MODAL STRUCTURES IN GROUPS AND VECTOR SPACES

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Abstract. Vector spaces contain a number of general structures that invite analysis in modal languages. The resulting logical systems provide an interesting counterpart to the much better-studied modal logics of topological spaces. In this programmatic paper, we investigate issues of definability and axiomatization using standard techniques for modal and hybrid languages. The analysis proceeds in stages. We first present a modal analysis of commutative groups which establishes our main techniques, next we introduce a new modal logic of linear dependence and independence in vector spaces, and finally, we study a modal logic for describing full-fledged vector spaces. While still far from covering every basic aspect of linear algebra, our discussion identifies several leads for more systematic research.

1. Introduction

Vector spaces and techniques from linear algebra are ubiquitous in applied mathematics and physics, but they also occur in areas such as cognitive science [28], machine learning [17], computational linguistics [32], the social sciences [6, 47], and formal philosophy [45]. There is also a body of logical work on vector spaces, in the first-order model-theoretic tradition [40, 52], in relevant logic [55], and in modal logics of space [12]. This paper offers a further exploration from the perspective of modal logic, in the broad spirit of [36, 37]. As it happens, such connections between logic and mathematics can be pursued in two directions, both present in the cited literature. Vector spaces have been applied as a source of new semantic models for existing independently motivated logical languages and axiom systems. But one can also put the focus on vector spaces themselves, asking which notions from linear algebra can be captured in which specially designed new logics. The latter approach will be our main interest in what follows, though the two directions are of course not incompatible.

Here are the structures to be analyzed step by step in this paper.

Definition 1.1. *A* vector space over a field F *is a set* V *with operations* $+ : V^2 \to V$ *of vector addition and a scalar multiplication* $\cdot : F \times V \to V$ *such that for each* $u, v, w \in V$ *and* $a, b, c \in F$*:*

(1) $u + (v + w) = (u + v) + w$
(2) $u + v = v + u$
(3) *there exists a 'zero vector'* $0 \in V$ *such that* $v + 0 = v$ *for all* $v \in V$
(4) *for every* $v \in V$*, there is an 'additive inverse'* $-v \in V$ *such that* $v + (-v) = 0$
(5) $a \cdot (b \cdot v) = (a.b) \cdot v$
(6) *the multiplicative unit* 1 *of the field* F *satisfies* $1 \cdot v = v$ *for all* $v \in V$
(7) $a \cdot (u + v) = a \cdot u + a \cdot v$
(8) $(a + b) \cdot v = a \cdot v + b \cdot v$

Vector spaces are the subject of Linear Algebra [39], whose notions and results set a target for logical analysis. This paper makes a modest start, proceeding in stages. First we analyze commutative groups, described by clauses (1) – (4) in the preceding definition. Next we focus on one key feature of vector spaces, viz. their notions of dependence and independence, in line with the current interest in 'dependence logics', [56], [7]. Finally, we deal with the field scalars explicitly, as governed by the above clauses (5) – (8).

More in detail, in Section 2 we introduce modal languages and an appropriate notion of bisimulation for commutative groups to study definability of various natural notions in vector spaces such as the Minkowski operations. In Section 3 we provide a complete axiomatization using techniques from hybrid logic. In Section 4, we define a new logic of subgroup closure in the style of propositional dynamic logic on neighborhood models and show its completeness. Next, going beyond commutative groups, in Section 5 we present a new modal logic for dependence which takes on board the key principle of Steinitz Exchange, and we explore connections with an alternative approach taking independence as a primitive linking up with Matroid Theory. In Section 6 we then introduce modal logics that explicitly describe field structure, while Section 7 explores further ways in which modal logics might deal with basic structures in Linear Algebra such as linear transformations and matrices. These are the main lines: the paper also identifies a number of connections of our systems with other fields of research, in particular, substructural logics of conjunction and implication, and contemporary dependence logics. Section 8 summarizes the findings of our exploration.

It may be good to state at the start what this paper does and does not offer. We do not propose modal syntax as an alternative for rewriting what works perfectly well in Linear Algebra, or for redoing foundational insights already obtained in Model Theory. Modal languages are a vehicle for describing subsets or 'patterns' in spaces of interest with a mixture of restricted kinds of 'guarded quantification' over objects plus unquantified unary second-order variables for sets of objects, offering an abstraction level that can help identify basic laws. We do not explain the basics of modal logic in this paper, but a knowledge of the standard textbook [18] should suffice. We believe that our results show the potential of a modal style of thinking about groups and vector spaces, which brings to light new abstract patterns and topics to pursue. In addition, in many applications, the benefit of modal languages is their relatively low computational complexity for satisfiability and model checking. Whether this is also true in our present setting is something we leave to further investigation.

Related Work. Our approach is indebted to several existing lines of work. We already mentioned [36, 37] on modal patterns in algebras of sets as an inspiration, as well as the vector semantics of relevant logic [55], categorial logic [16], complex algebra [41, 36], and dependence logics [56, 7]. Also, the "Handbook of Spatial Logics" [1] contains chapters that are relevant to the present study, in particular, [12] on modal logics for topology (with an excursion to vector spaces and their 'arrow logic'), and [4] on logics for geometries. A further interface are substructural categorial logics with vector space semantics in distributional semantics for linguistic corpora [46, 54]. A congenial recent approach in [45] encodes axioms for vector spaces in a non-classical propositional logic with names for real numbers, and then explores further notions in linear algebra, all the way to inner products. Leitgeb's logic is decidable through an embedding in Tarski's decidable first-order theory of the reals with addition and multiplication. This connection with classical model theory is not a coincidence, and vector spaces have also been studied in several formats in that area, e.g., [40, 52], though the connection with the modal languages presented in this paper remains to be clarified.

2. A modal logic for commutative groups

For convenience, we recall the definition of a group. For more specialized later algebraic notions, the classic textbook [58] remains a good source for definitions and facts.

Definition 2.1. *A* group *is a structure* $(G, +, -, 0)$ *where* $+$ *is a binary operation on* G *and* $-$ *is a unary operation on* G *and* $0 \in G$ *is a constant (0-ary operation) such that for all* $a, b, c \in G$*:*

(1) $a + (b + c) = (a + b) + c$;
(2) $a + 0 = 0 + a = a$;
(3) $a + (-a) = (-a) + a = 0$.

A group G *is* commutative *if for each* $a, b \in G$*:* $a + b = b + a$.

In this section, we will discuss commutative groups from a modal standpoint, though most of what we say also applies to groups in general. Once again, here and henceforth, we presuppose the basics of modal logic as set out in the standard textbook [18]. Many of our examples will be presented in a spatial phrasing, but arithmetical or algebraic interpretations are also possible, given the abstract nature of groups and vector spaces.

2.1. **Language and semantics.** Our main tool is the following modal language.

Definition 2.2. *Let* Prop *be a set of propositional variables,* Nom *a set of nominals and* $\mathsf{0}$ *a hybrid constant. The* modal group language MGL *is generated by the following grammar:*

$$\varphi := \bot \mid p \mid \mathsf{i} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\varphi \mid \langle - \rangle \varphi \mid \langle + \rangle (\varphi, \varphi) \mid \mathsf{0},$$

where $p \in \mathsf{Prop}$, $\mathsf{i} \in \mathsf{Nom}$*. We will also employ defined modalities* $\mathsf{U}\varphi = \neg\mathsf{E}\neg\varphi$ *and* $[+](\varphi_1, \varphi_2) = \neg\langle + \rangle(\neg\varphi_1, \neg\varphi_2)$*. Henceforth,* $\varphi \oplus \psi$ *will be used as shorthand for* $\langle + \rangle(\varphi, \psi)$ *for ease in reading formulas.*

A word of explanation. A 'pure' modal language for groups would just use proposition letters and our modalities, where formulas denote sets of objects, in line with practice in algebraic logic, [61]. The additional nominals allow us to also talk about single objects, using tools of hybrid logic, [3], which sits in between modal logic and first-order logic. Our reasons for choosing this formalism are several: it seems needed for some of the results that we wish to have, it makes for more perspicuous formulations of basic laws of reasoning, and reference to specific objects fits with practice in group theory and linear algebra.

