Chapter 7

Distributive Lattices with Quantifier: Topological Representation

Nick Bezhanishvili

Department of Foundations of Mathematics, Tbilisi State University

Abstract.

We give a representation of distributive lattices with the existential quantifier in terms of spectral spaces, which is an alternative to Cignoli's representation in terms of Priestley spaces. Then we describe dual spectral spaces of subdirectly irreducible and simple Q-distributive lattices and prove that the variety **QDist** of Q-distributive lattices does not have the congruence extension property and that there exist non-surjective epimorphisms in **QDist**.

7.1 Introduction

Distributive lattices with the existential quantifier, abbreviated as Q-distributive lattices, were first introduced and investigated by Cignoli ([3], [4]) as a natural extension of Halmos' monadic Boolean algebras [5]. In particular, Cignoli has proved a representation of Q-distributive lattices in terms of Priestley spaces. Several other results on Q-distributive lattices were obtained by Petrovich [7]. In this note we will give a representation of Q-distributive lattices in terms of spectral spaces which serves as an alternative to Cignoli's representation. Then we will describe the dual spectral spaces of subdirectly irreducible and simple Q-distributive lattices, and prove that the variety **QDist** of Q-distributive lattices does not have the congruence extension property and that there exist non-surjective epimorphisms in **QDist**.

7.2 Q-distributive lattices

Definition 1 *(See Cignoli [3]) A pair* $(D, \exists)$ *is said to be a* Q-distributive lattice, *if D is a bounded distributive lattice and $\exists$ is a unary operator on D satisfying the following conditions for all $a, b \in D$:*

1) $\exists 0 = 0$;

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2) $a \leq \exists a$*;*
3) $\exists(a \vee b) = \exists a \vee \exists b$*;*
4) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$*.*

Denote by **QDist** the variety of all Q-distributive lattices and corresponding homomorphisms. **Dist** will denote the variety of all bounded distributive lattices and bounded distributive homomorphisms.

For every $(D, \exists) \in$ **QDist**, denote by D_0 the set $\{a \in D : \exists a = a\}$ of all fixed-points of $\exists$. It is a routine to check that $D_0 = \{\exists a : a \in D\}$ and that D_0 forms a *relatively complete* sublattice of D, that is the set $\{b \in D_0 : a \leq b\}$ has the least element, for every $a \in D$, which we denote by $Inf a$. Conversely, every couple (D, D_0), where D_0 is a relatively complete sublattice of D, defines an operator C on D by putting $Ca = Inf a$. Unfortunatelly, C is not always a quantifier. Actually we have the following

Theorem 2 *(See Balbes and Dwinger [1]) For a given distributive lattice* D*, there exists a one-to-one correspondence between closure operators on* D[1] *and relatively complete sublattices of* D*.* □

Now we will a little bit extend the previous theorem. Consider a function $h : D_0 \to D$. A function $C_h : D \to D_0$ is said to be the *left adjoint* to h, if $C_h b \leq a$ iff $b \leq ha$, for any $a \in D_0$ and $b \in D$.

Theorem 3 *For a given distributive lattice* D*, there exists a one-to-one correspondence between*
1) closure operators on D*;*
2) relatively complete sublattices of D*;*
3) canonical embeddings $h : D_0 \hookrightarrow D$ *having the left adjoint.*

Proof (Sketch) 1) $\Leftrightarrow$ 2) see Theorem 2.

2) $\Rightarrow$ 3) For a given pair (D, D_0), where D_0 is a relatively complete sublattice of D, define $C_h : D \to D_0$ by putting $C_h(a) = Inf a$, for every $a \in D$. It is easy to check that C_h is the left adjoint to the canonical embeding $h : D_0 \hookrightarrow D$.

3) $\Rightarrow$ 2) Suppose D_0 is a sublattice of D, and the canonical embedding $h : D_0 \hookrightarrow D$ has the left adjoint function C_h. Then $Inf a = C_h(a)$, for every $a \in D$, and D_0 is a relatively complete sublattice of D.

