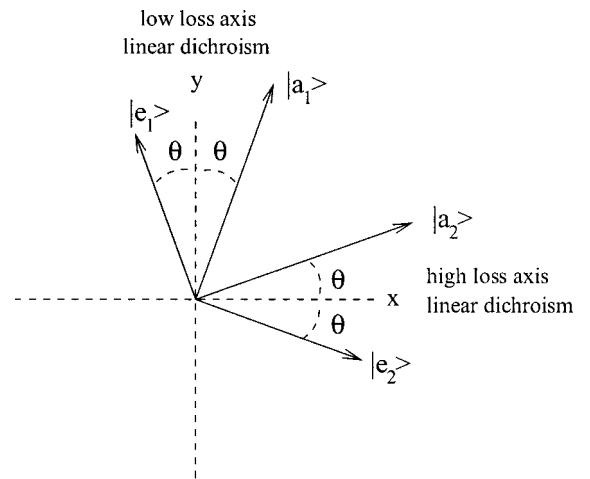

Fig. 1. Schematic picture of the polarization eigenmodes and adjoint modes of the resonator. They are linearly polarized, and the vectors show the polarization directions of the eigenmodes $|e_i\rangle$ and the corresponding adjoint modes $|a_i\rangle$ (with $\langle e_i|e_i\rangle = 1$ and $\langle a_i|a_i\rangle = K$). The nonorthogonality of the eigenmodes is determined by angle 2θ.

threshold is determined by that of the low-loss mode, which is the place where $\xi \to \Delta$.

To derive an expression for $s_{tot} \equiv E_x{}^2 + E_y{}^2$, which is the number of photons in the cavity in both x- and y-polarized modes, we use the matrix relation $\exp(L\tau) = \Sigma_i \exp(\lambda_i t)|e_i\rangle\langle a_i|$ and take the trace of the correlation matrix of Eq. (5) at $\tau = 0$:

$$\text{Tr}[C(0)] = \sum_{ij} \frac{-R_{sp}}{\lambda_i + \lambda_j{}^*} |e_i\rangle\langle a_i|a_j\rangle\langle e_j|, \qquad (7)$$

leading to

$$s_{tot} = \frac{R_{sp}}{2}\left[\frac{K}{\xi - \Delta} + \frac{K}{\xi + \Delta} + \frac{2(1-K)}{\xi}\right]. \qquad (8)$$

In Eq. (8) one might associate the first two terms with the two (polarization) eigenmodes and the third term with their interference. Note that the contributions of both eigenmodes are enhanced by excess-noise factor K but that the interference term has a negative value and can thereby cancel the enhancement. Two situations are easily analyzed: Far below threshold, when the hot cavity's loss rate 2ξ is much larger than spectral coloring rate 2Δ, i.e., the frequency/loss difference between the eigenmodes, the cancellation from the interference is maximum and the noise enhancement disappears. The opposite situation occurs close to threshold, when the hot-cavity loss rate of the dominant mode is much smaller than the coloring rate, i.e., when $\xi/\Delta - 1 \ll 1$. The first term in Eq. (8) then dominates the others, and the amount of spontaneous emission that ends up in the lasing mode is K times enhanced.

To obtain the input–output curve for the laser with nonorthogonal modes we have to include the polarization isotropic saturated gain ξ, which we do by substituting Eq. (4) into Eq. (8) and taking R_{sp} to be approximately constant, as it is close to threshold. We arrive at the following 4th-order equation in s_{tot}:

$$\left[\beta s_{tot}{}^2 + (1 - M_{iso} - 2\beta)s_{tot} - 2\right]\left[(1 + \beta_{tot} - M_{iso})^2\right.$$

$$\left. - \frac{4A^2}{\Gamma_{c,iso}{}^2 K}(1 + \beta s_{tot})^2\right] - \frac{8A^2}{\Gamma_{c,iso}{}^2}(1 + \beta s_{tot})^3 = 0.$$

$$(9)$$

Equation (9) is numerically solved. In Fig. 2 we have plotted the results for a laser with $\beta = 10^{-6}$ and $2A/\Gamma_c = 0.025$ for two values of the K factor, $K = 1$ and $K = 5.3$. We used linear scaling to transform M_{iso} into M [$M = M_{iso}/(1 - \Delta)$], so the threshold for all curves in Fig. 2 is at $M = 1$. For comparison we also plotted an input–output curve (short-dashed curve) for the hypothetical case when the spontaneous emission in the lasing mode would be enhanced by $K = 5.3$ on any time scale (i.e., neglecting the coloring).