Definition 2.3. *A* group model *is a tuple* $(S, +, V)$*, with a commutative group* $(S, +, -, 0)$ *and a valuation map* $V : \mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(S)$ *s.t.* $|V(\mathsf{i})| = 1$ *for all nominals* i *and* $V(\mathsf{0}) = \{0\}$ *is the unit of the group.*

Formulas φ of our formal language describe properties of elements in the domain of the group model according to the following truth definition:

Definition 2.4. *Let* $\mathfrak{M} = (S, +, V)$ *be a group model,* $s \in S$*:*

$\mathfrak{M}, s \models \bot$ *never*
$\mathfrak{M}, s \models \mathsf{i}$ *iff* $s \in V(\mathsf{i})$
$\mathfrak{M}, s \models p$ *iff* $s \in V(p)$
$\mathfrak{M}, s \models \varphi \vee \psi$ *iff* $\mathfrak{M}, s \models \varphi$ *or* $\mathfrak{M}, s \models \psi$

$\mathfrak{M}, s \models \neg\varphi$ *iff not* $\mathfrak{M}, s \models \varphi$
$\mathfrak{M}, s \models \mathsf{E}\varphi$ *iff* $\mathfrak{M}, t \models \varphi$, *for some* $t \in G$
$\mathfrak{M}, s \models \mathsf{0}$ *iff* $s \in V(\mathsf{0})$
$\mathfrak{M}, s \models \varphi \oplus \psi$ *iff* $\exists s_1, s_2$ *s.t.* $\mathfrak{M}, s_1 \models \varphi$, $\mathfrak{M}, s_2 \models \psi$ *and* $s = s_1 + s_2$
$\mathfrak{M}, s \models \langle - \rangle\varphi$ *iff* $\mathfrak{M}, -s \models \varphi$

The meaning of our defined modalities follows from these explanations. For instance, $[+](\varphi, \psi)$ says at point s that for each pair t, u with $s = t + u$, either $t \models \varphi$ or $u \models \psi$.

Remark 2.5 (Binary operations)**.** In standard modal logic, a modality $\langle f \rangle\varphi$ for a function is true at a point s if the function value $f(s)$ makes φ true. However, this does not lift to binary operations, and the above truth condition for $\oplus$ works in the *reverse* direction: the current s stands for the function value, and the 'ternary successors' s_1, s_2 for two arguments. We could have done the same for the inverse modality $\langle - \rangle\varphi$, but we followed standard practice there as the difference does not matter thanks to the valid identity $-(-(x)) = x$ in groups.

We say that a subset of a group model $\mathfrak{M}$ is *definable* if it is the denotation of some formula. With a spatial interpretation, one can think of these subsets as 'patterns' in space – though, as stated above, quite different interpretations are also possible. Here are some illustrations of what our language can define concretely.

Example 2.6 (Definable sets)**.** (a) Consider the group model based on the set of integers $(\mathbb{Z}, +, -, 0)$ with $V(p) = \{1\}$, and no nominals interpreted. Each set $\{z\}$ for a positive integer z is definable, by 'summing' p z times. Moreover, using the modality $-$, all singletons $\{-z\}$ are definable. Using disjunctions and negation, then, every finite and every cofinite subset of $\mathbb{Z}$ is definable. The crucial final observation is that we get no more definable sets, since this collection is closed under the operations $0, -, +$ defined above plus the Boolean operations. For instance, it is easy to see that the set of all sums of numbers from two cofinite sets is itself cofinite, and so on. A similar analysis of definable sets works for the two-dimensional model $(\mathbb{Z} \times \mathbb{Z}, +, -, (0, 0))$ with the valuation $V(p) = \{(0, 1)\}, V(q) = \{(1, 0)\}$. Z (b) In case our initial denotations have other special invariance properties, these may constrain what is definable. For an illustration, let $V(p) = \{(0, 1), (1, 0)\}$. Can we define, say, the single point $(1, 1)$? We first list some further sets that are definable from $V(p)$ using the operations for the Booleans and modalities in our language:

$\langle - \rangle p$ defines $\{(0, -1), (-1, 0)\}$, $\quad p \oplus p : \{(0, 2), (1, 1), (2, 0)\}$,
$p \oplus \langle - \rangle p : \{(0, 0), (-1, 1), (1, -1)\}$, $\quad p \oplus (p \oplus \langle - \rangle p) : \{(-1, 2), (0, 1), (1, 0), (2, -1)\}$,
$(p \oplus (p \oplus \langle - \rangle p)) \wedge \neg p : \{(-1, 2), (2, -1)\}$ Call this last formula ψ.
Now the single point $(1, 1)$ is definable by the formula $(\psi \oplus \psi) \wedge (p \oplus p)$.

The preceding sets and in fact all definable sets here have the invariance property of being *reflective*: they are closed under the map sending points (x, y) to their reflections (y, x) along the diagonal $x = y$. The key here is an inductive proof that this property is preserved under our operations. E.g., closure for sums is shown as follows. Let $(x_1, y_1) \in A, (x_2, y_2) \in B$ and consider $(x_1 + x_2, y_1 + y_2) \in A \oplus B$. Reflecting this gives $(y_1 + y_2, x_1 + x_2) = (y_1, x_1) + (y_2, x_2)$: a sum of two reflections which are in A, B by assumption. In fact the definable sets are precisely the finite and cofinite reflexive sets.[1] (c) Similar illustrations work just as well for dense spaces like $\mathbb{Q}$ or $\mathbb{Q} \times \mathbb{Q}$, and examples will be found below.

Determining all modally definable sets amounts to describing the subalgebra of the power set of a group generated by the valuation values $V(p)$ for atomic propositions and then applying the operations for inverse and product as well as the Boolean operations.

The semantics introduced here is in fact an instance of a much more general standard relational semantics on abstract models, with a well-known clause for binary modalities, [18]. In our setting, the relevant structures are as follows.

Definition 2.7. *A* relational model *is a tuple* $(S, R, I, 0, V)$*, of a non-empty set of points* S*, a ternary relation* $Rstu$*, a unary function* I *and a distinguished object* 0*, this triple is called a* frame*, plus a valuation map* $V : \mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(S)$ *s.t.* $|V(\mathsf{i})| = 1$ *and* $V(\mathsf{0}) = \{0\}$*.*

The truth definition on these models is as above for proposition letters and the Boolean operations. For the two modalities in our language, the truth conditions are

$$\begin{array}{lll} \mathfrak{M}, s \models \varphi \oplus \psi & \text{iff} & \exists s_1, s_2 \text{ s.t. } Rss_1s_2 \text{ and } \mathfrak{M}, s_1 \models \varphi,\ \mathfrak{M}, s_2 \models \psi \\ \mathfrak{M}, s \models \langle - \rangle \varphi & \text{iff} & \mathfrak{M}, I(s) \models \varphi \end{array}$$

We will use relational models at various places for useful background to the behavior of our group models. All standard notions and results from modal logic apply to these structures, such as bisimulations, translation into first-order logic or other standard logics, p-morphisms, frame correspondence for special modal axioms, and the like. We will use these without extensive further explanation in what follows.

2.2. **Definable notions and links to other logics.** In addition to definability of sets in specific models like above, there is generic definability of notions across models. As we will see now, our modal language contains several generic notions of interest, including general logical devices, well-known spatial operations, and connections to other families of logics.

Global modalities and nominals. For a start, the global existence modality of our language can in fact be defined in group models $\mathfrak{M}$.

Fact 2.8. $\mathsf{E}\varphi \leftrightarrow \varphi \oplus \top$ *is valid in group models.*

Proof. First let $s \models \varphi \oplus \top$. Then there are points v, u s.t. $s = v + u$ and $v \models \varphi$. In particular, there is a point where φ is true, and so $\mathsf{E}\varphi$ is true at s. Conversely, let $s \models \mathsf{E}\varphi$. Then φ is true at some point t, and we can use the fact that in any group: $s = t + (-t + s)$ to see that $s \models \varphi \oplus \top$. □

Even so, we will keep the global modalities in our language as they make principles easier to grasp, and rewriting them according to the preceding equivalence leads to obscure formulations. Next, we consider reference to single points, a device for which we used nominals.

[1]Peter van Emde Boas, p.c,, pointed out that this claim stated in an earlier version of this paper follows from the observations so far. First define the set $\{(-1, 1), (1, -1)\}(A)$ from what we have above. Next, given the definition of $(1, 1)$, we can define all points on the diagonal $x = y$ uniquely by taking sums and inverses. Then every reflective pair $\{(x, y), (y, x)\}$ can be defined by moving along lines $x + y = 2k$ using sums with the definable set A and excluding reflective pairs already defined. The same analysis applies to lines of the form $x + y = 2k + 1$ by first adding the set p. All finite reflective sets are finite unions of reflective pairs, hence definable. Finally, it suffices to note that the finite and cofinite reflective sets are closed under the algebraic operatuons for our language. In particular, the sum of two infinite cofinite sets is an infinite cofinite set.

Fact 2.9. *The formula $p \oplus \langle - \rangle p \leftrightarrow \mathsf{0}$ holds in a group model iff the set $V(p)$ is a singleton.*

Proof. From right to left, this is clear because of the axiom for group inverse. From left to right, suppose that $V(p)$ contains objects s, t. Then $p \oplus \langle - \rangle p$ contains $s + -t$. Given the assumption, $s + -t = 0$, and in groups it follows that $s = t$. Therefore, $V(p)$ contains at most one object. However, $V(p)$ cannot be empty, since in that case $p \oplus \langle - \rangle p$ would denote the empty set, which is different from $\{\mathsf{0}\}$. □

Using Fact 2.9 we could avoid nominals in axioms, as we can replace them by ordinary proposition letters p satisfying the above singleton assumption in an added antecedent clause, [36]. However, the nominal syntax is more perspicuous, so we will keep using it.