Now it is a routine to check that this correspondence is one-to-one. □

Now, as a corollary, we characterize those closure operators on D which serve as quantifiers. $D_0 \subseteq D$ is said to be a *quantifier sublattice* of D, if D_0 is a relatively complete sublattice of D and $a \wedge Inf b = Inf(a \wedge b)$, for every $a \in D_0$ and $b \in D$. C_h is said to satisfy the *Frobenius condition*, if $C_h(ha \wedge b) = a \wedge C_h b$, for every $a \in D_0$ and $b \in D$.

[1]Recall that an operator C on D is said to be a *closure operator*, if it satisfies Kuratowski's identities $C0 = 0$, $a \leq Ca$, $CCa \leq Ca$ and $C(a \vee b) = Ca \vee Cb$.

Corollary 4 *For a given distributive lattice D, there exists a one-to-one correspondence between*

1) quantifiers on D;

2) quantifier sublattices of D;

3) canonical embeddings $h : D_0 \hookrightarrow D$ having the left adjoint satisfying the Frobenius condition.

Proof. Observe that on the base of Theorem 3 the following three conditions are mutually equivalent: $\exists(\exists a \wedge b) = \exists a \wedge \exists b$, for every $a, b \in D$, $a \wedge Inf\, b = Inf(a \wedge b)$, for every $a \in D_0$ and $b \in D$, $C_h(ha \wedge b) = a \wedge C_h b$, for every $a \in D_0$ and $b \in D$. □

Now we are in a position to extend this correspondence to an equivalence of the corresponding categories. Denote by **CDist** the category of distributive lattices with a closure operator and corresponding homomorphisms. It is obvious that **QDist** is a (full) subcategory of **CDist**. Also denote by $\mathbf{Dist}^2$ the category whose objects are (D, D_0) pairs of distributive lattices, where an injective **Dist**-homomorphism $h : D_0 \to D$ has the left adjoint function, and whose morphisms are pairs of functions $(f, f_0) : (D, D_0) \to (D', D'_0)$ such that f is a **Dist**-homomorphism, $f \circ h = h' \circ f_0$ and $f_0 \circ C_h = C_{h'} \circ f$. We denote by $\mathbf{Dist}^2_\exists$ the (full) subcategory of $\mathbf{Dist}^2$ whose objects satisfy the Frobenius condition.

Theorem 5 *1).* **CDist** *is equivalent to* $\mathbf{Dist}^2$.

2). **QDist** *is equivalent to* $\mathbf{Dist}^2_\exists$.

Proof (Sketch) 1). Define the functors $\phi : \mathbf{CDist} \to \mathbf{Dist^2}$ and $\psi : \mathbf{Dist^2} \to \mathbf{CDist}$ by putting $\phi(D, C) = (D, D_0)$, where D_0 is the set of all fixed-points of C, and $\phi(f) = (f, f\mid_{D_0})$, and $\psi(D, D_0) = (D, C)$, where $C = h \circ C_h$, and $\psi(f, f_0) = f$. Now it is a routine to check that the so-defined functors set an equivalence between **CDist** and $\mathbf{Dist}^2$.

2) easily follows from 1) and Corollary 4. □

7.3 Topological Representation

Definition 6 *(See e.g. Balbes and Dwinger [1] and Johnstone [9]) A topological space (X, Ω) is said to be* spectral *if*

1) (X, Ω) is compact and T_0;

2) The set $\mathcal{C}(X)$ of all compact-open subsets of (X, Ω) is a sublattice of Ω and a base for the topology;

3) (X, Ω) is sober, *that is any closed set F which is not a closure of a singleton $\{x\}$ is a union of two closed sets which differ from F.*

Denote by **Spec** the category of all spectral spaces and all strongly continuous maps. Here a map is said to be *strongly continuous* if an inverse image of a compact open set is compact open. Stone's well-known theorem now establishes that **Dist** is dually equivalent to **Spec**. In particular, every distributive lattice can be represented as the set $\mathcal{C}(X)$ of all compact open sets of the corresponding spectral space (X, Ω)[2].

Another representation of distributive lattices can be obtained in terms of Priestley spaces. Let (X, R) be a partially ordered set. $A \subseteq X$ is said to be an *(upper) cone* of X, if $x \in A$ and xRy imply $y \in A$. A triple (X, Ω, R) is said to be a *Priestley space* if

1) (X, Ω) is a Stone space (that is a 0-dimensional, Hausdorff and compact space);
2) R is a partial order on X satisfying *Priestley separation axiom*: if $\neg(xRy)$, then there exists a *clopen* (simultaneously closed and open) upper cone A such that $x \in A$ and $y \notin A$.

