Extendible formulas in two variables in intuitionistic logic

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Dedicated to the memory of Leo Esakia

Abstract

We give alternative characterizations of exact, extendible and projective formulas in intuitionistic propositional calculus **IPC** in terms of n-universal models. From these characterizations we derive a new syntactic description of all extendible formulas of **IPC** in two variables. For the formulas in two variables we also give an alternative proof of Ghilardi's theorem that every extendible formula is projective.

1 Introduction

Exactly provable formulas (*exact formulas* for short) were introduced in [13] for Heyting arithmetic. These are the formulas that axiomatize propositional theories of substitutions. The definition of exact formulas directly transfers to the intuitionistic propositional calculus **IPC** (Definition 4.1). These formulas are called exactly provable, because given such a formula φ, there exists a substitution under which φ is provable, and also no propositional formula stronger than φ is provable. (For example, a proper disjunction like $\neg p \vee \neg\neg p$ can never be exactly provable because then, by the disjunction property, one of its disjuncts would be provable as well, and these disjuncts are clearly stronger than the original formula.) In **IPC** exact formulas also admit an algebraic characterization - an exact formula is the least element of the kernel of a homomorphism between free algebras. It follows from Pitts' uniform interpolation theorem [18] that in fact the theory of every **IPC**-substitution is axiomatized by an exact formula. Thus, exact formulas are exactly those formulas that axiomatize the theories of substitutions. This was first noted in [14]. De Jongh [13] described all (five up to equivalence) exact formulas in one variable in Heyting arithmetic and consequently in **IPC**. The characterization of exact formulas in two and more variables, however, was left open. This is a more complex task as even for two variables there are infinitely many non-equivalent exact formulas.

In order to reason about exact formulas de Jongh and Visser [15] introduced the notion of an *extendible formula* (Definition 4.7). They proved that every exact formula is extendible and conjectured that the converse is true as well. Unlike exact formulas, extendible formulas are defined semantically, and thus are easier to work with. For example, as it is illustrated in this paper, it is much easier to verify that a given formula is extendible than that it is exact.

Ghilardi [9] motivated by the theory of unification introduced *projective formulas* (Definition 4.14). Projective formulas are those that admit the most general unifiers. They are also closely connected to projective algebras as the quotient of a free algebra by a principal filter generated by a projective formula is a projective algebra. It is not hard to observe that every projective formula is exact and hence extendible. However, Ghilardi [9] also proved that every extendible formula is projective, thus closing the circle and confirming the de Jongh and Visser conjecture. So the three notions of exact, extendible and projective formulas coincide for **IPC**.

In this paper we discuss exact, extendible and projective formulas in the context of n-universal models. These models are dual to free algebras. n-universal models were thoroughly investigated by a number of authors [11], [20], [2], [19] (see [6, Sec. 8] and [3, Sec. 3] for an overview). Every formula in **IPC** corresponds to a particular subset of n-universal model, which we call *definable*. The algebra of all definable subsets of the n-universal model is isomorphic to the Lindenbaum-Tarski (free) algebra of **IPC** on n-generators. n-universal models can also be seen as "upper parts" of the n-canonical models of **IPC**.

We give alternative characterizations of exact, extendible and projective formulas using n-universal models and definable p-morphisms between them. For formulas in two variables this allows us to provide an alternative proof of Ghilardi's theorem that every extendible formula is projective. We also give a complete description of definable sets corresponding to these formulas in 2-universal model, and as a result, derive a syntactic description of all (infinitely many) exact, extendible and projective formulas in two variables.

Finally, we would like to mention that although Leo Esakia did not work on the particular topics studied in this paper, the duality of Heyting algebras developed by him is our crucial tool. One of the earliest works on dual characterizations of finitely generated modal and Heyting algebras was [8]. This paper gives a criterion for a Heyting or modal algebra to be finitely generated in terms of their dual frames. These results together with [12] precede all the forthcoming work on n-universal models and free Heyting algebras. Next to n-universal models, an important ingredient of our proofs throughout this paper is a dual correspondence between substitutions (Heyting algebra homomorphisms) and definable p-morphisms. This correspondence is also a part of Esakia duality for Heyting algebras. Thus, our work essentially applies the ideas developed by Leo Esakia.

The paper is organized as follows: In Section 2 we recall the Kripke semantics of **IPC** and basic operations on Kripke frames. In Section 3 we recall the structure of n-universal models. We also overview the connection between substitutions and definable p-morphisms. Section 4 gives characterizations of exact, extendible and projective formulas in terms of universal models. In Section 5 we give an alternative proof that every extendible formula is projective for formulas in two variables and provide a syntactic description of all extendible formulas in two variables. We finish the paper with some concluding remarks.

2 Preliminaries

For the definition and basic facts about intuitionistic propositional calculus **IPC** we refer to [6] or [7]. Here we briefly recall the Kripke semantics of intuitionistic logic.

Let $\mathcal{L}$ denote a *propositional language* consisting of

- infinitely many propositional variables (letters) $p_0, p_1, \ldots,$
- propositional connectives $\wedge$, $\vee$, $\rightarrow$,
- a propositional constant $\bot$.

We denote by Prop the set of all propositional variables. Formulas in $\mathcal{L}$ are defined as usual. Denote by Form($\mathcal{L}$) (or simply by Form) the set of all wellformed formulas in the language $\mathcal{L}$. We assume that $p, q, r, \dots$ range over propositional variables and $\varphi, \psi, \chi, \dots$ range over arbitrary formulas. For every formula φ and ψ we let $\neg\varphi$ abbreviate $\varphi \to \bot$ and $\varphi \leftrightarrow \psi$ abbreviate $(\varphi \to \psi) \wedge (\psi \to \varphi)$. We also let $\top$ abbreviate $\neg\bot$.

We now quickly recall the Kripke semantics for intuitionistic logic. Let R be a binary relation on a set W. For every $w, v \in W$ we write wRv if $(w, v) \in R$ and we write $\neg(wRv)$ if $(w, v) \notin R$.

Definition 2.1.

1. *An* intuitionistic Kripke frame *is a pair* $\mathfrak{F} = (W, R)$*, where* $W \neq \emptyset$ *and* R *is a partial order; that is, a reflexive, transitive and anti-symmetric relation on* W*.*

2. *An* intuitionistic Kripke model *is a pair* $\mathfrak{M} = (\mathfrak{F}, V)$ *such that* $\mathfrak{F}$ *is an intuitionistic Kripke frame and* V *is an* intuitionistic valuation; *that is, a map* $V : \textsc{Prop} \to \mathcal{P}(W)$*,*[1] *satisfying the condition:*

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p).$$

The definition of the satisfaction relation $\mathfrak{M}, w \models \varphi$, where $\mathfrak{M} = (W, R, V)$ is an intuitionistic Kripke model, $w \in W$ and $\varphi \in \textsc{Form}$ is given in the usual manner (see e.g., [6] or [7]). We will write $V(\varphi)$ for $\{w \in W | w \models \varphi\}$. The notions $\mathfrak{M} \models \varphi$ and $\mathfrak{F} \models \varphi$ (where $\mathfrak{F}$ is a Kripke frame) are also introduced as usual.

Let $\mathfrak{F} = (W, R)$ be a Kripke frame. $\mathfrak{F}$ is called *rooted* if there exists $w \in W$ such that for every $v \in W$ we have wRv. It is well known that **IPC** is complete with respect to finite rooted frames; see, e.g., [6, Thm. 5.12].

Next we recall the main operations on Kripke frames and models. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. For every $w \in W$ and $U \subseteq W$ let $R(w) = \{v \in W : wRv\}$, $R^{-1}(w) = \{v \in W : vRw\}$, $R(U) = \bigcup_{w \in U} R(w)$, and $R^{-1}(U) = \bigcup_{w \in U} R^{-1}(w)$. A subset $U \subseteq W$ is called an *upset of* $\mathfrak{F}$ if for every $w, v \in W$ we have that $w \in U$ and wRv imply $v \in U$. A frame $\mathfrak{F}' = (U, R')$ is called a *generated subframe of* $\mathfrak{F}$ if $U \subseteq W$, U is an upset of $\mathfrak{F}$ and R' is the restriction of R to U, i.e., $R' = R \cap U^2$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a Kripke model. A model $\mathfrak{M}' = (\mathfrak{F}', V')$ is called a *generated submodel of* $\mathfrak{M}$ if $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$ and V' is the restriction of V to U, i.e., $V'(p) = V(p) \cap U$.

[1] By $\mathcal{P}(W)$ we denote the powerset of W.

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be Kripke frames. A map $f : W \to W'$ is called a *p-morphism* [2] *between* $\mathfrak{F}$ *and* $\mathfrak{F}'$ if for every $w, v \in W$ and $w' \in W'$:

1. wRv implies $f(w)R'f(v)$,
2. $f(w)R'w'$ implies that there exists $u \in W$ such that wRu and $f(u) = w'$.