Minkowski operations. Our second illustration comes from Mathematical Morphology [19], where the following two operations are used to describe shapes and operations on shapes in vector spaces. However, the notions involved apply to groups in general.

Definition 2.10. *The Minkowski operations on subsets of groups are* addition $A \oplus B = \{x + y \mid x \in A \text{ and } y \in B\}$ *and* difference $A \ominus B = \{x \mid \text{for all } y \in B : x + y \in A\}$.

Obviously, the corresponding modal operator for Minkowski addition is our vector sum modality. But Minkowski difference is definable just as well.

Fact 2.11. *Minkowski difference $A \ominus B$ is definable by the modal formula $\neg(\langle - \rangle B \oplus \neg A))$.*

Proof. When true for a group element s, the modal formula $\neg(\langle - \rangle B \oplus \neg A))$ says that s cannot be written as a sum of a point $t \in \neg A$ and a point $-u$ with $u \in B$. But transforming this sum $s = t + -u$ into the equivalent form $t = s + u$, the stated condition says that, if we add any B-point to s, the result must be in A. □

Versatile modalities. The key to Fact 2.11 is that a decomposition of a group element s into two objects $s = t + u$ can also be described in terms of decomposing the elements t or u using suitable inverses. This feature makes our modal language *versatile* in the sense of [50]. If we think of an abstract ternary decomposition relation $Rx\,yz$ as interpreting the modality $\langle + \rangle$, then the matching binary modalities for its two naturally associated relations

$$R'y\,xz := Rx\,yz, \quad R''z\,xy := Rx\,yz$$

are definable in our language. Modal languages that support such definable shifts in perspective are known to be well-behaved in various ways, [59, Section 2.7].

Substructural logics of product and implication. Finally, our modal logic for groups relates to a more general literature. Minkowski addition has the same flavor as the *product conjunction* of substructural logics such as relevant logic, [55], categorial logic, [44], or linear logic, [31]. In particular, a product conjunction $p \oplus p$ differs from a single p, witness our spatial examples: the number of occurrences of p matters. But also, by the above definition, Minkowski difference $A \ominus B$ behaves like an *implication* $B \Rightarrow A$ true at those points which become A upon combination with any B-point, a ubiquitous truth condition in substructural logics, [53]. In fact, as shown in [2], group models and vector spaces validate all principles of the basic substructural commutative *Lambek Calculus*, which lies embedded inside our language, and thus, such logics generate valid principles of Mathematical Morphology. For further details of these connections, cf. the Handbook chapter [12], which also presents links with *arrow logic*, a modal logic of abstract 'transitions' that generalizes relational algebra. Thus, the present system lies at an interface with a range of other logical traditions.

2.3. **Bisimulations and invariance.** As is common in modal logics, definability can be analyzed in terms of a structural invariance, which in the present case can be seen as a relational form of the basic invariance under linear transformations in Linear Algebra.

Definition 2.12. *(Bisimulation) Let* $\mathfrak{M} = (G, +, -, 0, V)$, $\mathfrak{M}' = (G', +', -', 0', V')$ *be group models. A binary relation* $\mathsf{Z} \subseteq G \times G'$ *is a* bisimulation *if for every two points* $w \in G$ *and* $w' \in G'$ *with* $w\mathsf{Z}w'$, *it holds that:*

(1) $w \in V(p)$ *iff* $w' \in V'(p)$, $p \in \mathsf{Prop}$
(2) $w \in V(\mathsf{i})$ *iff* $w' \in V'(\mathsf{i})$, $\mathsf{i} \in \mathsf{Nom}$, *and also* $0\mathsf{Z}0'$
(3) $w = u + v$ *implies there are* $u', v' \in G'$ *such that* $u\mathsf{Z}u'$, $v\mathsf{Z}v'$ *and* $w' = u' + v'$
(4) $w' = u' + v'$ *implies there are* $u, v \in G$ *such that* $u\mathsf{Z}u'$, $v\mathsf{Z}v'$ *and* $w = u + v$
(5) $-w\mathsf{Z} -' w'$
(6) $\forall v \exists v'$ *such that* $v\mathsf{Z}v'$ *and* $\forall v' \exists v$ *such that* $v\mathsf{Z}v'$.

We say that w *and* w' *are* bismilar *if they are linked by some bisimulation.*

Remark 2.13. The back-and-forth clauses for group product in the preceding definition are a special case of those for general modal bisimulations on models with a ternary accessibility relation, [18]. In the latter setting, one views $s = t + u$ as an abstract ternary relation $Rstu$ that need not be functional, and postulates clauses such as the following

$$\text{if } u\mathsf{Z}u', \text{ and } Rust \text{ in } G, \text{ then there are } u', v' \in G' \text{ s.t. } Ru's't', s\mathsf{Z}s', \text{ and } t\mathsf{Z}t'.$$

The following invariance result is easy to prove by induction on our modal formulas, for group models, but also for arbitrary relational models:

Fact 2.14. *If* Z *is a bisimulation between two models* $\mathfrak{M}$, $\mathfrak{M}'$ *with* $u\mathsf{Z}u'$, *then we have that* $\mathfrak{M}, u \models \varphi$ *iff* $\mathfrak{M}', u' \models \varphi$, *for all formulas* φ *of our language.*

It follows that bisimulation as defined here does not just preserve structure of a group or vector space: it also preserves what may be called *patterns*: modally definable subsets in models based on these groups or spaces. These patterns are often essential in applications: e.g., 'shapes' in Mathematical Morphology are defined subsets of vector spaces.

The above is just one standard result from modal logic that applies here, and we will not attempt a complete survey. Instead, here is one more illustration. A common use of bisimulations is in contracting a given model to a smallest modally equivalent structure.

Example 2.15 (A finite bisimulation contraction)**.** Instead of the earlier examples with integers, we now consider the dense order $\mathbb{Q}$ of the rationals. Consider the model $\mathfrak{M} = (\mathbb{Q} \times \mathbb{Q}, +, -, (0,0))$ with $V(p) = \{(a,b) \in \mathbb{Q} \times \mathbb{Q} : a, b \geq 0\}$. It is easy to see, by the preceding definition and the symmetry in this model w.r.t. the diagonal, that two points (a,b) and (a',b') are bisimilar in this model if one of the following four conditions holds:

- $a = b = a' = b' = 0$, the equivalence class A,
- $a, b, a', b' \geq 0$ and $(a,b) \neq (0,0) \neq (a',b')$, the equivalence class B,
- $a, b, a', b' \leq 0$ and $(a,b) \neq (0,0) \neq (a',b')$, the equivalence class C,
- $a < 0, b > 0$ or $a' > 0, b' < 0$, otherwise: the equivalence class D.

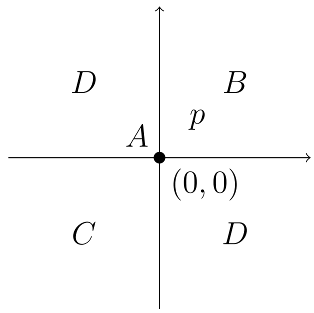

Figure 1. $\mathbb{Q} \times \mathbb{Q}$ and its bisimulation classes

Contracting these regions to single points via a function f from $\mathfrak{M}$ into a four-element relational structure, we obtain the *bisimulation contraction* of the preceding model, with (i) a binary relation Ist for the unary modality which holds just when there exist $y = -x$ in $\mathfrak{M}$ with $f(y) = t, f(x) = s$, (ii) a ternary relation $Rstu$ for the binary modality which holds when there exist $x = y + z$ in $\mathfrak{M}$ with $f(x) = s, f(y) = t, f(z) = u$. Here it is easy to see that the relation I becomes a function on the four-element contracted structure sending the set images $f[A]$ and $f[D]$ to themselves and interchanging the points $f[A], f[D]$. As for the ternary relation R, we just list some examples. On some pairs of arguments, R is functional, e.g., $R(B, B, B)$ holds but there is no further relation $R(X, B, B)$ with X different from B. On the other hand, it is easy to see by considering various vector sums in our picture that our definition generates the triples $R(X, B, C)$ for all four points X, so the ternary relation in the bisimulation contraction is no longer a function.

However, our definition does not yield: $R(A, B, B)$, $R(A, C, C)$, $R(A, B, D)$, $R(D, A, B)$ – and likewise, the other regions lack certain decompositions.