Denote the set of all clopen upper cones of X by $\mathcal{CON}X$. Also denote by **Priest** the category of Priestley spaces and monotonous continous maps. Then Priestley duality establishes that **Dist** is dually equivalent to **Priest**. In particular, every distributive lattice can be represented as the set $\mathcal{CON}X$ of all clopen cones of the corresponding Priestley space (X, Ω, R).

Cornish [2] has shown that these two dualities are in fact two sides of the same coin by establishing that **Spec** is isomorphic to **Priest**. Now we will do the same for the case of Q-distributive lattices.

Call a quadriple (X, Ω, R, E) a *Cignoli space*, if (X, Ω, R) is a Priestley space and E is an equivalence relation on X such that

a) $A \in \mathcal{CON}X \Rightarrow E(A) \in \mathcal{CON}X$. (Here $E(A) = \bigcup_{x \in A} E(x)$ and $E(x) = \{y \in X : yEx\}$.)
b) $E(x)$ is a closed set for any $x \in X$.

Denote by **Cign** the category of Cignoli spaces and R-monotonous continous maps $f : X_1 \to X_2$ such that for any $A \in \mathcal{CON}X_2$, $f^{-1}E_2(A) = E_1(f^{-1}A)$[3].

Theorem 7 *(See Cignoli [3])* **QDist** *is dually equivalent to* **Cign**. □

In particular, every Q-distributive lattice can be represented as the pair $(\mathcal{CON}X, E)$ for the corresponding Cignoli space (X, Ω, R, E).

Denote by **Priest**^{2} the category whose objects are (X, π, X_0) triples, where X and X_0 are Priestley spaces, $\pi : X \to X_0$ is a **Priest**-morphism and $\pi(A) \in \mathcal{CON}X_0$ for every $A \in \mathcal{CON}X$, and whose morphisms are

[2] Note that the notion of a spectral space is the same as in Hochster [6] and is widly used in ring theory. In particular, the dual spaces of rings are exactly the spectral spaces.
[3] It is worth noting that this last condition is equivalent to the following one on points of X_1 and X_2: for every $x \in X_1$ and $y \in X_2$, if $f(x)Ey$, then there exists $z \in X_1$ such that xEz and $yRf(z)$.

$(f, f_0) : (X, X_0) \to (X', X'_0)$ pairs, where $f : X \to X'$ is a **Priest**-morphism and $\pi' \circ f = f_0 \circ \pi$.

Theorem 8 **Cign** *is equivalent to* $\mathbf{Priest}^2$.

Proof. First note that for every Cignoli space (X, Ω, R, E), the factor-space $(X/_E, \Omega_E, R_E)$ is a Priestley space (follows from Cignoli [3]). Further, define the functor $\phi : \mathbf{Cign} \to \mathbf{Priest}^2$ by putting $\phi(X, \Omega, R, E) = (X, \pi, X/_E)$, where $\pi(x) = E(x)$, and $\phi(f) = (f, f_0)$, where $f_0(E(x)) = E'f(x)$. It is a routine to check that ϕ is defined correctly. Now define $\psi : \mathbf{Priest}^2 \to \mathbf{Cign}$ by putting $\psi(X, \pi, X_0) = (X, E)$, where xEy iff $\pi(x) = \pi(y)$, and $\psi(f, f_0) = f$. It is also a routine to check that ψ is defined correctly as well. Now one can easily check that these functors set an equivalence between **Cign** and $\mathbf{Priest}^2$. □

Corollary 9 $\mathbf{QDist} \simeq \mathbf{Dist}^2_\exists \simeq \mathbf{Cign}^{op} \simeq (\mathbf{Priest}^2)^{op}$. *(Here $\mathcal{K}^{op}$ denotes the opposite of the category $\mathcal{K}$.)* □

Definition 10 *A triple (X, Ω, E) is said to be an* augmented spectral space *if (X, Ω) is a spectral space and E is an equivalence relation on X such that*
a) $A \in \mathcal{C}(X) \Rightarrow E(A) \in \mathcal{C}(X)$;
b) $(X/_E, \Omega_E)$ is a T_0 space.