We call the conditions (1) and (2) the "forth" and "back" conditions, respectively. We say that f is *order-preserving* if it satisfies the forth condition. It is easy to see that f is a p-morphism iff for each $w \in W$ we have $f(R(w)) = R(f(w))$. If f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}'$, then $\mathfrak{F}'$ is called a *p-morphic image of* $\mathfrak{F}$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ and $\mathfrak{M}' = (\mathfrak{F}', V')$ be Kripke models. A map $f : W \to W'$ is called a *p-morphism between* $\mathfrak{M}$ *and* $\mathfrak{M}'$ if f is a p-morphism between $\mathfrak{F}$ and $\mathfrak{F}'$ and for every $w \in W$ and $p \in \text{Prop}$:

$$\mathfrak{M}, w \models p \text{ iff } \mathfrak{M}', f(w) \models p.$$

If f is surjective, then $\mathfrak{M}$ is called a *p-morphic image of* $\mathfrak{M}'$.

Next we recall the definition of general frames; see, e.g., [6, Sec. 8.1, 8.4].

Definition 2.2. *An* intuitionistic general frame *or simply a* general frame *is a triple* $\mathfrak{F} = (W, R, \mathcal{P})$*, where* (W, R) *is an intuitionistic Kripke frame and* $\mathcal{P}$ *is a set of upsets such that* $\emptyset$ *and* W *belong to* $\mathcal{P}$*, and* $\mathcal{P}$ *is closed under* $\cup$, $\cap$ *and* $\to$ *defined by*

$$U_1 \to U_2 := \{w \in W : \forall v(wRv \wedge v \in U_1 \to v \in U_2)\} = W - R^{-1}(U_1 - U_2).$$

Note that every Kripke frame can be seen as a general frame where $\mathcal{P}$ is the set of all upsets of $\mathfrak{F} = (W, R, \mathcal{P})$. A *valuation* on a general frame is a map $V : \text{Prop} \to \mathcal{P}$. The pair $(\mathfrak{F}, V)$ is called a *general model*. The validity of formulas in general models is defined exactly the same way as for Kripke models.

3 n-universal models of intuitionistic logic

In this section we recall (from e.g., [6, Sec. 8] and [3, Sec. 3]) the definition of the n-universal models of **IPC**. For $n \in \omega$ let $\mathcal{L}_n$ be the propositional language built on a finite set of propositional letters $\text{Prop}_n = \{p_1, \ldots, p_n\}$. Let Form_n denote the set of all formulas of $\mathcal{L}_n$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a (general) model.

[2] Some authors call such maps *bounded morphisms* [4] or *reductions* [6].

Definition 3.1. *With every point w of $\mathfrak{M}$, we associate a sequence $i_1 \dots i_n$ such that for $k = 1, \dots, n$:*

$$i_k = \begin{cases} 1 & if\ w \models p_k, \\ 0 & if\ w \not\models p_k. \end{cases}$$

We call the sequence $i_1 \dots i_n$ associated with w the color *of w, and denote it by $col(w)$.*

We define an order on the set of colors of length n.

Definition 3.2. *Let $i_1 \dots i_n$ and $j_1 \dots j_n$ be two colors. We write*

$$i_1 \dots i_n \leq j_1 \dots j_n \quad \textit{iff} \quad i_k \leq j_k \textit{ for each } k = 1, \dots, n.$$

We also write $i_1 \dots i_n < j_1 \dots j_n$ if $i_1 \dots i_n \leq j_1 \dots j_n$ and $i_1 \dots i_n \neq j_1 \dots j_n$.

Thus, the set of colors of length n ordered by $\leq$ forms a 2^n-element Boolean algebra. Let $\mathfrak{F} = (W, R)$ be a frame, and let U be a subset of W. A point $x \in U$ is called *U-maximal* (*U-minimal*) if for every $y \in W$ we have that xRy (yRx) and $x \neq y$ imply $y \notin U$. W-maximal and W-minimal points are simply called *maximal points* and *minimal points*. For every $U \subseteq W$ we let $max(U)$ and $min(U)$ denote the sets of all U-maximal and U-minimal points of U, respectively.

For a frame $\mathfrak{F} = (W, R)$ and $w, v \in W$, we say that a point w is an *immediate successor* of a point v if vRw, $w \neq v$, and there are no intervening points, i.e., for every $u \in W$ such that vRu and uRw we have $u = v$ or $u = w$. We call v an *immediate predecessor* of w if w is an immediate successor of v. We say that a set $A \subseteq W$ *totally covers* a point v and write $v \prec A$ if A is the set of all immediate successors of v. Note that $\prec$ is a relation relating points and sets. We will use the shorthand $v \prec w$ for $v \prec \{w\}$. Thus, $v \prec w$ means not only that w is an immediate successor of v, but that w is the only immediate successor of v. It is easy to see that if every point of W has only finitely many successors, then R is the reflexive and transitive closure of the immediate successor relation. Therefore, if (W, R) is such that every point of W has only finitely many successors, then R is uniquely defined by the immediate successor relation and vice versa. Thus, to define such a frame (W, R), it suffices to define the relation $\prec$. A set $A \subseteq W$ is called an *antichain* if $|A| > 1$ and for each $w, v \in A$, $w \neq v$ implies $\neg(wRv)$ and $\neg(vRw)$. Therefore, to define the n-universal model $\mathcal{U}(n) = (U(n), R, V)$, it

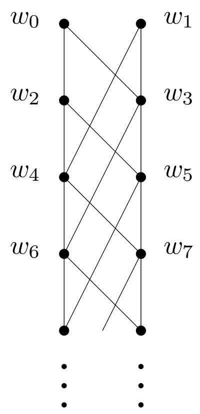

Figure 1: The 1-universal model

is sufficient to define the set $U(n)$, the relation $\prec$ relating points and sets, and the valuation V on $U(n)$. Let P be a property of Kripke models. We say that a model $\mathfrak{M}$ is a *minimal model with property* P if $\mathfrak{M}$ satisfies P and no proper submodel of $\mathfrak{M}$ satisfies P.

Definition 3.3. *The* n-universal model $\mathcal{U}(n)$ *is the minimal model satisfying the following three conditions.*

1. *max*$(\mathcal{U}(n))$ *consists of* 2^n *points of distinct colors.*
2. *For every* $w \in U(n)$ *and every color* $i_1 \dots i_n < col(w)$*, there exists a unique* $v \in U(n)$ *such that* $v \prec w$ *and* $col(v) = i_1 \dots i_n$.
3. *For every finite antichain* A *in* $U(n)$ *and every color* $i_1 \dots i_n$ *with* $i_1 \dots i_n \leq col(u)$ *for all* $u \in A$*, there exists a unique* $v \in U(n)$ *such that* $v \prec A$ *and* $col(v) = i_1 \dots i_n$.

The underlying frame of $\mathcal{U}(n)$ *is called the* n-universal frame.

It is well known (see, e.g., [3, Thm. 3.2.2]) that for every $n \in \omega$ the n-universal model of **IPC** exists and is unique up to isomorphism. The 1-universal model of **IPC** is shown in Figure 1, where $col(w_0) = 1$ and $col(w_n) = 0$ for each $n > 0$. The 1-universal model is often called the *Rieger-Nishimura ladder*. More generally, for each $n > 1$, one can think of the n-universal model of **IPC** as it is shown in Figure 2.

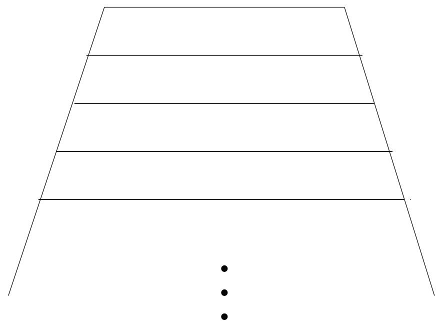

Figure 2: The n-universal model

There is a close connection between the universal models and the canonical models of intuitionistic logic. In fact, the universal models are "upper parts" of the canonical models. For the details see [6, Sec. 8.7] and [3, Sec. 3.2.2].

Next we recall the definition of a *depth* of a frame and point. Proofs involving n-universal models often use inductive arguments on the depths of points of the model.

Definition 3.4. *Let $\mathfrak{F}$ be a (general or Kripke) frame.*

1. *We say that $\mathfrak{F}$ is of* depth $n < \omega$, *denoted $d(\mathfrak{F}) = n$, if there is a chain of n points in $\mathfrak{F}$ and no other chain in $\mathfrak{F}$ contains more than n points.*

2. *We say that $\mathfrak{F}$ is of* infinite depth, *denoted $d(\mathfrak{F}) = \omega$, if for every $n \in \omega$, $\mathfrak{F}$ contains a chain consisting of n points. The frame $\mathfrak{F}$ is* of finite depth if $d(\mathfrak{F}) < \omega$.

3. *The* depth *of a point $w \in W$ is the depth of the subframe of $\mathfrak{F}$ generated by w. We denote the depth of w by $d(w)$.*

The following properties of n-universal models are well known see, e.g., [3, Thm. 3.2.16] and [6, Sec. 8.7].

Lemma 3.5.

1. *For every finite rooted model* $\mathfrak{M}$ *for* n *propositional variables, there exists a unique node* w *in* $\mathcal{U}(n)$ *such that the submodel of* $\mathcal{U}(n)$ *generated by* w *is a p-morphic image of* $\mathfrak{M}$.

2. *For every finite frame* $\mathfrak{F}$, *there exists a valuation* V *on* $\mathfrak{F}$ *and* $n \leq |\mathfrak{F}|$ *such that* $(\mathfrak{F}, V)$ *is a generated submodel of* $\mathcal{U}(n)$.