These features are enough to guarantee the required back-and-forth clauses between the original model and the contracted one. Call the above contraction map f. (a) First, the forth condition of bisimulation is immediate by the above definition of the relations on the values of f. (b) Next as for the back condition, we do only one illustrative case, all others are dealt in a similar way. Suppose that $f((a, b)) = B$ with $a > 0$ and $b > 0$, and consider the ternary relational fact $R(B, D, D)$ in the contracted model. Looking at the points $(2a, -b)$ and $(-a, 2b)$ in the region D of the original model, their sum is (a, b) and that $f(a, -2b) = D$ and $f(-2a, b) = D$ – see Figure 2 for an illustration. The boundary case of points $(a, 0)$ with $a > 0$ can be dealt with by working with $(2a, -1), (-a, +1)$.

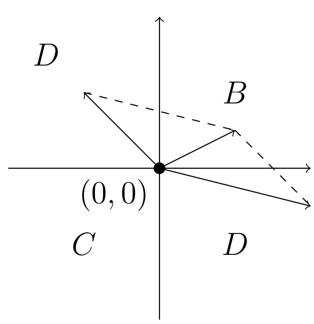

Figure 2. Decomposition for a backward clause in the contraction bisimulation

Remark 2.16. Note that the contracted model defined in the preceding example is no longer a group model, since its ternary relation is not the graph of a function $+$. This is not necessarily a drawback, but rather a possible advantage of taking a modal perspective. Group models can be bisimilar to, sometimes much simpler, generalized models that contain the same modal information. Thus we enrich the original class of models for Group Theory with new ones, a familair process of abstraction also in mathematics, witness, say, the transition from Linear Algebra to Matroid Theory, [62].

Remark 2.17 (Further bisimulation topics)**.** One converse to the earlier invariance Fact 2.14 is easy to prove by general modal techniques, (see, e.g., [18, Sec. 2.2]). For any two points w, v in two *finite* group models that satisfy the same modal formulas, there exists a bisimulation between the models connecting w to v. Much stronger results of this sort exist in modal logic, but here we just point at a feature of group models without a standard modal counterpart. Suppose we have sets of generators G_1, G_2 for the groups in two models $\mathfrak{M}_1, \mathfrak{M}_2$, and a total relation Z between G_1, G_2. Does Z extend to a bisimulation on the whole models? To illustrate this, consider $\mathbb{Z}_3 = \{0, 1, 2\}$ and $\mathbb{Z}_5 = \{0', 1', 2', 3', 4'\}$. For a start, we must match $0Z0'$, as this link for a nominal is unique. Let $\mathsf{Z} = \{(0, 0'), (1, 1')\}$. We now follow the clauses for a bisimulation to place further links as required. First, since $1\mathsf{Z}1'$ and $1 = 2 + 2$ and $1' = 3' + 3'$ we must have $2\mathsf{Z}3'$ [there are no other ways of writing $1'$ as a sum $x + x$.] Then using the clause for inverses, we must also put $2, 4', 1, 2'$. However, this is not a bisimulation yet. Consider the link $1, 1'$ and the fact that $1' = 2' + 4'$. To witness this in $\mathbb{Z}_3$, the only candidate is $1 = 2 + 2$, and therefore, we must put $2, 2'$. Continuing in the way, we see that the smallest bisimulation between the two given models contains all links between non-unit elements. This procedure can clearly be turned into an algorithm for a bisimulation extension of a given relation between finite sets of generators.

2.4. **Frame definability and p-morphisms.** In addition to analyzing definable subsets of models, modal logic can also define properties of the pure underlying group structure. Recall that a *frame* was a group model stripped of its valuation. In this case, the relevant notion is *frame truth* of modal formulas φ at points s in a frame F, which is defined by saying that φ must be true at s in every model obtained by endowing F, with a valuation map for atomic propositions. When analyzing frame truth, an important notion is the following functional version of a bisimulation as defined above, dropping the requirement on valuations.

Definition 2.18. *A map f is a* p-morphism *from frame F_1 to F_2 if it sends points in F_1 to points in F_2 subject to the following conditions:*

(1) *f maps the zero element of F_1 to that of F_2,*
(2) *$f(v + v') = f(v) + f(v')$ and $f(-v) = -f(v)$,*
(3) *If $f(v) = v_1' + v_2'$ then there are v_1, v_2 with $v = v_1 + v_2$, $f(v_1) = v_1'$ and $f(v_2) = v_2'$.*

Again this is a special case of the standard notion of p-morphisms between general relational frames. A map f is a *p-morphism* from a general relational frame $\mathsf{F}_1 = (S_1, R_1, I_1, 0_1)$ to a general relational frame $\mathsf{F}_2 = (S_2, R_2, I_2, 0_2)$ if it sends points in F_1 to points in F_2 subject to the following conditions: (a) f maps the zero element of F_1 to that of F_2, (b) $R_1(v_1, v_2, v_3)$ implies $R_2(f(v_1), f(v_2), f(v_3))$ and $f(I_1(v)) = I_2(f(v))$, (c) If $R_2(f(v), v_1', v_2')$ then there are v_1, v_2 with $R(v, v_1, v_2)$ s.t. $f(v_1) = v_1'$ and $f(v_2) = v_2'$.

Using the earlier invariance of modal formulas under bisimulations between either group models or their relational generalizations, it is now easy to see that

Fact 2.19. *If F_1 and F_2 are relational or group models and f is a p-morphism from frame F_1 to F_2 and $\mathsf{F}_1, s \models \varphi$, then $\mathsf{F}_2, f(s) \models \varphi$.*

That is, p-morphisms preserve modal frame truth at specific points, and it follows at once that they also preserve global truth at all points in a frame. As an illustration, we show that no formula of our modal language, viewed as globally frame true, can define the property of group frames of having dimension 2 when viewed as a vector space over the rationals.

Example 2.20 (Modal undefinability of dimension)**.** Consider the group frames $(\mathbb{Q} \times \mathbb{Q}, +, -, (0,0))$ and $(\mathbb{Q}, +, -, (0,0))$ of dimensions 2 and 1, respectively. The projection map sending (x, y) to $(x, 0)$ is a p-morphism in the above sense. Going from $(\mathbb{Q} \times \mathbb{Q}, +, -, (0,0))$ to $(\mathbb{Q}, +, -, (0,0))$, the 'forth' clauses are obvious from the definition of the projection map. In the opposite direction, for the 'back' clause, let $(x, 0) = (x_1, 0) + (x_2, 0)$. Then there clearly exist $(x_1, y_1), (x_2, y_2)$ such that $(x_1, y_1) + (x_2, y_2) = (x, y)$, and the projections of these two vectors are $(x_1, 0), (x_2, 0)$ as required.

Modal p-morphisms may come from modal logic, but they fit with group theory in that, on group models, they are just the surjective homomorphisms. The proof of this fact reflects our earlier observation that the universal modality is definable in our language.

Fact 2.21. *Given Condition (2), Condition (3) for a p-morphism f as defined above is equivalent to surjectivity of the function f.*

Proof. Let the function f satisfy Condition (2). Assuming Condition (3), we show that f is surjective. For any y in the range and x in the domain of f, we can write $f(x) = y + (f(x) - y)$. Therefore, by Condition (3) for p-morphisms, there is a v such that $f(v) = y$.

Conversely, let f be surjective and $f(x) = y + z$ (i). We choose y', z' s.t. $f(y') = y$ and $f(z') = z$. Now, $x = y' + (z' + (x - (y' + z')))$. For proving Condition (3) on p-morphisms, it then remains to show that $f(y') = y$ (ii) and $f(z' + (x - (y' + z'))) = z$ (iii). Here (ii) is true by assumption, and as for (iii), by Condition (2) for p-morphisms $f(z' + (x - (y' + z'))) = z + (f(x) - (y + z)) =$ [using (i)] $z + 0 = z$. □

Our final topic is the role of nominals denoting single points in frame definability results, illustrated for the property of functionality of the group product. The following correspondence fact is easy to prove, even on general relational frames as introduced above for arbitrary binary modalities over ternary relations.

Fact 2.22. *Functionality of the binary operation $+$ is defined on relational frames by the modal formula* $((\mathsf{n} \wedge \mathsf{Em}) \to \mathsf{E}(\mathsf{m} \oplus \mathsf{n})) \wedge (\mathsf{E}(\mathsf{n} \wedge \mathsf{m} \oplus \mathsf{k}) \to \mathsf{U}(\mathsf{m} \oplus \mathsf{k} \to \mathsf{n}))$.

What happens when we drop nominals from our language? In the frames underlying group models, we can still express functionality, because of Fact 2.9, which showed how to define singleton sets $V(p)$. The preceding definition will still work when we replace the nominals by proposition letters, and add antecedent formulas forcing these proposition letters to denote singletons. However, this trick is not available in general relational models, and functionality is not definable there. This fact already follows from Example 2.15, but we find the next example informative about the more abstract models that our language supports.