Note that since $\mathcal{C}(X)$ forms a base for the topology, a) implies that E is an open equivalence relation, but not every open equivalence relation satisfies a).

Denote by **ASpec** the category of augmented spectral spaces and strongly continuous maps $f : X_1 \to X_2$ such that for any $A \in \mathcal{C}(X)$, $f^{-1}E_2A = E_1f^{-1}A$.

Theorem 11 **ASpec** *is isomorphic to* **Cign**.

Proof. First we recall an equivalence between spectral spaces and Priestly spaces shown in Cornish [2]. With any spectral space (X, Ω) is associated the Priestly space (X, Ω^+, R_Ω), where R_Ω is the specialization order on X (that is a partial order R_Ω defined by putting $xR_\Omega y$ iff $x \in C(y)$, where C is the topological closure operator), and a base for the topology Ω^+ is the Boolean closure of the set $\mathcal{C}(X)$ of all compact open subsets of (X, Ω). Moreover, the set of all clopen cones of (X, Ω^+, R_Ω) is precisely the set of all compact open subsets of (X, Ω). Conversely, with every Priestley space (X, Ω, R) is associated the spectral space (X, Ω^-), where $\mathcal{CON}X$ is taken as a base for Ω^-. Moreover, the set of all compact open subsets of (X, Ω^-) is precisely the set of all clopen cones of (X, Ω, R). **Spec**-morphisms are precisely **Priest**-morphisms and vise versa.

In our case it remains to show that if (X, Ω, E) is an augmented spectral space, then $(X, \Omega^+, R_\Omega, E)$ is a Cignoli space, and conversely, if (X, Ω, R, E) is a Cignoli space, then (X, Ω^-, E) is an augmented spectral space. Indeed, from the definition of Cignoli spaces, augmented spectral spaces and the fact that clopen cones are precisely compact open sets of the corresponding Priestly and spectral spaces it follows directly that $A \in \mathcal{CON}X \Rightarrow E(A) \in \mathcal{CON}X$ is equivalent to $A \in \mathcal{C}(X) \Rightarrow E(A) \in \mathcal{C}(X)$. Further, let us show that if $X/_E$ is a T_0-space, then $E(x)$ is closed in (X, Ω^+). From $y \notin E(x)$ it follows that $E(y) \cap E(x) = \emptyset$. Since $X/_E$ is T_0 and E-saturated compact open sets of X form a base for the topology on $X/_E$, there exists a compact open set $U = E(U)$ such that say $E(y) \subseteq U$ and $E(x) \cap U = \emptyset$. Since U is compact open, U is a clopen cone in (X, Ω^+). Therefore, there exists a clopen U such that $y \in U$ and $E(x) \cap U = \emptyset$, and hence $E(x)$ is a closed set. Conversely, if $E(x)$ is closed for every $x \in X$, then $(X/_E, \Omega_E^-)$ is a T_0-space. Indeed, as was mentioned before, for any Cignoli space (X, Ω, R, E), the factor space $(X/_E, \Omega_E, R_E)$ is a Priestly space, and hence $(X/_E, \Omega_E^-)$ is a T_0-space (even a spectral space). Finally, it is a routine to check that **ASpec**-morphisms correspond to **Cign**-morphisms and vice versa. Hence **ASpec** is isomorphic to **Cign**. □

Corollary 12 **QDist** *is dually equivalent to* **ASpec**. □

In particular, every Q-distributive lattice can be represented as $(\mathcal{C}(X), E)$ for the corresponding augmented spectral space (X, Ω, E). It is worth noting that this fact together with the last corollary can be obtained directly without using Theorem 11.

Denote by $\mathbf{Spec}^2$ the category whose objects are (X, π, X_0) triples, where X and X_0 are spectral spaces, π is a **Spec**-morphism and $\pi(U) \in \mathcal{C}(X_0)$ for every $U \in \mathcal{C}(X)$, and whose morphisms are $(f, f_0) : (X, X_0) \to (X', X'_0)$ pairs, where $f : X \to X'$ is a **Spec**-morphism and $\pi' \circ f = f_0 \circ \pi$.