We can repeat this calculation to obtain the input–output curve of a laser with N modes, which could be a model description of the transverse K factor of an unstable cavity for which many modes are involved.[17] By combining Eq. (4) with the N-mode version of Eq. (8) we find the input–output curve to be an $[N^2 + 1]$th-order
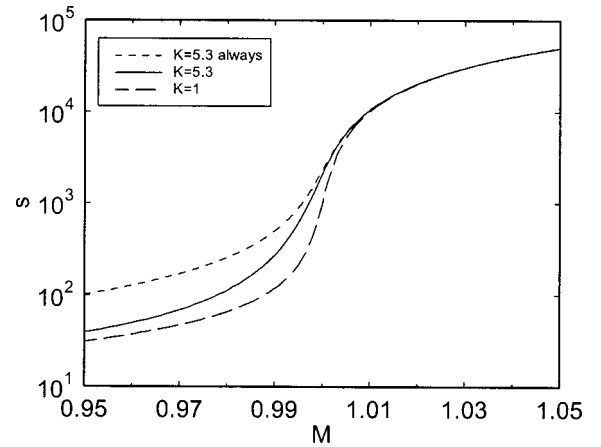

Fig. 2. Input–output curve for a two-mode laser with $K = 5.3$ and input–output curve for $K = 1$. Both curves were obtained by numerical solution of Eq. (9) with $\beta = 1 \times 10^{-6}$ and $2A/\Gamma_c = 0.025$. The short-dashed curve is the input–output curve for a hypothetical laser with a $K = 5.3$ times enhanced spontaneous emission rate for all pump rates (i.e., coloring is neglected).

equation in s_{tot}, so it will become difficult to calculate the input–output curve (see Appendix B). In this case the asymptotic behavior close to threshold and that far below threshold should still be the same as for a two-mode laser, namely, a K-times-enhanced spontaneous emission rate and a normal spontaneous emission rate, respectively. The transition between these two asymptotic regimes is governed by the dynamics of all modes.

3. EXPERIMENTAL SETUP

The experiments were performed with a He–Xe gas laser with a quasi-Brewster plate to create linear dichroism and an axial magnetic field in the He–Xe gain tube to induce circular birefringence (see Ref. 15 for calibration of these polarization anisotropies). The cavity consisted of a plane gold mirror and a concave dielectric mirror with a radius of curvature of -30 cm and a reflectivity of 27%. The mirrors were separated by 7 cm. A dressed cavity loss rate of $\Gamma_c/2\pi = 200$ MHz was achieved ("dressed" means that bad-cavity effects are included[18]).

We varied the laser gain by changing the amplitude V_{RF} of the rf power supply that drives the He–Xe gas discharge. The variation of V_{RF} remained small enough ($\lesssim 10\%$) to make the gain linearly proportional to this variation. In general the normalized pump parameter is given by $M - 1 = a(V_{RF}/V_{th} - 1)$, where V_{th} is the value of the rf voltage at laser threshold. Experimentally the lasing threshold is found as the rf voltage at which the relative increase in the output power is largest. We experimentally calibrated this proportionality constant a by changing the cavity losses and measuring the corresponding shift in threshold pump power, V_{RF}, of the laser, which yielded $a \approx 1$. This result can be understood as follows: The physics of the rf excited discharge is almost the same as that of a dc discharge, as the rf field still varies slowly with respect to the collision dynamics in the discharge. In a dc discharge the voltage remains approximately constant, so the electrical power that is dissipated in the discharge, which corresponds to the pump

rate, is linearly proportional to the current.[19]   As a re-
sult, $a = 1$ when the voltage over the ballast resistor (in
the rf case this is a ballast capacitance) is much larger
than over the discharge such that $M \propto I \propto V_{RF}$.