Fact 2.23. *Functionality of + is not expressible in relational frames in our modal language of groups without nominals.*

Proof. Let $\mathbb{N}$ and $\mathbb{N}'$ be two isomorphic copies of the natural numbers. Let $\mathfrak{M} = \mathbb{N} \cup \mathbb{N}' \cup \{\alpha\}$. We define a binary function $+$ by setting

$$x + y = \begin{cases} n', & \text{if } x = n \text{ and } y = n+1, \\ \alpha, & \text{otherwise.} \end{cases}$$

We also set $0 = \alpha$ and $-x = x$ for each $x \in X$. In the figure for this model included below, if $x = y + z$, we draw lines from y and z to x.

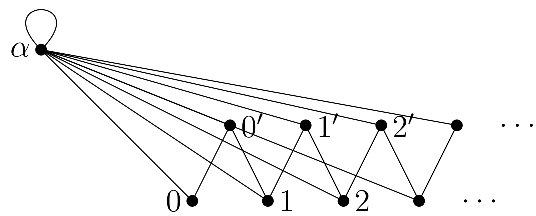

FIGURE 3. The model $\mathfrak{M}$

Next we define a model $\mathfrak{N} = \{a, b, c, d, \beta\}$. We let R be a ternary relation on Y such that $R = \{(a, c, d), (b, c, d)\} \cup \{(\beta, x, y) : x, y \in Y\}$. We also set $0 = \beta$ and $-y = y$ for each $y \in Y$. Again see the figure below for a concrete depiction of this model, where we have omitted displaying the direct line from c to β for ease of drawing.

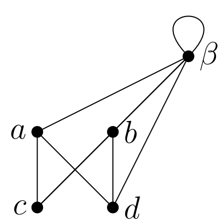

FIGURE 4. The model $\mathfrak{N}$

Now, we define a map $f : \mathfrak{M} \to \mathfrak{N}$ by setting

$$f(x) = \begin{cases} c, & \text{if } x \in \mathbb{N} \text{ is odd,} \\ d, & \text{if } x \in \mathbb{N} \text{ is even,} \\ a, & \text{if } x \in \mathbb{N}' \text{ is odd,} \\ b, & \text{if } x \in \mathbb{N}' \text{ is even,} \\ \beta, & \text{otherwise} \end{cases}$$

It is easy to check that f is a p-morphism of relational models for our modal language. Now the model $\mathfrak{M}$ is functional for $+$ while R is not a functional relation on $\mathfrak{N}$. If functionality of ternary relations were modally definable then, as p-morphisms preserve frame truth of formulas, we would have a contradiction. $\square$

3. Modal validities, axiom system and completeness

Our next topic is determining the valid principles of reasoning in our formalism. What the modal language with nominals captures is reasoning of two kinds: about objects (group elements, vectors) and sets of objects (which stand for the earlier-mentioned 'patterns'). Accordingly, our proof system will have two components: (a) purely modal principles expressing basic reasoning about sets, and (b) hybrid logic principles allowing us to use nominals as names of specific objects, like individual constants in first-order logic, and formalize the simple equational-style first-order arguments we have employed before. One can see this as serving two purposes: in reasoning about objects, we stay close to mathematical practice, but in adding the modal set principles, we extend the scope of the patterns that we can describe and identify interesting laws that are not specific to objects.[2]

We will describe the validities that can be stated in our modal language in several steps. First we present an axiomatic proof system that mixes principles using nominals with principles in a pure modal format. We illustrate in some detail how this system works in deriving formal theorems in practice. Then we prove the meta-property of completeness, based on known methods in hybrid and first-order logic, noting that a more austere (though in our view, less informative) purely nominal version would also be complete. Next, we return to the more general modal laws encoded inside the system and how these can be made explicit. This connects to the field of 'Complex Algebra', [34, 36, 37, 41], as will be explained. The outcome is a mixture of reasoning about points and about sets, and we end with some discussion of further issues and open problems. Many of our findings are not confined to just groups, commutative or otherwise, but we will not pursue generalizations.

3.1. **A proof system.**

Definition 3.1. *The* Modal Logic of Commutative Groups LCG *has the following axioms:*

(1) *All tautologies of classical propositional logic*
(2) $(\varphi \vee \psi) \oplus \chi \leftrightarrow (\varphi \oplus \chi) \vee (\psi \oplus \chi)$
(3) $\langle - \rangle(\varphi \vee \psi) \leftrightarrow \langle - \rangle\varphi \vee \langle - \rangle\psi$
(4) $\varphi \to \mathsf{E}\varphi$
(5) $\mathsf{EE}\varphi \to \mathsf{E}\varphi$
(6) $\varphi \to \mathsf{UE}\varphi$
(7) $\varphi \oplus \psi \to \mathsf{E}\varphi$
(8) $\langle - \rangle\varphi \to \mathsf{E}\varphi$
(9) $E\mathsf{i}$ *where* i *ranges over all nominals in our language*

[2]The following background may be helpful. As noted earlier, our modal language on models is a fragment of a first-order language describing the group structure plus subsets denoted by unary predicate variables, but without quantification over subsets. This puts its expressive power in between a pure first-order theory of groups, and the monadic second-order theory of groups. One striking special case occurs in spatial logics: Tarski's Elementary Geometry is a decidable first-order theory of Euclidean space, while the monadic second-order theory of even just betweenness in Euclidean space is Π^1_1-complete, [43]. For modal theories of affine and metric structure in Euclidean space, and projective geometry cf. [4].

(10) $\mathsf{E}(\varphi \wedge \mathsf{i}) \to \mathsf{U}(\mathsf{i} \to \varphi)$

(11) $\langle - \rangle \neg\varphi \leftrightarrow \neg\langle - \rangle\varphi$

(12) $(\varphi \wedge \mathsf{E}\psi) \to \mathsf{E}(\varphi \oplus \psi)$

(13) $(\varphi \wedge \mathsf{m} \oplus \mathsf{n}) \to \mathsf{U}(\mathsf{m} \oplus \mathsf{n} \to \varphi)$

(14) $\varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$

(15) $\varphi \oplus \psi \leftrightarrow \psi \oplus \varphi$

(16) $\varphi \oplus \mathsf{0} \leftrightarrow \varphi$

(17) $\mathsf{i} \oplus \langle - \rangle\mathsf{i} \leftrightarrow \mathsf{0}$

The rules of inference for LCG *are*

Modus Ponens, Replacement of Provable Equivalents, plus this Substitution Rule:

arbitrary formulas can be substituted for formula variables φ,
while for nominals, we can only substitute nominal terms
formed from nominals using the functional modalities $\langle - \rangle$, $\oplus$.

Moreover, we have the following rules governing (and entangling) nominals and modalities:

Necessitation Rules: $\frac{\neg\varphi}{\neg(\varphi\oplus\psi)}$, $\frac{\neg\varphi}{\neg\langle - \rangle\varphi}$, $\frac{\neg\varphi}{\neg\mathsf{E}\varphi}$

Naming Rule: $\frac{\mathsf{j}\to\theta}{\theta}$ *and*

Witness Rules: $\frac{\mathsf{E}(\mathsf{j}\wedge\varphi)\to\theta}{\mathsf{E}\varphi\to\theta}$, $\frac{\langle - \rangle(\mathsf{j}\wedge\varphi)\to\theta}{\langle - \rangle\varphi\to\theta}$, $\frac{(\mathsf{j}\oplus\mathsf{k}\wedge\mathsf{E}(\mathsf{j}\wedge\varphi)\wedge\mathsf{E}(\mathsf{k}\wedge\psi))\to\theta}{(\varphi\oplus\psi)\to\theta}$.

In the Naming and Witness rules, j, k *are nominals that do not occur in any of the formulas φ, ψ or θ.*

The first axioms in this system are standard for normal existential modalities, next comes the basic hybrid machinery of global modalities and nominals, then principles fixing the functional structure for inverse and addition, and finally some specific axioms transcribing the definition of commutative groups. The derivation rules are standard, and they allow us, in particular, to carry out elementary equational reasoning about groups using nominal terms to denote objects. Moreover, the Naming and Witness rules allow us to reason with nominals like with individual constants in first-order logic, and also, as we will see, lift nominal principles to more general modal ones. Notice also that all our axioms have modal Sahlqvist form, which simplifies modal completeness arguments, [24] and [23, Section 5.4]. The deductive power of this calculus will become clearer with the examples to follow.

Theorem 3.2. *All theorems of the system* LCG *are valid on group models.*

Proof. The preceding explanations should largely suffice. More specifically, Axiom (12) says that if we have φ true at any point x and ψ at a point y, then there exists a point, namely $x + y$ where $\varphi \oplus \psi$ is true. In fact, this axiom enforces precisely that addition is defined on any two points. In addition, Axiom (13) then says that these function values are unique. As another example, the Witness Rule for the $\oplus$ modality, read contrapositively, says the following. If we have $(\varphi \oplus \psi) \wedge \neg\theta$ true at any point x in any model, we can add fresh names for a witnessing pair y, z s.t. $x = y+z, y \models \varphi, z \models \psi$ and make the antecedent of the premise conditional true, while keeping θ false. This mirrors exactly how introducing witnesses for existential quantifiers works in completeness proofs for first-order logic, [26, 57]. □

Remark 3.3. Given the mentioned Sahlqvist form of all our axioms, it is easy to run a standard modal correspondence argument showing that frame truth of the above principles taken together forces general relational frames to be commutative groups.