Let U be a subset of $U(n)$. In the remainder of the paper we often use the notation $-U$ to denote $U(n) - U$.

Lemma 3.6. *For every upset* $U \subseteq U(n)$ *we have that* $U = -R^{-1}(S)$, *where* $S = max(-U)$.

Proof. The result follows immediately from the fact that, by construction, the n-universal model is dually well-founded. □

Then next theorem has a simple algebraic explanation/proof. However, the reader can verify it just using the construction of universal models.

Theorem 3.7. *For each* $i \leq n$ *the generated submodel of* $\mathcal{U}(n)$ *based on the upset* $V(p_i)$ *is isomorphic to* $\mathcal{U}(n-1)$.

The following result describes the crucial property of n-universal models. It is an immediate consequence of Lemma 3.5 and the fact that **IPC** is complete with respect to finite rooted frames.

Theorem 3.8. *For every formula* φ *in the language* $\mathcal{L}_n$, *we have*

$$\mathbf{IPC} \vdash \varphi \quad \textit{iff} \quad \mathcal{U}(n) \models \varphi.$$

Obviously, every formula in n variables defines an upset of the n-universal model. Moreover, every upset of the 1-universal model is defined by a formula in 1 variable (see, e.g., Section 5). It is well known, however, that for $n > 1$ not every upset of the n-universal model is defined by a formula, e.g., Section 5. This leads to the following important definition.

Definition 3.9. *We call a set* $U \subseteq U(n)$ definable *if there is a formula* $\varphi(p_1, \ldots, p_n)$ *such that* $U = \{w \in U(n) : w \models \varphi\}$.

We will close this section by recalling the connection of n-universal models and Lindenbaum-Tarski algebras of intuitionistic logic.

Let $F(n)$ be the Lindenbaum-Tarski algebra of **IPC** of the formulas in n variables; that is, the algebra of all formulas in n variables modulo **IPC**-equivalence. This algebra is also called the *n-generated free Heyting algebra.* Let $Up(\mathcal{U}(n))$ denote the algebra of all definable subsets of $\mathcal{U}(n)$, respectively. The next theorem spells out the crucial connection between n-universal models and Lindenbaum-Tarski algebras. For the proof we refer to either of [6, Sec. 8.6 and 8.7], [11, §2], [2], [20] and [19], [3, Thm. 3.2.20], [16].

Theorem 3.10. *The algebra of all definable upsets of the n-universal model is isomorphic to the Lindenbaum-Tarski algebra of* **IPC** *on n generators; that is, $Up(\mathcal{U}(n))$ are isomorphic to $F(n)$.*

We have the following useful consequence of Theorem 3.10.

Corollary 3.11. *Let φ and ψ be formulas in n variables. Then*

$$\varphi \vdash \psi \text{ iff } w \in V(\varphi) \Rightarrow w \in V(\psi) \text{ for each } w \in U(n).$$

It is well known that every substitution σ can be seen as a homomorphism between free algebras. The next theorem characterizes those maps between universal frames that correspond to substitutions.

Definition 3.12. *A frame p-morphism $f : \mathcal{U}(m) \to \mathcal{U}(n)$ is called* definable *if for every definable upset $U \subseteq \mathcal{U}(n)$ the upset $f^{-1}(U)$ is definable.*

Obviously, for $f : \mathcal{U}(m) \to \mathcal{U}(n)$ to be a definable p-morphism it suffices that $f^{-1}(V(p_i))$ is definable for each $i \leq n$. The fact that the inverse image of an upset is an upset follows from the fact that f is order-preserving. We are now ready to spell out the connection between substitutions and definable p-morphisms. The next theorem is a direct consequence of the duality of Heyting algebras. For the details see e.g., [6, Thm. 8.57 and 8.59], or [3, Thm. 2.3.25].

Theorem 3.13.

1. *For every substitution $\sigma : F(n) \to F(m)$ there exists a definable frame p-morphism $f : \mathcal{U}(m) \to \mathcal{U}(n)$ such that for every formula φ in n variables we have*
$$V(\sigma\varphi) = f^{-1}(V(\varphi)).$$
where V is the valuation of $\mathcal{U}(n)$.

2. *For every definable frame p-morphism* $f : \mathcal{U}(n) \to \mathcal{U}(m)$ *the map* $\sigma : F(m) \to F(n)$ *defined by*

$$\sigma\varphi = \psi \text{ iff } f^{-1}(V(\varphi)) = V(\psi)$$

is a substitution.

In the second part of this paper we will make a substantial use of this theorem.

Finally, we recall that every point generated upset of the n-universal model is definable. We also recall the particular structure of the formulas that define them. This will be used consequently in the paper.

Definition 3.14 (de Jongh [12])**.** *Let* w *be a point in the* n*-universal model. The formulas* φ_w *and* ψ_w *are defined inductively. If* $d(w) = 0$*, then let*

$$\varphi_w = \bigwedge\{p_k : w \models p_k\} \wedge \bigwedge\{\neg p_j : w \not\models p_j\} \text{ for each } k, j = 1, \ldots, n$$

and

$$\psi_w = \neg\varphi_w.$$

If $d(w) > 0$*, then let* $\{w_1, \ldots, w_m\}$ *be the set of all immediate successors of* w*. We let*

$$prop(w) = \{p_k : w \models p_k\}$$

and

$$newprop(w) = \{p_k : w \not\models p_k \text{ and } w_i \models p_k \text{ for each } i \text{ such that } 1 \leq i \leq m\}.$$

The formulas φ_w *and* ψ_w *are defined by*

$$\varphi_w := \bigwedge prop(w) \wedge \left((\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i}) \to \bigvee_{i=1}^{m} \varphi_{w_i} \right)$$

and

$$\psi_w = \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}$$

The formulas φ_w *and* ψ_w *are called the* de Jongh formulas.

For the proof of the next theorem the reader is referred to [12] and [3, Thm. 3.2.2].

Theorem 3.15. *For every $w \in U(n)$ we have*

- $R(w) = \{v \in U(n) : v \models \varphi_w\}$, *i.e.,* $V(\varphi_w) = R(w)$.
- $U(n) - R^{-1}(w) = \{v \in U(n) : v \models \psi_w\}$, *i.e.,* $V(\psi_w) = U(n) - R^{-1}(w)$.

Using this theorem it was shown in [16] that the isomorphism of $\mathcal{U}(n)$ to the upper part of the n-canonical model is given by $f(w) = Cn_n(\varphi_w)$ where $Cn_n(\varphi_w) = \{\psi(p_1, \ldots, p_n) | \varphi_w \vdash \psi\}$.

4 Exact, extendible and projective formulas

In this section we recall the definitions of exact, extendible and projective formulas and give alternative characterizations of these formulas in terms of n-universal models.

4.1 Exact formulas

We start by discussing exact formulas.

Definition 4.1 (de Jongh [13])**.** *A formula φ is called an* exact formula *if there is a substitution σ such that*

1. $\mathbf{IPC} \vdash \sigma(\varphi)$,
2. *For any formula ψ, if* $\mathbf{IPC} \vdash \sigma(\psi)$, *then* $\varphi \vdash \psi$.

We first look into an algebraic characterization of exact formulas, which also provides an additional motivation for introducing them. Let $\sigma : F(n) \to F(m)$ be a substitution. The *theory of* σ is the filter $\sigma^{-1}(\top)$. The theory of σ is finitely axiomatizable if $\sigma^{-1}(\top)$ is a principal filter; that is, if there exists a formula $\varphi \in F(n)$ such that $\sigma^{-1}(\top) = [\varphi)$. The next proposition shows that exact formulas are exactly those formulas that axiomatize the theories of substitutions.

Proposition 4.2. *A formula φ is exact iff there is a substitution $\sigma : F(n) \to F(m)$, such that φ axiomatizes the theory of σ.*

Proof. The proof is just spelling out the definitions. $\square$

Next we discuss the question whether the theory of every substitution is finitely axiomatizable. In fact, its positive answer is a direct consequence of Pitts' Uniform Interpolation Theorem. We formulate this result in its more general form; see e.g., [18, 10].

Theorem 4.3 (Pitts [18])**.** *Every substitution $\sigma : F(n) \to F(m)$ possesses right and left adjoints $\exists_\sigma, \forall_\sigma : F(m) \to F(n)$. That is, for any formula $\varphi(p_1, \dots, p_m)$, there are formulas $\exists_\sigma \varphi$ and $\forall_\sigma \varphi$ in n variables such that for any $\psi(p_1, \dots, p_n)$*

$$(1) \quad \varphi \vdash \sigma(\psi) \text{ iff } \exists_\sigma \varphi \vdash \psi$$
$$(2) \quad \sigma(\psi) \vdash \varphi \text{ iff } \psi \vdash \forall_\sigma \varphi.$$

Pitts' theorem immediately implies the following result (see also [14]).

Corollary 4.4.

1. *For every substitution $\sigma : F(n) \to F(m)$, the theory of σ is finitely axiomatizable by the formula $\exists_\sigma \top$.*

2. *A formula φ is exact iff there is a substitution $\sigma : F(n) \to F(n)$ such that φ is equivalent to $\exists_\sigma \top$.*

Proof. The proof is spelling out the definitions. □

Let $f : \mathcal{U}(m) \to \mathcal{U}(n)$ be the definable p-morphism corresponding to a substitution σ. Then $\exists_\sigma$ and $\forall_\sigma$ can be seen as maps from $Up(U(m))$ to $Up(U(n))$. The next proposition, which is well known, characterizes these maps in terms of f. We give the proof to make the paper more self contained.