The pump strength could be adjusted to better than
$\Delta M = 0.001$.   The overall reproducibility of the adjusted
gain from measurement to measurement was not so good
as this.   This result can be partly attributed to the Xe
cleanup effect[20]; i.e., the gain changed during the mea-
surement.   Another cause of this larger uncertainty is
that the actual length of the discharge in the gain tube is
slightly dependent on the rf voltage and moreover shows
hysteresis behavior.   The input–output curve measured
by increasing the gain is slightly different from that mea-
sured by decreasing the gain.   By making the measure-
ment of the input–output curve within 15 min and by
measuring always from high to low rf voltage we found a
reproducibility of $\Delta M = 0.004$.   We used a chopper and a
lock-in amplifier to measure more accurately the low-
output powers below threshold.

4. MEASUREMENT OF THE INPUT–OUTPUT CURVE

To measure the input–output curve we start by determin-
ing parameter β of the laser.   To this end we measure the
input–output curve of the low-loss mode for a laser with
orthogonal modes, $K = 1$.   The low-loss mode is filtered
from the total output of the laser by a polarizer.   Figure 3
shows the output power as a function of normalized pump
parameter M.   The experimental data were fitted with
Eq. (1) and yielded $\beta = 9 \times 10^{-7}$.

Now we address the experimental input–output curve
for a laser with nonorthogonal modes.   We measured
the total power to compare the result with theory [cf.
Eqs. (8) and (9)].   Figure 4(a) shows the raw data ob-
tained for a laser with $K = 5$ and $2A/2\pi = 5$ MHz as a
function of the rf voltage, V_{RF}; for comparison we also
show the data for a laser with $K = 1$.   In Fig. 4(a) the
shift in threshold between the curves as a result of the de-
pendence of the polarization anisotropic loss Δ on K ob-
scures the modified threshold behavior.[21]   Therefore we
have plotted in Fig. 4(b) the input–output curves as a
function of their normalized pump parameter, M.   This
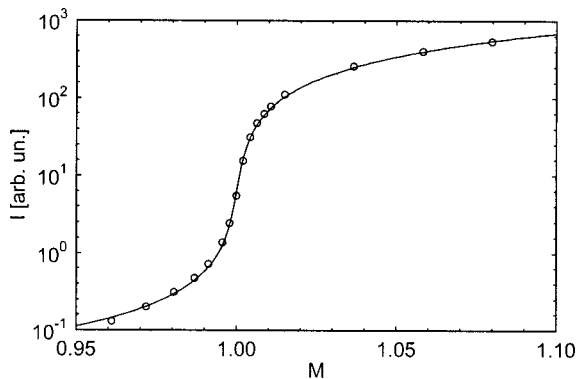result shows clearly how the threshold behavior differs

Fig. 3.   Input–output curve of the lasing mode measured behind
a polarizer for $K = 1$.   From the fit curve we can determine
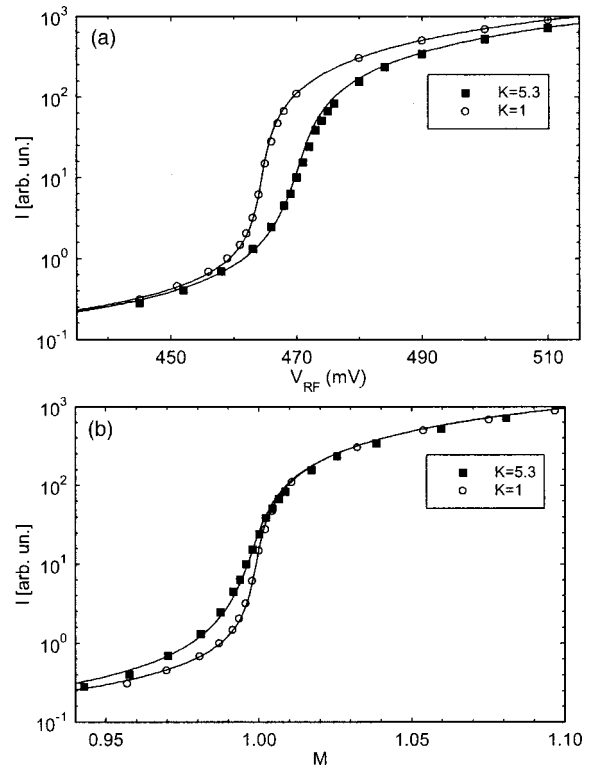that $\beta \approx 9 \times 10^{-7}$.