More concrete information about the system LCG arises in formal derivations of some facts shown semantically in the preceding section.

Example 3.4. The formula $\mathsf{i} \to ((\mathsf{Ei} \wedge \varphi) \leftrightarrow \varphi)$ is provable. First, $\mathsf{i} \wedge \varphi$ implies $\mathsf{E}(\mathsf{i} \wedge \varphi)$ by Axiom (4). Vice versa, $\mathsf{E}(\mathsf{i} \wedge \varphi)$ implies $\mathsf{U}(\mathsf{i} \to \varphi)$ by Axiom (10), and together with the assumption i, this implies φ.

Example 3.5 (Surjectivity of addition)**.** The formula $\top \oplus \top$ says that each point is a sum of two points. This is provable from Axiom (16) by instantiating φ to obtain $\top \to (\top \oplus \mathsf{0})$, and then applying the upward monotonicity of $\oplus$ [a standard consequence of Axiom (2) – suitably modified by Axiom (15)] to the occurrence of $\mathsf{0}$ triggered by the propositionally provable implication $\mathsf{0} \to \top$.

Example 3.6 (Definition of the existential modality)**.** The equivalence $\mathsf{E}\varphi \leftrightarrow \varphi \oplus \top$ stated at the start of this paper can be proved as follows. From right to left, using Axiom (7), $\varphi \oplus \top$ immediately implies $\mathsf{E}\varphi$. From left to right, using nominals, we first represent the earlier argument for Fact 2.7 in our proof system. (a) $\mathsf{i} \to \mathsf{j} \oplus \top$ is provable as follows for any two nominals [we will not state all the obvious uses of LCG principles being used here]: i implies $\mathsf{i} \oplus \mathsf{0}$, which implies $\mathsf{i} \oplus (\mathsf{j} \oplus \langle - \rangle \mathsf{j})$, which implies $\mathsf{j} \oplus (\mathsf{i} \oplus \langle - \rangle \mathsf{j})$, which implies $\mathsf{j} \oplus \top$. (b) Then we also get, as in the preceding example, that $(\mathsf{i} \wedge \mathsf{E}(\mathsf{j} \wedge \varphi)) \to \varphi \oplus \top$, or rearranged in propositional logic: $(\mathsf{E}(\mathsf{j} \wedge \varphi)) \to (\mathsf{i} \to \varphi \oplus \top)$. (c) Now we use the first Witness Rule to conclude $\mathsf{E}\varphi \to (\mathsf{i} \to \varphi \oplus \top)$, and rearranging once more, we get $\mathsf{i} \to (\mathsf{E}\varphi \to \varphi \oplus \top)$. (d) Finally, we apply the Naming Rule to obtain the desired formula $\mathsf{E}\varphi \to \varphi \oplus \top$.

Finally, we discuss how our proof system lifts basic facts derivable in the elementary equational logic of groups to modal 'pattern validities'.

Example 3.7 (Exporting equational validities to LCG)**.** As is well-known, the group axioms imply that inverse is self-dual. For instance, in equational logic, one can use the following simple chain of equality statements:

$$x + -x = 0, \;\; (x + -x) + - - x = 0 + - - x$$

$$x + (-x + - - x) = - - x, \;\; x + 0 = - - x, \;\; x = - - x$$

Essentially this same reasoning can be represented as a proof of the nominal validity

$$\mathsf{n} \leftrightarrow \langle - \rangle \langle - \rangle \mathsf{n}$$

where we use the associativity axiom plus Replacement of Equivalents plus the Substitution Rule for nominal terms in LCG. Next, using the Naming and Witnessing rules as above, we show how to lift this equivalence to arbitrary formulas. For instance, we can first prove the following implications using nominals as witnesses for the two iterated inversion modalities, and appealing to the relevant axioms:

$(\mathsf{n} \wedge \langle - \rangle \mathsf{k} \wedge \mathsf{E}(\mathsf{k} \wedge \langle - \rangle \mathsf{l}) \wedge \mathsf{E}(\mathsf{l} \wedge \varphi)) \to (\mathsf{n} \wedge \langle - \rangle \langle - \rangle \mathsf{l} \wedge \mathsf{E}(\mathsf{l} \wedge \varphi)) \to$
$(\mathsf{n} \wedge \mathsf{l}) \wedge \mathsf{E}(\mathsf{l} \wedge \varphi)$ *[using the derived equivalence for nominals]* $\to \varphi$

Now two applications of the Witness rule derive $\langle - \rangle \langle - \rangle \varphi \to \varphi$. For deriving the converse direction, one can appeal to the principle just proved with $\neg\varphi$ instead of φ and then use the functionality axiom plus propositional contraposition. Finally, combining these two results, we have proved the following modal principle as a theorem:

$$\varphi \leftrightarrow \langle - \rangle \langle - \rangle \varphi$$

Similar proof techniques can be used to derive the following modal version of one more basic equational law for commutative groups:

$$\langle - \rangle (\varphi \oplus \psi) \leftrightarrow (\langle - \rangle \varphi \oplus \langle - \rangle \psi)$$

More can be said about the styles of reasoning available in our proof system LCG, and we will return to this with further illustrations in the final part of this section.

3.2. **Completeness.**

Theorem 3.8. LCG *is complete with respect to models over commutative groups.*

Proof. Completeness is proved by showing that any consistent set of formulas is satisfied at some point in some group model. For this, one can largely use the standard canonical model argument for hybrid logic with global modalities and nominals, cf. [18, 3] for details. We merely outline the main steps of the construction and some key assertions in what follows. Hybrid completeness proofs are also close to the standard Henkin-style completeness proof for first-order logic, [26, 57], which saturates and completes a single given consistent set of formulas, and this, too, will show in our presentation which makes use of one 'guidebook set' $\Sigma^\bullet$ in leading up to the crucial Truth Lemma formulated below.[3]

First, starting from a given consistent set Σ in the above language, we consider all possible maximally consistent sets, at least one of which contains Σ. Next, suitably extending the language, (a) for each set we add a fresh nominal providing a unique name for it [the result is a family of consistent sets, by using the Naming Rule of LCG], (b) we then add fresh nominals witnessing each existential modal formula in each set, so that for instance, with $\varphi \oplus \psi \in \Delta$, we add the formulas $\mathsf{i} \oplus \mathsf{j}, \mathsf{E}(\mathsf{i} \wedge \varphi), \mathsf{E}(\mathsf{j} \wedge \psi)$ to Δ [again this can be done consistently using the three Witness Rules of of LCG] for existential modalities]. Now we have a family of consistent sets in an extended language. Next, we consider all maximally consistent sets extending these, repeat the naming and witnessing process, and so on. We then take the countable limit of this construction, where we note that, since our maximally consistent sets were named uniquely by nominals, they cannot 'split' into two different sets at the next witnessing level. In this way, we find a family of maximally consistent sets for an extended language L where each maximally consistent set is *named*: it contains at least one nominal denoting it uniquely, and these sets are also *witnessing* in the following sense:

[3] Our exposition uses formulas $\mathsf{E}(\mathsf{n} \wedge \varphi)$ for what is often written as $@_{\mathsf{n}}\varphi$ in hybrid notation. In first-order terms, this amounts to describing substitution instances $\varphi(\mathsf{n}/x)$ where n is an individual constant denoting an object, and x the running variable of our modal formulas (now viewed as first-order formulas via the well-known standard translation) which are always interpreted local to one point at at a time.