Proposition 4.5. *Let $\sigma : F(n) \to F(m)$ be a substitution and $f : \mathcal{U}(m) \to \mathcal{U}(n)$ the corresponding definable p-morphism. Then for every formula φ in m variables we have:*

$$V(\exists_\sigma(\varphi)) = f(V(\varphi))$$
$$V(\forall_\sigma(\varphi)) = -R^{-1} f(-V(\varphi)).$$

Proof. It is well known that if an adjoint exists, then it is unique. Therefore, to prove the proposition it is sufficient to show that for each definable upset $U \subseteq U(m)$ and $V \subseteq U(n)$ the following holds:

1. $f(U) \subseteq V$ iff $U \subseteq f^{-1}(V)$

2. $f^{-1}(V) \subseteq U$ iff $V \subseteq -R^{-1}f_\sigma(-U)$.

(1) Suppose $f(U) \subseteq V$. Then $f^{-1}(f(U)) \subseteq f^{-1}(V)$. Since $U \subseteq f^{-1}(f(U))$, we obtain that $U \subseteq f^{-1}(V)$. Conversely, let $U \subseteq f^{-1}(V)$. Then $f(U) \subseteq f(f^{-1}(V))$. Since $f(f^{-1}(V)) \subseteq V$, we obtain that $f(U) \subseteq V$.

(2) Now let $f^{-1}(V) \subseteq U$ and suppose $x \notin -R^{-1}f(-U)$. Then $x \in R^{-1}f(-U)$. Therefore, there exists $y \in f(-U)$ such that xRy. The fact that $y \in f(-U)$ means that there is $z \notin U$ such that $f(z) = y$. Since $f^{-1}(V) \subseteq U$, we have that $z \notin f^{-1}(V)$. On the other hand, if $x \in V$ we also have that $y \in V$ (V is an upset). Therefore, $z \in f^{-1}(V)$, a contradiction. Thus, $x \notin V$ and $V \subseteq -R^{-1}f(-U)$.

Conversely, suppose $V \subseteq -R^{-1}f(-U)$ and $x \in f^{-1}(V)$, then $f(x) \in V$. If $x \notin U$, then $f(x) \in f(-U)$. Since R is reflexive we have $f(x) \in R^{-1}f(-U)$. On the other hand, $V \subseteq -R^{-1}f(-U)$ implies $f(x) \notin R^{-1}f(-U)$. This is a contradiction. Therefore, $f^{-1}(V) \subseteq U$. □

This result gives us the following characterization of exact formulas in terms of universal models.

Corollary 4.6. *A formula $\varphi(p_1, \ldots, p_n)$ is exact iff there exists a definable p-morphism $f : \mathcal{U}(n) \to \mathcal{U}(n)$ such that $V(\varphi) = f(U(n))$.*

Proof. The result follows immediately from Corollary 4.4(2) and Proposition 4.5. □

4.2 Extendible formulas

It this section we define extendible formulas and prove that every exact formula is extendible.

Definition 4.7 (de Jongh, Visser [15])**.** *A formula φ is called* extendible *if any finite disjoint union of finite rooted Kripke models validating φ can be extended to a Kripke model validating φ by adding a new root to this disjoint union of Kripke models.*

This concept was introduced in an attempt to find a semantic characterization of exact formulas later given by Ghilardi. It is easy to give some examples of extendible formulas.

Lemma 4.8. *For every formula φ, the formula $\varphi \to p$ is extendible.*

Proof. Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ be such that $\mathfrak{M}_i \models \varphi \to p$ for each $i \leq n$. Let $\mathfrak{F}$ be a frame obtained by adding a new root w to the disjoint union of $\mathfrak{M}_i$'s. To prove that $\varphi \to p$ us extendible we need to define a valuation on w such that $w \models \varphi \to p$. If $\mathfrak{M}_i \models p$, for every $i \leq n$, then we let $w \models p$, which obviously implies that $w \models \varphi \to p$. Now suppose there exists $i \leq n$ such that $\mathfrak{M}_i \not\models p$. Then, we let $w \not\models p$. Since $\mathfrak{M}_i \models \varphi \to p$, we have that $\mathfrak{M}_i \not\models \varphi$. Therefore, $w \not\models \varphi$, and we again obtain that $w \models \varphi \to p$. Thus, $\varphi \to p$ is an extendible formula. $\square$

Next we will characterize extendible formulas in terms of universal models.

Definition 4.9. *An upset U of the n-universal model $\mathcal{U}(n)$ is called* extendible *if for every finite antichain $\Delta \subseteq U$, there exists a point $w \in U$ such that $w \prec \Delta$.*

In fact, there is a one-to-one correspondence between definable extendible upsets of the n-universal model and extendible formulas in n-variables. The next theorem was first proved in [1, Thm. 12].

Theorem 4.10. *A formula $\varphi(p_1, \ldots, p_n)$ is extendible iff $V(\varphi)$ is an extendible subset of the n-universal model $\mathcal{U}(n)$.*

Proof. The direction from left to right is obvious. So, assume $V(\varphi)$ is an extendible subset of the n-universal model $\mathcal{U}(n)$. Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ be such that $\mathfrak{M}_i \models \varphi$. Then, by Lemma 3.5(1), there are unique p-morphisms f_i mapping the $\mathfrak{M}_i$ onto $R(w_i)$ for some elements w_i in $\mathcal{U}(n)$. By the property of p-morphisms, the w_i are elements of $V(\varphi)$. As $V(\varphi)$ is an extendible subset of $\mathcal{U}(n)$ there exists a $w \in V(\varphi)$ such that $w \prec \{w_1, \ldots, w_k\}$. We add a new root v below the disjoint union of $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ with $col(v) = col(w)$ to obtain a new model $\mathfrak{M}$. It is easy to see that the p-morphisms f_i can be joined to one p-morphism f from $\mathfrak{M}$ onto $R(w)$ by taking $f(v) = w$. Again by the property of p-morphisms, $\mathfrak{M} \models \varphi$. $\square$

We are now ready to give an alternative proof of the fact that every exact formula is extendible using universal models.

Theorem 4.11 (de Jongh and Visser [15])**.** *If a formula φ is exact, then it is extendible.*

Proof. Let φ be an exact formula in n variables. Then by Corollary 4.6, there is a definable p-morphism $f : \mathcal{U}(n) \to \mathcal{U}(n)$ such that $V(\varphi) = f(U(n))$. We show that $V(\varphi)$ is an extendible upset. Let $\Delta \subseteq V(\varphi)$ be any finite antichain. Since f is order-preserving, $f^{-1}(\Delta)$ is an antichain of $\mathcal{U}(n)$. Let $x \in U(n)$ be such that $x \prec f^{-1}(\Delta)$ (by the construction of $\mathcal{U}(n)$ we know that such a point always exists). Then $f(x) \in V(\varphi)$ and it is easy to see that $f(x) \prec \Delta$. Therefore, $V(\varphi)$ is an extendible upset of $\mathcal{U}(n)$ and, by Theorem 4.10, φ is an extendible formula. $\square$

In the remainder of this section we characterize extendible upsets of n-universal models.

Definition 4.12. *For every upset U of the n-universal model, we call a point $x \in U(n)$ a U-*border point *if $x \notin U$, but for every proper successor y of x we have $y \in U$. For every upset U let $B(U)$ denote the set of all U-border points.*

In other words, w is a U-border point if $w \in max(-U)$. Let φ be any formula. The $V(\varphi)$-border points we simply call *φ-border points*. Let $B(\varphi)$ denote the set of all φ-border points. The points that belong to $V(\varphi)$ will be called φ-points. Let $x \in U(n)$ be totally covered by an antichain Δ; that is, $x \prec \Delta$. We call a point $y \in U(n)$ a *sister* of x if $x \neq y$ and $y \prec \Delta$. (If Δ is empty both points x and y are endpoints.)

Theorem 4.13. *Let U be an upset of the n-universal model $\mathcal{U}(n)$. Then U is extendible iff $U = -R^{-1}(S)$, where $S \subseteq \bigcup_{i=1}^{n} V(p_i) \cup \bigcup_{i=1}^{n} B(p_i)$, and if $x \in S$, then x has a sister $y \notin S$.*

Proof. By Lemma 3.6, we have that $U = -R^{-1}(S)$, for some (finite or infinite) antichain S in $U(n)$. Now assume U is extendible, and suppose there is $z \in S$ such that $z \notin \bigcup_{i=1}^{n} V(p_i) \cup \bigcup_{i=1}^{n} B(p_i)$. First note that since $z \notin \bigcup_{i=1}^{n} B(p_i)$, there is no u such that $z \prec u$; otherwise, by the structure of the n-universal model z would be a p_i-border point for some $i \leq n$. Let Δ be an antichain such that $z \prec \Delta$. It also follows from the structure of $\mathcal{U}(n)$ that there is no other point that is totally covered by Δ; otherwise z is a p_i point or a p_i-border point for some $i \leq n$. Since $z \in S$, we have that $\Delta \subseteq U$. This means that U is not extendible, which is a contradiction. Therefore, $S \subseteq \bigcup_{i=1}^{n} V(p_i) \cup \bigcup_{i=1}^{n} B(p_i)$. Finally, note that if $x \in S$ and all the sisters of x also belong to S, then the antichain Δ that totally covers x belongs to

U and every point y such that $y \prec \Delta$ is outside U, which again contradicts the extendibility of U.