Fig. 4.   (a) Total output power plotted versus V_{RF}.   (b) Total
output power as a function of M.   The strength of the linear di-
chroism was $2A/2\pi = 5$ MHz, and the dressed cavity loss rate
was $\Gamma_c/2\pi \approx 200$ MHz.   Solid curves, theoretical curves based on
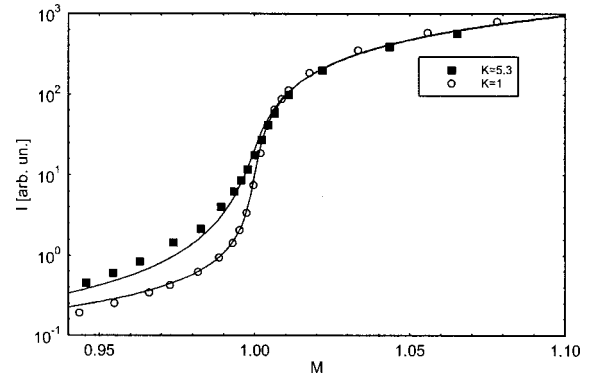Eq. (9).

Fig. 5.   Input–output curve of a laser with a larger strength of
the linear dichroism $2A/2\pi = 11.1$ MHz and a dressed cavity
loss rate $\Gamma_c/2\pi \approx 200$ MHz for values of $K = 5.3$ and $K = 1$.
Solid curves, theoretical curves based on Eq. (9).   Because of the
larger value of A the output power of the curve with $K = 5.3$ is
higher for the same M than the curve in Fig. 4(b) (the vertical
scale used for the intensities in Figs. 4 and 5 is the same).

for the laser with nonorthogonal modes.   The threshold
transition has become broader.   Sufficiently far below
threshold, i.e., when $M \lesssim 0.95$, the two curves are identi-
cal; for increasing M the excess spontaneous emission in
the low-loss mode can continue to build up such that the
output power of the laser with nonorthogonal modes is en-
hanced compared with that of a laser with orthogonal
modes.   The coloring rate, $\lambda_1 - \lambda_0$, is 1% of the cavity
loss rate, implying that at $M = 0.99$ the hot-cavity decay
rate is equal to the coloring rate.   So above $M = 0.99$ the

excess noise is fully developed. Above threshold, i.e., when $M > 1$, the stimulated emission saturates the gain and the two curves are identical again.

The solid curves in Fig. 4 are theoretical curves obtained from Eq. (9). To calculate the input–output curves we used the values $2A/\Gamma_c = 0.025$, $K = 5.3$, and $\beta = 1.0 \times 10^{-6}$, which were determined independently; β followed from the input–output curve of Fig. 3 and is slightly modified for a better fit ($\beta = 0.9 \times 10^{-6}$ from Fig. 3 and $\beta = 1.0 \times 10^{-6}$ used in fitting Fig. 4). A and K were determined by the polarization-rotation method discussed in Ref. 16, yielding A and Ω. To fit the solid curves to the experimental data we can use only horizontal and vertical scaling to link the lock-in voltage to the output power and the rf voltage to the pump parameter. Note the good agreement between theory and measurement.

As a next step we increase the linear dichroism by making the tilt angle of the glass plate larger, thus decreasing the coloring time of the excess quantum noise. Figure 5 shows the result for a laser whose strength of linear dichroism $2A/2\pi$ was 11.1 MHz; this means that the coloring time has become approximately two times shorter.

Again we measured the input–output curve for $K = 5.3$ and $K = 1$. The solid curves were calculated from Eq. (9) with $\beta = 1 \times 10^{-6}$ and $2A/\Gamma_c = 0.055$. When the input–output curves of Figs. 5 and 4(b) are compared, the consequence of the faster coloring time is obvious. The input–output curve of a laser with nonorthogonal modes $K = 5.3$ differs from that of a laser with $K = 1$ over a larger range of M.