- If $\mathsf{E}(\mathsf{n} \wedge \mathsf{E}\varphi) \in \Gamma$, then, for some nominal i not occurring in $\mathsf{n} \wedge \varphi$: $\mathsf{E}\mathsf{n} \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \in \Gamma$
- If $\mathsf{E}(\mathsf{n} \wedge \langle - \rangle\varphi) \in \Gamma$ then, for some some nominal i: $\mathsf{E}(\mathsf{n} \wedge \langle - \rangle \mathsf{i}) \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \in \Gamma$
- If $\mathsf{E}(\mathsf{n} \wedge \varphi \oplus \psi) \in \Gamma$, then for some nominals i, j: $\mathsf{E}(\mathsf{n} \wedge \mathsf{i} \oplus \mathsf{j}) \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \wedge \mathsf{E}(\mathsf{j} \wedge \psi) \in \Gamma$

For an illustration, the third closure condition listed here holds for the following reason. Suppose that $\mathsf{E}(\mathsf{n} \wedge \varphi \oplus \psi) \in \Gamma$. Then by the limit construction described above, this formula already belongs to some set $\Gamma_k \subseteq \Gamma$ produced at some finite stage k, which means that there was a maximally consistent set Δ_{k+1} containing $\mathsf{n} \wedge \varphi \oplus \psi$ by the next witnessing stage $k+1$, and by stage $k+2$, the formula $\varphi \oplus \psi$ will have been witnessed in an extension Δ_{k+2} using new nominals i, j such that $\mathsf{i} \oplus \mathsf{j}, \mathsf{E}(\mathsf{i} \wedge \varphi), \mathsf{E}(\mathsf{j} \wedge \psi) \in \Delta_{k+2}$, which implies that the formula $\mathsf{E}(\mathsf{n} \wedge \mathsf{i} \oplus \mathsf{j}) \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \wedge \mathsf{E}(\mathsf{j} \wedge \psi)$ is in Γ_{k+2} and hence also in Γ.

On this universe, we define the usual accessibility relation for the existential modality E:

$$\Sigma R_{\mathsf{E}} \Delta \text{ if for every formula } \alpha \in \Delta : \mathsf{E}\alpha \in \Sigma.$$

Given our axioms for E, this is an equivalence relation.

Now we fix one particular maximally consistent set $\Sigma^\bullet$ containing the original set Σ for which we want to find a model. This set contains all the information that we need, and hence we shall often refer to $\Sigma^\bullet$ as the *guidebook*. Moreover, we restrict attention to all maximally consistent sets reachable from $\Sigma^\bullet$ in the relation R_{E}: this standard move to a 'generated submodel' ensures that the global existential modality will get its correct interpretation. In particular, all sets that remain contain the same formulas of the form $\mathsf{E}\varphi$. The resulting family is the *domain* of our group model $\mathfrak{M}$. A *valuation* for generic proposition letters and nominals can now be read off by letting them denote all sets in which they occur. Here the axioms for nominals ensure that their denotations are singleton sets. In particular, we have an interpretation for the nominal $\mathsf{0}$ since we have $\mathsf{E0}$ present by Axiom (9).

Next, we introduce the *relational structure* for the modalities. Recall that each maximally consistent set in our domain contains a unique nominal, and we will often use this nominal instead of set notation in what follows. We now define:

$$R_{\langle - \rangle}(\mathsf{i}, \mathsf{j}) \text{ if } \mathsf{E}(\mathsf{i} \wedge \langle - \rangle \mathsf{j}) \in \Sigma^\bullet$$

$$R_{\oplus}(\mathsf{n}, \mathsf{i}, \mathsf{j}) \text{ if } \mathsf{E}(\mathsf{n} \wedge (\mathsf{i} \oplus \mathsf{j})) \in \Sigma^\bullet$$

Here by the axioms (7), (8), these relations for inverse and addition stay inside the R_{E}-equivalence class of $\Sigma^\bullet$, as these relations are included in R_{E}. Also in a standard manner, Axiom (11) enforces that the relation for inverse is a function f, [4] where $f \circ f$ is the identity map by Axiom (17). In addition, using the presence of the Axioms (12), (13) in the system LCG, we can show that the ternary sum relation is in fact *functional*.

Finally Axioms (14)–(17) enforce via a standard argument that the functions defined here satisfy the conditions for a commutative group. Thus, we have defined a group model $\mathfrak{M}$.

It remains to show the usual Truth Lemma in a suitable form: truth in the model thus defined at a maximally consistent set viewed as a point is in harmony with the syntactic content of that set: it 'does' what it 'says'. The following result makes use of the Boolean

[4] As an illustration, here is how Axiom (13) yields partial functionality of the relation $R_{\oplus}$. Let $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ be maximally consistent sets named by $\mathsf{n}_1, \mathsf{n}_2, \mathsf{i}, \mathsf{j}$, respectively. By definition then, $\mathsf{E}(\mathsf{n}_1 \wedge \mathsf{j} \oplus \mathsf{k})$ and $\mathsf{E}(\mathsf{n}_2 \wedge \mathsf{j} \oplus \mathsf{k})$ are both in $\Sigma^\bullet$. But then, using Axiom (13) and the deductive closure of maximally consistent sets, we get that $\mathsf{E}(\mathsf{n}_1 \wedge \mathsf{n}_2) \in \Sigma^\bullet$. But this implies that $\mathsf{n}_1, \mathsf{n}_2$ must name the same set.

decomposition properties of maximally consistent sets, denoted by their names for simplicity, plus the special witnessing properties for our existential modalities put in place above.

Fact 3.9. *For all formulas φ and all sets n in $\mathfrak{M}$, the following are equivalent:*

(*a*) $\mathfrak{M}, \mathsf{n} \models \varphi$, (*b*) $\mathsf{E}(\mathsf{n} \wedge \varphi) \in \Sigma^\bullet$.

Proof. For an illustration, consider the case $\varphi = \psi \oplus \chi$. *From (a) to (b).* Let $\mathfrak{M}, \mathsf{n} \models \psi \oplus \chi$. Then there are nominals i and j with $R_\oplus(\mathsf{n}, \mathsf{i}, \mathsf{j})$, $\mathfrak{M}, \mathsf{i} \models \psi$ and $\mathfrak{M}, \mathsf{j} \models \chi$. By the inductive hypothesis, we have $\mathsf{E}(\mathsf{i} \wedge \psi), \mathsf{E}(\mathsf{j} \wedge \chi) \in \Sigma^\bullet$, while the definition of $R_\oplus$ gives $\mathsf{E}(\mathsf{n} \wedge \mathsf{i} \oplus \mathsf{j}) \in \Sigma^\bullet$. These formulas imply $\mathsf{E}(\mathsf{n} \wedge \psi \oplus \chi)$ using the principles for nominals and the existential modality in LCG, and so, since $\Sigma^\bullet$ is maximally consistent, $\mathsf{E}(\mathsf{n} \wedge \psi \oplus \chi) \in \Sigma^\bullet$.

From (b) to (a). $\mathsf{E}(\mathsf{n} \wedge \psi \oplus \chi) \in \Sigma^\bullet$ implies by the Witnessing properties of our model that for some nominals i, j, $\mathsf{E}(\mathsf{n} \wedge \mathsf{i} \oplus \mathsf{j}), \mathsf{E}(\mathsf{i} \wedge \psi), \mathsf{E}(\mathsf{j} \wedge \chi) \in \Sigma^\bullet$. Thus, $R_\oplus(\mathsf{n}, \mathsf{i}, \mathsf{j})$ and also, by the inductive hypothesis, $\mathfrak{M}, \mathsf{i} \models \psi$ and $\mathfrak{M}, \mathsf{j} \models \chi$. It follows that $\mathfrak{M}, \mathsf{n} \models \psi \oplus \chi$. ☐

This concludes our outline of the proof that each LCG-consistent set is satisfiable in a group model, and hence of the completeness theorem for our proof system. ☐

Inspecting the details of the above proof, we can also see the following.

Fact 3.10. LCG *remains complete with Axioms (14)–(16) just in their nominal versions.*

We chose to define our proof system with a mix of nominal and modal principles to bring better out its power, and we conclude with some further observations about this interplay.

Remark 3.11. Our completeness proof also yields *strong completeness* in the same way as with the Henkin completeness proof for first-order logic, or standard completeness proofs for hybrid logics. The same applies to the completeness proof for vector spaces in Section 6 – though not to our analysis in Section 4 of logics with fixed-point operators for closure.

Remark 3.12. A different perspective on our completeness proof might arise from the analysis in [23, 38] The canonicity of modal Sahlqvist axioms can be seen as coming from their *d-persistence* in passing from so-called 'descriptive general frames' to full frames. In hybrid logic, given the Witness rules, it suffices to work with '*di*-persistence' with respect to 'discrete general frames' where every singleton is definable. Now suitably simple Sahlqvist formulas are *di*-persistent, and so are 'shallow modal formulas'. The axioms of our logic LCG have these simplicity properties, except for (13), but this might be overcome by the versatility of our language noted earlier, using the results in [60]. We leave matters here, but one outcome of this type of analysis would be that we also keep completeness for extensions of our system to special kinds of groups that can be defined by suitably simple modal axioms.

3.3. **Point and set principles in the modal logic of groups.** The proof system LCG can represent first-order equational reasoning by using only nominal terms, and transcribing the group axioms. But as said earlier, the modal formulas of the language extend the analysis from objects to sets of objects. In particular, some validities for nominals are in fact valid for arbitrary modal formulas. For instance, $\mathsf{n} \oplus 0 \leftrightarrow \mathsf{n}$ is valid, but so is $\varphi \oplus 0 \leftrightarrow \varphi$, which is why we put the latter version as an axiom. Here is one more illustration of the proof techniques involved here, which shows the power of our proof system. Of course, by the completeness theorem for the purely nominal version of LCG, the following formal derivations must exist. However, a completeness proof alone seldom gives direct information for specific cases.