The converse implication is similar. $\square$

In Section 5 we will give a complete characterization of the definable extendible upsets of the 2-universal model $\mathcal{U}(2)$.

4.3 Projective formulas

In this section we recall the definition of projective formulas and give an alternative proof using universal models that every projective formulas is exact and extendible.

Definition 4.14 (Ghilardi [9]). *A formula φ is called* projective *if there is a substitution σ (called a* projective substitution*) such that*

1. $\mathbf{IPC} \vdash \sigma(\varphi)$,
2. *For any formula ψ, we have $\varphi \vdash \psi \leftrightarrow \sigma(\psi)$.*

For the motivation for introducing projective formulas and their connection with the theory of unification we refer to [9]. We will just briefly mention the connection between projective formulas and projective Heyting algebras. For the definition of projective and finitely presentable algebras see [5]. For the next theorem see Grigolia [11] and Ghilardi [9]; although neither of the references state the result explicitly. A direct proof can be found in [1, Cor. 9].

Theorem 4.15. *There is a one-to-one correspondence between projective formulas and finitely presentable, projective Heyting algebras. In particular, for every projective formula φ in n variables the algebra $F(n)/[\varphi)$ is a finitely presentable projective algebra, and for every finitely presentable projective Heyting algebra A there exists a projective formula φ such that A is isomorphic to $F(n)/[\varphi)$.*

We will now explore the connection of projective formulas with the exact and extendible formulas. We first note the following simple fact.

Theorem 4.16. *Every projective formula is exact.*

Proof. In fact the substitution σ that makes φ projective will also make it exact. For this all one needs to observe is that if $\vdash \sigma(\psi)$, then $\varphi \vdash \psi \leftrightarrow \sigma(\psi)$ implies $\varphi \vdash \psi$. $\square$

Next we give a characterization of projective formulas in terms of the universal models.

Theorem 4.17. *A formula φ is projective iff there exists a definable p-morphism $f : \mathcal{U}(n) \to \mathcal{U}(n)$ such that $V(\varphi) = f(U(n))$ and $f(x) = x$, for every $x \in V(\varphi)$.*

Proof. Suppose φ is such that there exists a definable p-morphism $f : \mathcal{U}(n) \to \mathcal{U}(n)$ with $V(\varphi) = f(U(n))$ and $f(x) = x$, for every $x \in V(\varphi)$. Let σ be the substitution that corresponds to f. Then $f^{-1}(V(\varphi)) = U(n)$. Therefore, $\vdash \sigma(\varphi)$. Let $x \in V(\varphi)$ and let ψ be some formula. Then as $f(x) = x$, we have that $x \models \psi$ is equivalent to $f(x) \models \psi$, which is equivalent to $x \models \sigma(\psi)$. Thus, by Corollary 3.11, $\varphi \vdash \psi \leftrightarrow \sigma\psi$ and φ is projective.

Now assume φ is projective and let σ be its projective substitution. Let f be a definable p-morphism corresponding to σ. Then $\vdash \sigma(\varphi)$ implies that $f^{-1}(V(\varphi)) = U(n)$ and therefore $V(\varphi) \supseteq f(U(n))$. Next we show by induction on the depth of the elements of $V(\varphi)$ that for every $x \in V(\varphi)$ we have $f(x) = x$. Since f is a p-morphism, for every $x \in U(n)$ we have $f(R(x)) = R(f(x))$. So if $x \in V(\varphi)$ is a maximal point of $\mathcal{U}(n)$, then $f(x)$ is also a maximal point of $\mathcal{U}(n)$. Let $\varphi_{f(x)}$ be the de Jongh formula of the point $f(x)$. Then $f(x) \models \varphi_{f(x)}$ implies that $x \models \sigma(\varphi_{f(x)})$. Since $x \models \varphi$ and φ is projective, by Corollary 3.11, we have that $x \models \varphi_{f(x)}$. This, by Theorem 3.15, implies that $f(x)Rx$, which by the maximality of $f(x)$ is possible only if $f(x) = x$. Now assume that x is of depth $k > 0$ and the theorem is true for every point y of depth $< k$. That is, for every immediate successor y of x we have $f(y) = y$. This, together with $f(R(x)) = R(f(x))$, gives us that the set of immediate (in fact all proper) successors of x and the set of immediate (in fact all proper) successors of $f(x)$ are the same. Therefore, x and $f(x)$ are totally covered by the same antichain. Now exactly the same argument as above using the de Jongh formula of $f(x)$ shows that $f(x)Rx$, which implies $f(x) = x$. So we showed that for each $x \in V(\varphi)$ we have $f(x) = x$. This of course also proves that $V(\varphi) \subseteq f(U(n))$. Therefore, $V(\varphi) = f(U(n))$ and $f(x) = x$ for every $x \in V(\varphi)$. $\square$

Remark 4.18. The property of projective formulas φ discussed in Theorem 4.15, in fact, means that $V(\varphi)$ is a retract of $\mathcal{U}(n)$. This, in its turn means that the algebra $F(n)/[\varphi)$ that corresponds to the upset $V(\varphi)$ is a retract of $F(n)$. It is known (see, e.g., [5]) that an algebra A is a retract of a free algebra iff it is projective. This gives us another proof of the fact that φ is a projective formula iff $F(n)/[\varphi)$ is a projective algebra. We will skip the details.

5 Extendible formulas in two variables

In this section we will concentrate on formulas in two variables. In the first subsection we give a complete description of the definable extendible upsets of the 2-universal model. After that we give an alternative proof of Ghilardi's theorem for formulas in two variables. In the second subsection we describe explicitly all the extendible formulas in two variables.

5.1 Extendible formulas in 2 variables are projective

We start with the characterization of the definable extendible subsets of the 2-universal model. We will use the propositional letters p and q instead of p_1 and p_2. First we will prove an auxiliary lemma. We call the sets $F \subseteq B(p)$ and $G \subseteq B(q)$ cofinite, if the sets $B(p) - F$ and $B(q) - G$ are finite.

Lemma 5.1. *Let φ be some formula in 2 variables. Then the sets $V(\varphi) \cap B(p)$ and $V(\varphi) \cap B(q)$ are either finite or cofinite.*

Proof. We prove the lemma by induction on the complexity of φ. If φ is p or q, then obviously the set of border points that satisfy φ is empty. Now suppose φ has the form $\psi \wedge \chi$. Then, $\{w \in B(p) : w \models \psi \wedge \chi\} = \{w \in B(p) : w \models \psi\} \cap \{w \in B(p) : w \models \chi\}$. By the induction hypothesis, $\{w \in B(p) : w \models \psi\}$ and $\{w \in B(p) : w \models \chi\}$ are finite or cofinite. Therefore, their intersection is also finite or cofinite. The case of $\vee$ is similar. Finally, suppose φ has the form $\psi \to \chi$. If there exists a point $v \in V(p)$ such that $v \not\models \psi \to \chi$, then every p-border point which is below v also does not satisfy $\psi \to \chi$. Thus, the set $\{w \in B(p) : w \models \varphi\}$ is finite. Now, assume for each $v \in V(p)$, we have that $v \models \psi \to \chi$, then obviously $\{w \in B(p) : w \models \psi \to \chi\} = \{w \in B(p) : w \not\models \psi \text{ or } w \models \chi\} = (B(p) - \{w \in B(p) : w \models \psi\}) \cup \{w \in B(p) : w \models \chi\}$. By the induction hypothesis, both sets are finite or cofinite. Therefore, $\{w \in B(p) : w \models \varphi\}$ is finite or cofinite as well. □

The next corollary gives a necessary condition for extendible upsets of $\mathcal{U}(2)$ to be definable.

Corollary 5.2. *Let U be an upset of the 2-universal model $\mathcal{U}(2)$. If U is definable and extendible, then $U = -R^{-1}(S)$, where $S \subseteq V(p) \cup V(q) \cup B(p) \cup B(q)$, the sets $S \cap B(p)$ and $S \cap B(q)$ are finite or cofinite subsets of $B(p)$ and $B(q)$, respectively, and if $x \in S$, then there exists a sister y of x such that $y \notin S$.*

Proof. Assume $U \subseteq U(2)$ is extendible and definable. By Theorem 4.13, all we need to show is that $S \cap B(p)$ and $S \cap B(q)$ are finite or cofinite. Since U is definable, there exists a formula φ such that $U = V(\varphi)$. Then $B(p) - (S \cap B(p)) \subseteq V(\varphi)$ and $B(q) - (S \cap B(q)) \subseteq V(\varphi)$. By Lemma 5.1, these sets are finite or cofinite, which implies that $S \cap B(p)$ and $S \cap B(q)$ are also finite or cofinite.

□

In the last section of the paper we will show that in fact the converse of Corollary 5.2 also holds.

We are now ready to give an alternative proof (for the restricted case of formulas in two variables) of Ghilardi's theorem that the exact, extendible and projective formulas are the same.