5. CONCLUSIONS

We have experimentally and theoretically investigated the input–output curve of a laser with nonorthogonal polarization modes. The influence of the excess-noise factor on the input–output curve has been shown. Whereas just below threshold the spontaneous emission rate into the lasing mode is enhanced by geometrical excess-noise factor K, far below threshold the excess noise disappears. For our two-mode laser we can calculate the input–output curve explicitly, and the comparison between theory and experiment is good.[22]

In the case of transverse-mode nonorthogonality, for example, in an unstable resonator, the excess noise is caused by many more than two nonorthogonal modes. Explicit calculation of the input–output curve is almost impossible, as the dynamics of all modes is involved. However, it is expected that the asymptotic behavior will be qualitatively the same as in the two-mode case. Specifically, we predict that above and just below threshold, where the (hot-cavity) photon decay time is still much longer than the coloring time, the excess spontaneous emission rate in the lasing mode will be fully developed. In the opposite case, for which the photon decay time is much shorter than the coloring time, the excess spontaneous emission will disappear and the spontaneous emission rate into the lasing mode will be normal. That this behavior was not observed in the research reported in

Ref. 10 is so because the laser was not operated far enough below threshold. For the unstable resonator used in that research the ratio of coloring time and photon lifetime was smaller than in the experiment reported in this paper.[16] Therefore the measurements could be well described by a model that assumes a constant enhancement of the spontaneous emission by K, which is the asymptotic limit of the full theory close to threshold.

Recent speculations on use of the Petermann K factor to create a thresholdless laser[10] and modification of the spontaneous emission of a single atom in a cavity with nonorthogonal modes[23] have neglected the influence of the dynamics of the Petermann K factor. As we have shown in this paper, the influence of the nonorthogonality of the cavity modes depends crucially on the ratio of coloring time and the hot-cavity photon lifetime; i.e., the Petermann excess noise factor is not only geometry but also dynamics (see also Refs. 11, 12, and 16). The inclusion of the dynamics in the speculations mentioned above is therefore highly recommended, as it might (significantly) influence the outcome of the calculations.

APPENDIX A. POLARIZATION EIGENMODES OF THE RESONATOR

In this appendix we briefly summarize the polarization eigenmodes of a cavity with linear dichroism and circular birefringence that are needed for calculation of the output power. A more elaborate discussion of the evolution matrix and of the consequences of the nonorthogonal polarization eigenmodes can be found in Ref. 24. Our starting point is the evolution matrix of Eq. (2). The eigenvalues of this matrix are

$$\lambda_{1,2} = -\xi \pm \sqrt{A^2 - \Omega^2}. \tag{A1}$$

The real and the imaginary parts of the eigenvalues of the matrix give their damping rates and corresponding frequencies (relative to the carrier frequency), respectively. We observe that when $|\Omega| < |A|$ the eigenvalues are pure real, so the eigenmodes have the same frequency but different loss. This corresponds to a single-mode laser: The mode with the lowest loss will lase and clamp the inversion such that the other, more lossy, mode will always experience loss. The experiments presented in this paper were performed in this regime. In the other regime, $|\Omega| > |A|$, the eigenmodes have the same loss but a different frequency. In this case the laser is no longer single mode. See Ref. 7 for discussion of this regime.

In Fig. 6 we show the exact behavior of the eigenvalues of both polarization eigenmodes as a function of Ω/A. The dashed and the solid curves give the losses and the frequencies, respectively, of the eigenmodes. Also, the polarization ellipses of the eigenmodes are indicated. In the experiments presented in this paper we operated the laser such that $|\Omega| < A$. The polarization eigenmodes are linearly polarized, and the polarization angle $90°$ − 2θ between them is given by (see Fig. 1 for the definition of angle θ)
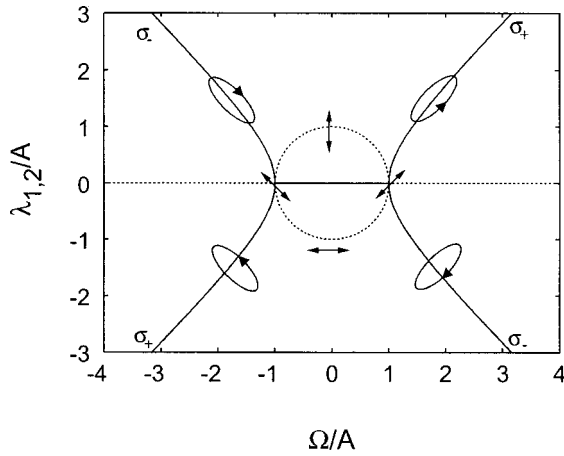
$$\sin(2\theta) = \Omega/A. \tag{A2}$$

Fig. 6. Frequencies (solid curves) and net gains (dashed curves) of the eigenmodes as functions of Ω/A. The polarization states of the eigenmodes are also indicated.