Theorem 13 **ASpec** *is equivalent to* $\mathbf{Spec}^2$.

Proof is similar to the one of Theorem 8 and rests on the fact that if (X, Ω, E) is an augmented spectral space, then the factor-space $(X/_E, \Omega_E)$ is a spectral space. □

Corollary 14 $\mathbf{QDist} \simeq \mathbf{Dist}^2_\exists \simeq \mathbf{ASpec}^{op} \simeq (\mathbf{Spec}^2)^{op} \simeq \mathbf{Cign}^{op} \simeq (\mathbf{Priest}^2)^{op}$. □

We will close out this section by mentioning a correspondence between Halmos' monadic Boolean algebras and Q-distributive lattices. It is easy to see that a Q-distributive lattice $(D, \exists)$ is a monadic Boolean algebra iff D is a Boolean algebra. From the point of view of Cignoli's duality it means that a Cignoli space is a Halmos space (see Halmos [5]) iff R is a descrete order. And from the point of view of our duality it means that an augmented spectral space is a Halmos space iff it is T_1.

7.4 Serpinski $\mathfrak{m}$-spaces

In [3] Cignoli has described the dual spaces of simple and subdirectly irreducible Q-distributive lattices in terms of Cignoli spaces. Now we are going to do it in terms of augmented spectral spaces. As we shall see below, the dual spaces of several classes of simple and subdirectly irreducible Q-distributive lattices can be obtained by a generalization of the well known concept of the Serpinski space.

Recall that a topological space $S = (X, \Omega)$ is said to be the *Serpinski space*, if $X = \{x, y\}$ and $\Omega = \{\emptyset, \{x\}, X\}$.

For any cardinal $\mathfrak{m}$, consider the topological sum $\overline{S} = \bigoplus_{\mathfrak{m}-times} S$. Define an equivalence relation E on $\overline{S}$ by identifying the closed points of $\overline{S}$. Denote $\overline{S}/_E$ by $\mathfrak{m}S$ and call it the *Serpinski* $\mathfrak{m}$*-space.* Define on the Serpinski $\mathfrak{m}$-space an anti-discrete equivalence relation E: xEy for all $x, y \in \mathfrak{m}S$. The obtained space denote by $(\mathfrak{m}S, E)$ and call it *the Serpinski* $\mathfrak{m}, E$*-space.*

Theorem 15 *1)* $\mathfrak{m}S$ *is a spectral space.*

2) $(\mathfrak{m}S, E)$ *is an augmented spectral space.*

3) If $\mathfrak{m} < \aleph_0$*, then the* Q*-distributive lattice* $(\mathcal{C}(\mathfrak{m}S), E)$ *is subdirectly irreducible.*

4) If $\aleph_0 \leq \mathfrak{m}$*, then* $(\mathcal{C}(\mathfrak{m}S), E)$ *is simple.*

Proof. 1) Observe that every point of $\mathfrak{m}S$ is open, except the one which is closed and will be denoted by ξ. Also observe that compact open sets of $\mathfrak{m}S$ are exactly the finite subsets of $\mathfrak{m}S$ not containing ξ, and that the closure of any subset A of $\mathfrak{m}S$ is $A \cup \{\xi\}$. Now it is a rather easy exercise to check that $\mathfrak{m}S$ is a spectral space.

2) Easily follows from 1).

3) Observe that if $\mathfrak{m} < \aleph_0$, then $\emptyset$, $\mathfrak{m}S - \{\xi\}$ and $\mathfrak{m}S$ are the only subsets of $(\mathfrak{m}S, E)$ which correspond to the congruences on $(\mathcal{C}(\mathfrak{m}S), E)$. Therefore, $\mathfrak{m}S - \{\xi\}$ corresponds to the least non-diagonal congruence on $(\mathcal{C}(\mathfrak{m}S), E)$, and hence $(\mathcal{C}(\mathfrak{m}S), E)$ is subdirectly irreducible.