Theorem 5.3 (Ghilardi [9])**.** *Let φ be a formula in 2 variables. The following three conditions are equivalent.*

1. *φ is projective,*

2. *φ is exact,*

3. *φ is extendible.*

Proof. (1) $\Rightarrow$ (2) is Theorem 4.16. and (2) $\Rightarrow$ (3) is Theorem 4.11. We now prove that (3) implies (1). In fact, this proof will also be a direct proof of (3) $\Rightarrow$ (2).

Let φ be an extendible formula. Then $V(\varphi)$ is an extendible upset of the 2-universal model. We assume that $V(\varphi)$ is nonempty. By Theorem 4.13, we have that $V(\varphi) = -R^{-1}(S)$, where S is an antichain that contains only p-points, q-points, p-border points and q-border points. We will construct a definable frame p-morphism $f : U(2) \to U(2)$, such that $f(U(2)) = V(\varphi)$, and $f(x) = x$, for every $x \in V(\varphi)$. By Theorem 4.17 this will imply that φ is a projective formula.

Note that by Corollary 5.2, for every x in S, x has a sister belonging to $V(\varphi)$. Intuitively speaking, f will be the the least definable p-morphism that maps the points in S to their sisters. The four maximal points of $\mathcal{U}(2)$ will play a somewhat special role. Let us name them m_{11}, m_{10}, m_{01} and m_{00} with the obvious meaning. Note that m_{11} is both a p-point and a q-point, that m_{10} is a p-point as well as a q-border point, analogously for m_{01}, and that m_{00} is both a p-border point and a q-border point. Let us note that $max(V(\varphi))$ is the subset of those 4 points that are members of $V(\varphi)$. If this set is a singleton the situation becomes very special, and we leave it to the end of the proof. So, for the main part of the proof we assume $max(V(\varphi))$ to contain at least two elements.

Two other points will play a special role. First, the point with color 10 which has m_{11} as its only proper successor: let us call it $m_{10\to 11}$. Similarly, $m_{01\to 11}$.

Let $S_p = (S \cap V(p)) - \{m_{10\to 11}\}$ and $S_q = (S \cap V(q)) - \{m_{01\to 11}\}$. Let also $T_{B(p)} = B(p) - V(\varphi)$ and $T_{B(q)} = B(q) - V(\varphi)$. The special role of $m_{10\to 11}$ is caused by the fact that this point is a p-point as well as a q-border point, and we prefer to treat it as the latter. The point $m_{01\to 11}$ gives similar problems of course.

We define f as the composition of four definable p-morphisms $f_{B(p)}$, $f_{B(q)}$, f_p, f_q constructed using these four sets. One or more of these four p-morphisms may be left completely undefined, and is then supposed to be left out of this composition.

Definition of $f_{B(p)}(x)$ by distinguishing cases depending on the relationship of x to $T_{B(p)}$. If neither m_{11} nor m_{10} is in $V(\varphi)$ (i.e. $V(\varphi) \cap V(p)$ is empty), then $f_{B(p)}$ is left undefined. Otherwise:

1. $x \in -R^{-1}(T_{B(p)})$. Then $f_{B(p)}(x) = x$.

2. $x \in T_{B(p)}$. We distinguish three cases:

   (a) x has one successor y in $V(p)$. Then $f_{B(p)}(x) = y$.

   (b) x has two successors in $V(p)$. Then $f_{B(p)}(x) = y$ where y is x's sister in $V(p)$.

(c) x has no successors: x is m_{01} or m_{00}. Then $f_{B(p)}(x) = m_{10}$ if m_{10} is in $V(\varphi)$, otherwise m_{11}.

Note that in each case $f_{B(p)}(x) \in V(p)$.

3. $x \in R^{-1}(T_{B(p)}) - T_{B(p)})$. We define $f_{B(p)}$ by induction on the layers of $R^{-1}(T_{B(p)})$. Let $x \prec \Delta$ and let $f_{B(p)}$ be already defined on the elements of Δ. If $min(f_{B(p)}(\Delta))$ consists of a single point z we define $f_{B(p)}(x) = z$. Otherwise, $min(f_{B(p)}(\Delta))$ is an antichain consisting of two points z and u with $z \models p$ and $u \not\models p$, and we let $f_{B(p)}(x) = z$. By induction, in all cases $f_{B(p)}(x) \in V(p)$.

Checking that $f_{B(p)}$ is a well-defined p-morphism is routine, it follows from the easily checked fact that, if $f_{B(p)}(x) \prec \Delta$, then the elements of Δ are $f_{B(p)}(y)$ for some y with xRy.

Claim 5.4. $f_{B(p)}(U(2)) = -R^{-1}(T_{B(p)})$.

Proof. Since $f_{B(p)}(x) = x$ for every $x \in -R^{-1}(T_{B(p)})$, we have $-R^{-1}(T_{B(p)}) \subseteq f_{B(p)}(U(2))$. The converse inclusion follows by induction on the layers of $R^{-1}(T_{B(p)})$. Note that Case 2 of the definition of $f_{B(p)}$ guarantees that $f(T_{B(p)}) \subseteq -R^{-1}(T_{B(p)})$ and the later f-values automatically get into $-R^{-1}(T_{B(p)})$ as well. □

Next we show that $f_{B(p)}$ is definable. We will prove that it is defined by the substitution $\sigma_{B(p)}$ given by $\sigma_{B(p)}(p) = \varphi \to p$, $\sigma_{B(p)}(q) = q$. We will need the following claim.

Claim 5.5. *If $f_{B(p)}(x)$ is defined, we have for every $x \in U(2)$*

$$x \models \varphi \to p \textit{ iff } f_{B(p)}(x) \models p.$$

Proof. $\Longleftarrow$: Assume this direction fails. Let x be a maximal point with the property $x \not\models \varphi \to p$ and $f_{B(p)}(x) \models p$. There exists y such that xRy, $y \models \varphi$ and $y \not\models p$. Since $f_{B(p)}$ is order-preserving we have $f_{B(p)}(y) \models p$. This, by the maximality of x, implies that $x = y$. Thus, $x \in V(\varphi)$ and $x \notin V(p)$. Since $V(\varphi) \subseteq -R^{-1}(T_{B(p)})$, by Case 1 of the definition of $f_{B(p)}(x)$, $f_{B(p)}(x) = x$, hence $f_{B(p)}(x) \models p$, a contradiction.

$\Longrightarrow$: Assume this direction fails. Let x be a maximal point with the property $f_{B(p)}(x) \not\models p$ and $x \models \varphi \to p$. Let Δ be such that $x \prec \Delta$. (Note that Δ is non-empty, since otherwise $x = m_{01}$ or $x = m_{00}$ and then $f_{B(p)}(x) \models p$ by the definition of f.) By the maximality of x, we have that $f_{B(p)}(\Delta) \subseteq V(p)$. If $x \in R^{-1}(T_{B(p)})$, then by (2) and (3) of the definition of $f_{B(p)}$ we have that $f_{B(p)}(x) \models p$, a contradiction. Therefore, $x \notin R^{-1}(T_{B(p)})$. Then, by (1) of the definition of $f_{B(p)}$, we have $f_{B(p)}(x) = x$ and $f_{B(p)}(\Delta) = \Delta$. Thus, $\Delta \subseteq V(p)$. This means that x is a p-border point. If $x \in V(\varphi)$, then $x \not\models \varphi \to p$, which is a direct contradicton. And if $x \notin V(\varphi)$, then $x \in T_{B(p)}$, which contradicts the established fact that $x \notin R^{-1}(T_{B(p)})$. □

Therefore, $x \in f^{-1}_{B(p)}(V(p))$ iff $x \in V(\varphi \to p)$. Thus, $f^{-1}_{B(p)}(V(p)) = V(\varphi \to p)$. This means that the p-morphism $f_{B(p)}$ corresponds to the substitution $\sigma_{B(p)}$.

Definable p-morphism $f_{B(q)}$ corresponding to the substitution $\sigma_{B(q)}(p) = p$, $\sigma_{B(q)}(q) = \varphi \to q$ is defined similarly to $f_{B(p)}$ by replacing everywhere p-border points by q-border points.

Definition of f_p:

1. $x \in -R^{-1}(S_p)$. Then $f_p(x) = x$.

2. $x \in S_p$. Then $f_p(x) = y$ for a sister y of x not in S, and hence in $V(\varphi)$, and not in p. By Theorem 4.13, such a y exists if x has more than one immediate successor. In the (improper) case that $x = m_{11}$ or $x = m_{10}$, by the assumption made at the start of the proof not both m_{01} and m_{00} are in S . We choose m_{00} if it is in $V(\varphi)$, otherwise m_{01}. The point x cannot be in $V(p)$ and have exactly one immediate successor, because then x would have to be $m_{10\to 11}$, and we did not take this to be a member of S_p.

3. $x \in R^{-1}(S_p) - S_p$. Then there exists at least one $y \in S_p$ such that xRy. We define f_p by induction on the layers of $R^{-1}(S_p)$. Let $x \prec \Delta$ and assume that f_p is already defined on the elements of Δ. Consider $min(f_p(\Delta))$. If $min(f_p(\Delta))$ is a single point z, then we put $f_p(x) = z$. Otherwise $min(f_p(\Delta))$ is an antichain. By (2), $f_p(y) \notin V(p)$. This, by the the fact that f_p is order-preserving, implies that there is $u \in min(f_p(\Delta))$ such that $u \notin V(p)$. Because φ is extendible there is $z \in V(\varphi)$ such that $z \prec min(f_p(\Delta))$. We define $f_p(x) = z$. If we have a choice we take z with the color 01.