So, when $\Omega = 0$, the eigenmodes are orthogonally polarized, but increasing $\Omega \to A$ makes the polarization eigenmodes increasingly more nonorthogonal. The corresponding adjoint modes $|a_i\rangle$ are found from the biorthogonality relation[1,2] $\langle e_j|a_i\rangle = \delta_{ij}$ (where $|e_j\rangle$ are the eigenmodes).

APPENDIX B. DERIVATION OF INPUT–OUTPUT CURVE FOR A LASER WITH *N*-NONORTHOGONAL MODES

The derivation of the input–output curve for a laser in the general case of N nonorthogonal modes is basically a matter of bookkeeping. Therefore we start with a slightly rewritten version of Eq. (7):

$$s_{\text{tot}} = \text{Tr}[C(0)] = \frac{R_{\text{sp}}}{\Gamma_{c,\text{iso}}} \sum_{i,j} A_{ij} \frac{\Gamma_{c,\text{iso}}}{2\xi + \gamma_{ij}}, \quad \text{(B1)}$$

where $A_{ij} = \langle e_i|e_j\rangle\langle a_i|a_j\rangle$ expresses the overlap among the nonorthogonal eigenmodes of the cavity. In the denominator we have separated from the two eigenvalues λ_i and λ_j^* the isotropic hot-cavity loss ξ and the anisotropic part of the eigenvalues $\gamma_{ij} = \lambda_i + \lambda_j^* - 2\xi$. Below, we assume that the ratio $R_{\text{sp}}/\Gamma_{c,\text{iso}}$ is constant and unity; this assumption is valid close to threshold. Equation (B1) can in the same way as Eq. (8) be subdivided into diagonal terms $(i = j)$ that correspond to the power in each eigenmode and into cross terms $(i \neq j)$ that describe the interference between the eigenmodes.

Hot-cavity loss ξ depends on the photon number [see Eq. (4)]:

$$2\xi = \frac{\Gamma_{c,\text{iso}}(1 + \beta s_{\text{tot}} - M_{\text{iso}})}{1 + \beta s_{\text{tot}}}. \quad \text{(B2)}$$

Substituting Eq. (B2) into Eq. (B1) and multiplying the denominators result in a rather complicated polynomial equation for s_{tot} of the order of $N(N + 1)/2 + 1$ for a specific N-mode case. The argument goes as follows: When

we assume that the eigenmodes differ from one another only in damping but not in frequency, the anisotropic loss rates γ_{ij} are real valued, and Eq. (B1) contains N terms with $i = j$ and two identical sets of $N(N - 1)/2$ cross terms with $i \neq j$. Bringing these terms under a common denominator gives a polynomial expression in ξ of the order $N(N + 1)/2$. The substitution of Eq. (B2) finally increases the order by 1 to the mentioned $[N(N + 1)/2 + 1]$-th order in s_{tot}. For $N = 2$ this is Eq. (9). When the N eigenmodes also differ in frequency, a similar argument shows that we are generally stuck with a polynomial expression of order $N^2 + 1$.

*Present address, Philips Research Laboratories, Prof. Holstlaan 4, 5656 AA Eindhoven, The Netherlands.

†Present address, Department of Applied Physics, Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands.

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13. We have chosen a symmetric form for absorption A in the evolution matrix in Eq. (2) to simplify the equations.

14. In our own experiment we have $\Omega < A$; in this case the polarization anisotropic saturation does not influence the dc output power at all.[15]   This result can be understood as follows: The strength of the anisotropic saturation is proportional to the ellipticity of the light [see Eq. (10) of Ref. 15], and the polarization eigenstates of the resonator are linearly polarized, so one will conclude that the anisotropic saturation vanishes.

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21. In the experiment we leave the value of A fixed, and we tune the magnetic field to change Ω and thus K [see Eq. (6)].   As depicted in Fig. 1, the low-loss eigenmode rotates away from the low-loss axis of the dichroism when the polarization eigenmodes are nonorthogonal; the loss of this mode increases when the nonorthogonality of the polarization modes increases.

22. Note that incorporation of population of the lower laser level into the theory [introducing N_{sp} (Ref. 20)] does not significantly alter the correspondence between theory and experiment.   The incorporation shows up mainly in a modified β,[10] which is also present in our calibration measurement of β.

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