4) If $\aleph_0 \leq \mathfrak{m}$, then $\mathfrak{m}S - \{\xi\}$ does not correspond to a congruence on $(\mathcal{C}(\mathfrak{m}S), E)$ any more. Hence the only subsets of $(\mathfrak{m}S, E)$ which correspond to the congruences on $(\mathcal{C}(\mathfrak{m}S), E)$ are $\emptyset$ and the whole $\mathfrak{m}S$. Therefore, there are only two congruences on $(\mathcal{C}(\mathfrak{m}S), E)$, the diagonal and the whole $\mathcal{C}(\mathfrak{m}S)$. Thus, $(\mathcal{C}(\mathfrak{m}S), E)$ is simple. □

Unfortunately, all subdirectly irreducible Q-distributive lattices can not be obtained in this way. However, every finite subdirectly irreducible Q-distributive lattice can be obtained by means of finite Serpinski n-spaces. For every natural numbers n and m, consider the topological sum $nS \bigoplus m$, where m denotes, up to homeomorphism, the discrete topological space containing exactly m points.

Theorem 16 *1) A finite Q-distributive lattice $(D, \exists)$ is subdirectly irreducible iff there exist natural n and m such that the dual of $(D, \exists)$ is homeomorphic to $nS \bigoplus m$.*

2) A finite Q-distributive lattice $(D, \exists)$ is simple iff there exists a natural m such that the dual of $(D, \exists)$ is homeomorphic to m.

Proof. 1) Similarly to Theorem 15 we have that $(\mathcal{C}(nS \bigoplus m), E)$ is subdirectly irreducible. Conversely, suppose the dual (X, E) of $(D, \exists)$ is not homeomorphic to $nS \bigoplus m$. Denote by X_0 the set of all open points of X. Observe that any set containing X_0 corresponds to a congruence on $(D, \exists)$, and that $X - X_0$ contains at least two points, say ξ_1 and ξ_2. Then $X_0 \cup \{\xi_1\}$ and $X - \{\xi_2\}$ correspond to such congruences θ_1 and θ_2 on $(D, \exists)$ that $\theta_1 \cap \theta_2 = \Delta$, where Δ denotes the diagonal congruence on $(D, \exists)$, and hence $(D, \exists)$ is subdirectly reducible.

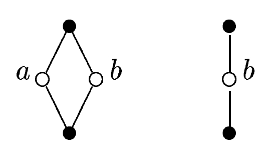
2) Obviously $(\mathcal{C}(m), E)$ is simple. Conversely, if the dual (X, E) of $(D, \exists)$ is not homeomorphic to m, then $X_0 \neq X$, and, as follows from 1), $(D, \exists)$ is not simple. □

As it is known from Cignoli [4], the variety of Q-distributive lattices is locally finite. Hence **QDist** and its every subvariety is generated by finite Q-distributive lattices. Therefore, the varieties of Q-distibutive lattices are characterized by means of finite Serpinski n-spaces.

7.5 Surjectiveness of epimorphisms and congruence extension

First note that, similarly to **Dist**, in **QDist** also exist epimorphisms which are not surjective. An example would be any inclusion map $i : D \hookrightarrow B$, where B is a Boolean algebra generated by its distributive sublattice D, and $\exists$ is discrete, that is $\exists a = a$, on both D and B, as well as an example shown below.

Now we show that, unlike **Dist**, **QDist** does not have the congruence extention property. Indeed, consider the following figure below:



Suppose $\mathcal{D}_1$ is the four element Q-distributive lattice, where $\exists 0 = 0$, $\exists a = \exists b = \exists 1 = 1$, while $\mathcal{D}_2$ - the three element Q-distributive lattice, where

$\exists 0 = 0$, $\exists b = \exists 1 = 1$. Obviously $\mathcal{D}_2$ is a Q-sublattice of $\mathcal{D}_1$. (Note that the dual of $\mathcal{D}_1$ is homeomorphic to 2, while the dual of $\mathcal{D}_2$ is homeomorphic to S.) Then it is a routine to check that the only congruences on $\mathcal{D}_1$ are the diagonal and the whole D_1^2. On the other hand, $\{(0,0),(b,b),(1,1),(b,1)\}$ is a congruence on $\mathcal{D}_2$ which is not extended to any congruence on $\mathcal{D}_1$. Therefore, we arrive at the following

Theorem 17 **QDist** *does not have the congruence extension property.* □

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