Checking that f_p is a p-morphism is routine. Moreover, $f_p(x) = x$ for each $x \in -R^{-1}(S_p)$. We prove that

Claim 5.6. $f_p(U(2)) = -R^{-1}(S_p)$, $f_p(-R^{-1}(S_p)) \subseteq V(\varphi)$.

Proof. Since $f_p(x) = x$, for every $x \in -R^{-1}(S_p)$, we have that $-R^{-1}(S_p) \subseteq f_p(U(2))$. All we need to check now is that if $x \in R^{-1}(S_p)$, then $f_p(x) \in -R^{-1}(S_p)$ and $f_p(x) \in V(\varphi)$. This is actually clear by (2) and (3) of the definition of f_p. □

Next we show that f_p is definable. For this it is sufficient to prove that $f_p^{-1}(V(p))$ and $f_p^{-1}(V(q))$ are definable. It is easy to see that $f_p^{-1}(V(q)) = V(q)$. Next we show that $f_p^{-1}(V(p)) = V(\varphi \wedge p)$; that is, the substitution σ_p defined by f_p is given by $\sigma_p(p) = \varphi \wedge p$ and $\sigma_p(q) = q$. Thus we need to show that

Claim 5.7. *For every* $x \in U(2)$*:*

$$x \models \varphi \wedge p \textit{ iff } f_p(x) \models p.$$

Proof. By (1) of the definition of f_p, if $x \models \varphi \wedge p$, then obviously $x = f_p(x) \models p$. And if $x \not\models \varphi \wedge p$, then by (2) and (3) of the definition of f_p, we have $f_p(x) \not\models p$. □

Thus, f_p is a definable p-morphism. Definable p-morphism f_q is defined similarly to f_p by replacing everywhere p-points by q-points. Let $f = f_{B(q)} \circ f_{B(p)} \circ f_q \circ f_p$. Then f is a composition of definable p-morphisms and hence is a definable p-morphism.

Claim 5.8. $f(U(2)) = f_q f_p f_{B(q)} f_{B(p)}(U(2)) = V(\varphi)$ *and* $f(x) = x$*, for every* $x \in V(\varphi)$.

Proof. First note that by the definition of each of the four defined p-morphisms g we have in case $x \models \varphi$ that $g(x) = x$, and hence that $f(x) = x$. Therefore, $V(\varphi) \subseteq f(U(2))$. For every $x \not\models \varphi$ we have that x is a predecessor of some node in S. We systematically took care that in such a case for the relevant p-morphisms g, $g(x) \in V(\varphi)$. The other p-morphisms, either are the identity on x or will not disturb its value $g(x)$ in $V(\varphi)$. This finishes the proof of the claim. □

Thus, f is a definable p-morphism such that $f(U(2)) = V(\varphi)$ and $f(x) = x$, for every $x \in V(\varphi)$. By Theorem 4.17, this means that φ is projective, and in fact, by Theorem 4.6, it is also exact. Moreover, the substitution σ corresponding to f is a composition of the substitutions given by the formulas

$\varphi \wedge p$, $\varphi \to p$, $\varphi \wedge q$ and $\varphi \to q$. This finishes the proof of the theorem except for the degenerate cases in which $max(V(\varphi))$ is a singleton. Let us treat these one by one.

1. $max(V(\varphi)) = m_{00}$. Then φ is (equivalent to) $\neg p \wedge \neg q$. We define only f_p and f_q. Now $f_p \circ f_q$ is equivalent to the substitution $\sigma(p) = (\neg p \wedge \neg q) \wedge p = \bot, \sigma(q) = (\neg p \wedge \neg q) \wedge q = \bot$.

2. $max(V(\varphi)) = m_{10}$. Then φ is $p \wedge \neg q$ or $\neg\neg p \wedge \neg q$. We define only $f_{B(p)}$ and f_q. Now $f_{B(p)} \circ f_q$ is equivalent to the substitutions $\sigma(p) = top, \sigma(q) = \bot$, and $\sigma(p) = \neg\neg p \to p, \sigma(q) = \bot$, respectively.

3. $max(V(\varphi)) = m_{01}$.This is symmetric to the previous case.

4. $max(V(\varphi)) = m_{11}$. This case is more complicated, but defining only $f_{B(p)}$ and $f_{B(q)}$ works out properly.

□

Remark 5.9. To adjust the proof to the case of formulas of n variables will not be easy. Ghilardi [9] does show that if a formula is extendible, then its projective substitution is the composition of the substitutions of the form $\varphi \to p_i$ and $\varphi \wedge p_i$. But such substitutions may have to be applied more than once. The fact that this is not necessary in the 2-variable case is in fact a corollary of our proof.

We also mention that in the light of Remark 4.18, Theorem 5.3 shows that A is a finitely presentable projective algebra iff its corresponding definable upset of the universal model is extendible.

5.2 Syntactic characterization

In this section we give a syntactic description of the extendible formulas in two variables. Some of these formulas have already appeared in [14]. First we recall the characterization of the extendible formulas in one variable.

Definition 5.10. *The* Rieger-Nishimura polynomials *are given by the following recursive definition:* 1. $g_0(p) = p$, 2. $g_1(p) = \neg p$, 3. $f_1(p) = p \vee \neg p$, 4. $g_2(p) = \neg\neg p$, 5. $g_3(p) = \neg\neg p \to p$, 6. $g_{n+4}(p) = g_{n+3}(p) \to (g_n(p) \vee g_{n+1}(p))$, 7. $f_{n+2}(p) = g_{n+2}(p) \vee g_{n+1}(p)$.

Recall the labeling of $\mathcal{U}(1)$ by w_k's as shown in Figure 1.

Lemma 5.11. *For every $k \in \omega$ we have:*

1. $R(w_k) = \{w \in U(1) : w \models g_k(p)\}$,
2. $R(w_k) \cup R(w_{k-1}) = \{w \in U(1) : w \models f_k(p)\}$.

Proof. The proof is a routine check. □

We are now ready to characterize the projective, exact and extendible formulas in one variable.

Theorem 5.12 (de Jongh [13])**.**

1. *The only extendible upsets of $\mathcal{U}(1)$ are $U(1)$, $\{w_0\}$, $\{w_1\}$, $R(w_2)$, and $R(w_3)$.*
2. *The only projective formulas in one variable are the formulas $p \to p$, p, $\neg p$, $\neg\neg p$, and $\neg\neg p \to p$.*

Proof. (1) The proof is a matter of a routine check.
(2) The result follows from (1) and Lemma 5.11. □

Now we move to the two-variable case. By Theorem 3.7, we know that $V(p) \subseteq U(2)$ and $V(q) \subseteq U(2)$ (with restricted order and valuation) are isomorphic to $\mathcal{U}(1)$. We assume that the points of $V(p)$ are labelled by w_k's as in Figure 1 and the points of $V(q)$ are labelled by w'_k's in the same way. This makes it obvious that

Lemma 5.13. *For every $k < \omega$ we have*

1. $R(w_k) = V(g_k(q) \wedge p)$.
2. $R(w'_k) = V(g_k(p) \wedge q)$.

Next we characterize those extendible formulas that define upsets generated by p-points and q-points. Note that the de Jongh formulas of type ψ_w have this property. But here we will find simpler formulas. We will need a few auxiliary lemmas.

Lemma 5.14. *Let $\mathfrak{M} = (W, R, V)$ be a general model. Then for every formula φ and ψ we have*

$$V(\varphi \to \psi) = -R^{-1}(V(\varphi) - V(\psi)).$$

Proof. The proof is a routine check. □

We assume that if $w \in U(2)$ is such that $w \models p$, then $w = w_k$ for some $k \in \omega$ and if $w \models q$, then $w = w'_k$ for some $k \in \omega$.

Lemma 5.15. *For every $k \in \omega$ we have*

1. *$-R^{-1}(w_k)$ is defined by the formula $r_n(p, q) = p \to g_{k+1}(q)$.*
2. *$-R^{-1}(w'_k)$ is defined by the formula $r_n(q, p) = q \to g_{k+1}(p)$.*

Proof. (1) By Lemma 5.14, we have $V(p \to g_{k+1}(q)) = -R^{-1}(V(p) - V(g_{k+1}(q))$. Note that $V(p) - R(w_{k+1}) = R^{-1}(w_k)$. So by Lemma 5.15, $V(p) - V(g_{k+1}(q)) = R^{-1}(w_k)$. Thus, $V(p \to g_{k+1}(q)) = -R^{-1}(R^{-1})(w_k) = -R^{-1}(w_k)$.

(2) is similar to (1). □

Next we characterize the extendible formulas that define upsets generated by a single p-border or q-border point. Again, the de Jongh formulas of type ψ_w do this, but we will simplify them here a bit (sometimes reducing $\varphi_w \to \bigvee \varphi_{w_i}$ to $\varphi_w \to p$).

Lemma 5.16. *Assume $w \in U(2)$.*

1. *Let w be a p-border point totally covered by a point w_k for some $k < \omega$. Then $-R^{-1}(w)$ is defined by a formula*

$$h_k(p, q) = ((p \vee (p \to g_{k+1}(q))) \to (p \wedge g_k(q))) \to p.$$

2. *Let w be a p-border point totally covered by a point w'_k for some $k < \omega$. Then $-R^{-1}(w)$ is defined by a formula*

$$h_k(q, p) = ((q \vee (q \to g_{k+1}(p))) \to (q \wedge g_k(p))) \to q.$$

3. *Let w be a p-border point totally covered by the points w_k and w_{k+1} for some $k < \omega$. Then $-R^{-1}(w)$ is defined by the formula*

$$j_k(p, q) = ((p \vee (p \to (g_{k+1}(q)) \vee (p \to g_{k+2}(q))) \to (p \wedge f_{k+2}(q))) \to p.$$

4. *Let w be a q-border point totally covered by the points w'_k and w'_{k+1} for some $k < \omega$. Then $-R^{-1}(w)$ is defined by the formula*

$$j_k(q,p) = ((q \vee (q \to (g_{k+1}(p)) \vee (q \to g_{k+2}(p))) \to (q \wedge f_{k+2}(p))) \to q.$$

Proof. (1) Recall that for every $w \in U(2)$ the de Jongh formula φ_w defines $R(w)$. Therefore, $V(\varphi_w) = R(w)$. Then $V(\varphi_w \to p) = -R^{-1}(V(\varphi_w) - V(p)) = -R^{-1}(w)$. The formula φ_w is described in Definition 3.14. Note that since w is p-border point we have that $col(w) = 00$, and therefore, $prop(w) = \emptyset$. Moreover, $notprop(w) = \{p\}$. By Lemma 5.15, the formula ψ_{w_k} is equivalent to $p \to g_k(q)$ and by Lemma 5.13, the formula φ_{w_k} is equivalent to $p \wedge g_k(q)$. Therefore, we obtain that $\varphi_w = (p \vee (p \to g_{k+1})) \to (p \wedge g_k(q))$. Thus, the upset $-R^{-1}(w)$ is defined by the formula $((p \vee (p \to g_{k+1})) \to (p \wedge g_k(q)) \to p$.

(2) is similar to (1).

(3) As in (1) the upset $-R^{-1}(w)$ is defined by the formula $\varphi_w \to p$. In this case $\varphi_w = (p \vee (p \to (g_{k+1}(q)) \vee (p \to g_{k+2}(q))) \to ((p \wedge g_k(q)) \vee (p \wedge g_{k+1}(q)))) = (p \vee (p \to (g_{k+1}(q)) \vee g_{k+2}(q))) \to ((p \wedge f_{k+2}(q)))$. This finishes the proof of (3).

(4) is similar to (3). □

Lemma 5.17.

1. *Let $w \in U(2)$ be a p-border point totally covered by a two element antichain $\Delta = \{x, y\}$ and let v be its sister, that is, $w, v \prec \Delta$. Then for every formula $\varphi(q)$ we have*

$$w \models \varphi(q) \text{ iff } v \models \varphi(q).$$

2. *Let $w \in U(2)$ be a q-border point totally covered by a two element antichain $\Delta = \{x, y\}$ and let v be its sister, that is, $w, v \prec \Delta$. Then for every formula $\varphi(p)$ we have*

$$w \models \varphi(p) \text{ iff } v \models \varphi(p).$$

Proof. (1) Just note that w and v are bisimilar in the language $\mathcal{L} - \{p\}$ that does not contain the variable p. (2) is similar to (1). □

Lemma 5.18. *For any formula φ and ψ and any model $\mathfrak{M} = (W, R, V)$ we have*

$$V(\varphi \to \psi) \cap B(\psi) = B(\psi) - V(\varphi).$$

Proof. The result follows immediately from that fact that if $w \in B(\psi)$, then $w \notin V(\varphi)$ if and only if $w \not\models \varphi \to \psi$. $\square$

Lemmas 5.15 and 5.16 show how to construct extendible formulas that define extendible upsets of the form $-R^{-1}(S)$ for finite S. Now we will introduce the formulas that define extendible upsets of the form $-R^{-1}(S)$ for a cofinite set S of p and q-border points. This togther with Lemmas 5.15 and 5.16 will immediately yieled a syntactic description of extendible formulas in two variables.

Let $F_k = \{w \in B(p) : w_i \text{ for some } i \leq k \text{ is an immediate successor of } w\}$ and $F'_k = \{w \in B(q) : w'_i \text{ for some } i \leq k \text{ is an immediate successor of } w\}$. Let also $G_k = B(p) - F_k$ and $G'_k = B(q) - F'_k$.

Lemma 5.19. *For each $n \in \omega$:*

1. *The upset $-R^{-1}(G_k)$ is defined by the formula*

$$a_k(p, q) = (g_k(q) \to p) \to p.$$

2. *The upset $-R^{-1}(G'_k)$ is defined by the formula*

$$a_k(q, p) = (g_k(p) \to q) \to q.$$

Proof. (1) By Lemma 4.8, the formula $(g_k(q) \to p) \to p$ is extendible, for every $k \in \omega$. Therefore, the definable upset U defined by $a_k(p, q)$ is extendible. By Theorem 4.13, we have that $U = -R^{-1}(S)$ and S may contain only p-points q-points, q-border points and q-border points. It is easy to see that every q-point and q-border point (except for the single point of $U(2)$, which is also a p-border point) satisfies $a_k(p, q)$. Obviously, if a point $u \in U(2)$ is such that $u \models p$, then $u \models a_k(p, q)$. So $S \subseteq B(p)$. By Lemma 5.15, a p-point v satisfies $g_k(q)$ iff vRw_i for some $i \leq k$. By Lemma 5.17, a p-border point w satisfies $g_k(q)$ if and only if wRw_i for some $i \leq k$. Therefore, $w \models g_k(q)$ iff $w \in B(p) - G_k$. So $B(p) \cap V(g_k(q)) = B(p) - G_k$. Then by Lemma 5.18, $V(g_k(q) \to p) \cap B(p) = B(p) - V(g_k(q)) = G_k$, and so $V(a_k(p, q)) \cap B(p) = B(p) - V(g_k(q) \to p) = B(p) - G_k$. Therefore, $G_k \subseteq S$ and $B(p) - G_k \subseteq U$. This means that $S = G_k$.

(2) follows from (1). $\square$

Corollary 5.20. *Let U be an upset of the 2-universal model $\mathcal{U}(2)$. Then U is definable and extendible, iff $U = -R^{-1}(S)$, where $S \subseteq V(p) \cup V(q)$, the sets $S \cap B(p)$ and $S \cap B(q)$ are finite or cofinite subsets of $B(p)$ or $B(q)$, and if $x \in S$, then there exists a sister y of x such that $y \notin S$.*

Proof. The result is an immediate consequence of Corollary 5.2 and Lemmas 5.16 and 5.19. □

Finally, we arrive at the following characterization of the extendible formulas in two variables.

Theorem 5.21. *If $\varphi(p,q)$ is an extendible formula in two variables, then it is equivalent to a formula $\varphi_1 \wedge \cdots \wedge \varphi_k$, $k \geq 1$, where each φ_i belongs to the set*

$$\{r_n(p,q),\ r_n(q,p),\ h_n(p,q),\ h_n(q,p),\\ j_n(p,q),\ j_n(q,p), a_n(p,q),\ a_n(q,p):\ n \in \omega\},$$

and for each $n \in \omega$, the formulas $r_n(p,q) \wedge j_n(p,q)$, $r_n(q,p) \wedge j_n(q,p)$, $r_n(p,q) \wedge a_n(p,q)$ and $r_n(q,p) \wedge a_n(q,p)$ are not part of $\varphi_1 \wedge \cdots \wedge \varphi_k$.

Proof. The result follows immediately from Corollary 5.20 and Lemmas 5.15, 5.16 and 5.19. □

One can make many restrictions on the forms that occur. For example, if $r_n(p,q)$ occurs as a φ_i, then no $r_m(p,q)$ and $h_m(p,q)$ need to occur for any $m > n+1$. Finding a projective substitution for the formulas occurring in Theorem 5.21 would provide an alternative proof of Theorem 5.3.

6 Conclusions

We gave characterizations of exact, extendible and projective formulas in terms of n-universal models and definable p-morphisms. For formulas in two variables, using these characterizations, we gave a proof of Ghilardi's theorem that every extendible formula is projective. We also gave a full recursive description of the infinitely many exact, extendible and projective formulas in two variables.

The question whether these results could be generalized to formulas in three and more variables in **IPC** is far from obvious. The recursive formulas constructed in this paper heavily depend on the Rieger-Nishimura polynomials and the characterization of extendible subsets of $\mathcal{U}(2)$ in terms of finite and cofinite p-border and q-border points. These characterizations do not transfer directly to the case of n-universal models for $n > 2$. Thus we leave it as an (interesting) open problem whether a complete characterization can be given to exact, extendible and projective formulas of **IPC** in three and more variables. Another topic for future work is to investigate connections of this approach with the now actively developing theory of projective formulas in other non-classical logics. In particular, connections between projective formulas in the Lukasiewicz logic and projective MV-algebras have already been made in [17]. However, a syntactic description of projective formulas in the Lukasiewicz logic is still open.

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