Example 3.13 (The modal associativity law)**.** Suppose we only have the associativity law for nominals, i.e., $\mathsf{n} \oplus (\mathsf{k} \oplus \mathsf{m}) \leftrightarrow (\mathsf{n} \oplus \mathsf{k}) \oplus \mathsf{m}$. Then Axiom (14) $\varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$ in its full generality can still be derived. We only consider the direction from left to right, and analyze what principles with nominals added would suffice by the Witness Rule for the $\oplus$ modality. First it suffices to show (a) $(\mathsf{k} \oplus \mathsf{l} \wedge \mathsf{E}(\mathsf{k} \wedge \varphi) \wedge \mathsf{E}(\mathsf{l} \wedge \psi \oplus \chi)) \to (\varphi \oplus \psi) \oplus \chi$ for some fresh nominals k, l. (b) Next we rearrange this to the task of proving the propositionally equivalent $\mathsf{E}(\mathsf{l} \wedge \psi \oplus \chi) \to ((\mathsf{k} \oplus \mathsf{l} \wedge \mathsf{E}(\mathsf{k} \wedge \varphi)) \to (\varphi \oplus \psi) \oplus \chi)$. (c) Next, we use the easily shown fact [by the axioms for the global modality] that LCG can prove an implication of the form $\mathsf{E}\alpha \to \beta$ iff it can prove $\alpha \to \mathsf{U}\beta$. This reduces the preceding task to proving $(\mathsf{l} \wedge \psi \oplus \chi) \to \mathsf{U}((\mathsf{k} \oplus \mathsf{l} \wedge \mathsf{E}(\mathsf{k} \wedge \varphi)) \to (\varphi \oplus \psi) \oplus \chi)$, or equivalently $\psi \oplus \chi \to (\mathsf{l} \to \mathsf{U}((\mathsf{k} \oplus \mathsf{l} \wedge \mathsf{E}(k \wedge \varphi)) \to (\varphi \oplus \psi) \oplus \chi))$. (d) By the Witness Rule once more, it suffices to derive $(\mathsf{m} \oplus \mathsf{n} \wedge \mathsf{E}(\mathsf{m} \wedge \psi) \wedge \mathsf{E}(\mathsf{n} \wedge \chi)) \to (\mathsf{l} \to \mathsf{U}((\mathsf{k} \oplus \mathsf{l} \wedge \mathsf{E}(\mathsf{k} \wedge \varphi)) \to (\varphi \oplus \psi) \oplus \chi))$, with fresh nominals m, n, or equivalently $(\mathsf{l} \wedge \mathsf{m} \oplus \mathsf{n} \wedge \mathsf{E}(\mathsf{m} \wedge \psi) \wedge \mathsf{E}(\mathsf{n} \wedge \chi)) \to \mathsf{U}((\mathsf{k} \oplus \mathsf{l} \wedge \mathsf{E}(\mathsf{k} \wedge \varphi)) \to (\varphi \oplus \psi) \oplus \chi)$. What remains to be addressed is a provability task with complete naming of all points involved, and the crux is then to note that $\mathsf{l} \wedge \mathsf{m} \oplus \mathsf{n}$ implies that $\mathsf{U}(\mathsf{l} \to \mathsf{m} \oplus \mathsf{n})$, so we can replace the subformula $\mathsf{k} \oplus \mathsf{l}$ in the antecedent of the consequent by $\mathsf{k} \oplus (\mathsf{m} \oplus \mathsf{n})$. Then we appeal to our assumed nominal version of Associativity to replace this by $(\mathsf{k} \oplus \mathsf{m}) \oplus \mathsf{n}$, and using further principles in LCG for nominals, we put the formulas φ, ψ, χ back in place to obtain the desired modal formula $(\varphi \oplus \psi) \oplus \chi$.

Behind this example lies a general issue of lifting laws for algebras to set versions.

Remark 3.14 (Complex algebra)**.** The theory of *complex algebras* studies set liftings of standard algebraic structures. A standard example is the set-lifting of standard Boolean algebras $(B, \wedge, \vee, \neg, 0)$. Define the complex algebra $(\mathcal{P}(B), \wedge, \vee, \neg, \{0\})$, where we set (a) $A \wedge B = \{a \wedge b : a \in A, b \in B\}$, (b) $A \vee B = \{a \vee b : a \in A, b \in B\}$, and (c) $\neg A = \{\neg a : a \in A\}$. The result is a new richer kind of algebra where the lifted conjunction and disjunction show clear similarities with our binary product modality, [11]. Complex algebras can be defined over any algebraic structure, and we refer to [34, 21] for applications and more details of their theory. One old question in the area is when valid equations for the underlying algebras transfer to the set-lifted versions of their operations. Gautam's Theorem [30] shows that this is rare: happening only when each variable occurs at most once on each side of the equation. This is precisely the sort of phenomenon we have noticed in the above.

The examples in this section of modal lifting of nominal principles worked because all axioms involved had Gautam form. But even in the absence of such a form, there may still be workarounds. The lifting need not be literal from equations to corresponding modal equivalences, but could also have other formats, as in the following illustration.

Example 3.15 (Group inverses)**.** We expressed the basic law $v + (-v) = 0$ as follows in our logic using nominals: $\mathsf{i} \oplus \langle - \rangle \mathsf{i} \leftrightarrow \mathsf{0}$. This cannot be lifted to sets in a direct manner, since $\varphi \oplus \langle - \rangle \varphi \leftrightarrow \mathsf{0}$ is clearly not a valid formula if the value of φ is not a singleton set, cf. Fact 2.9. However, there is a formula in our language that does the job without nominals except for the special nominal $\mathsf{0}$ for the zero vector, which can be viewed a a 0-ary modality in the signature. On the frames underlying group models, the following formula

$$\mathsf{E}\varphi \to \mathsf{E}((\varphi \oplus \langle - \rangle \varphi) \wedge \mathsf{0})$$

is easily seen to enforce the basic law of inverses we started with.

Again there is a background in Complex Algebra here, where axiomatizations in other non-literal formats have been proposed. In particular, [41] give a general game-based technique for turning equational axiomatizations for classes of algebras into axiomatizations of the theory of their complex algebras. Such results may also apply here, though we do not just have the complex algebra by itself, but also the standard Booleans in our language. However, as emphasized in the introduction to this paper, the most relevant analogy for the approach in this paper is the following approach to complex algebra by Goranko & Vakarelov [36].

Remark 3.16 (A congenial modal approach)**.** In [36] the authors axiomatize the complex algebra of Boolean algebras [with the standard Booleans thrown in, now as standard modal operations at the set level] using a modal logic based on a *difference modality* saying that a formula is true at some point different from the current one. Amongst other things, this device allows for defining truth in single points, as pioneered in [29], while also facilitating the formulation of a family of new derivation rules that ensure completeness. An interesting feature of this approach is that the difference modality is definable via the standard Boolean connectives of classical logic plus the modal operator $\langle\rightarrow\rangle$ obtained by lifting the Boolean implication $\rightarrow$. The authors suggest that this approach is also available for groups, since a difference modality can be defined there as well. Indeed, the formula $\varphi \oplus \neg \mathsf{0}$ in our language defines the difference modality. If it is true for a point s, then $s = y + u$ for some t, u with $t \models \varphi$ and $u \neq 0$, and by the group axioms it follows that $s \neq t$. Our axiomatization is in the above spirit, but the use of nominals leads to gains in perspicuity.

From a model-theoretic perspective, Gautam equations translate into formulas where the proposition letters or formula variables for the algebraic variables are in *distributive* position, which allows for taking them out by single existential quantifiers.[5] From a proof-theoretic perspective, our earlier examples illustrate why equations in Gautam form allow for set lifting, and indeed, it can be shown that every such lifting is available in LCG. More general questions arise here, but they are beyond the scope of this paper.

However, one can also drop this all-or-nothing lifting perspective, and ask which *mixed principles* with nominals as needed and formula variables where possible are derivable. One can start from a purely nominal principle and test for φ-type generalizations of some nominals, or start with an invalid purely modal principle, and ask which replacements of formula variables by nominals might lead to validity. A case where this makes sense goes back to Section 2.2 about the Minkowski operations definable in our modal language.

Example 3.17 (Minkowski operations revisited)**.** In Mathematical Morphology, some laws for shapes are valid in our basic modal logic without nominals, whereas others need more. Here is an illustration which also highlights the earlier-mentioned analogy between a Minkowski difference $A \ominus B$ and a substructural implication with antecedent B and consequent A. Accordingly, we switch to the notation $\psi \Rightarrow \varphi$ with formulas standing for subsets while the implication symbol $\Rightarrow$ abbreviates our complex modal definition of Minkowski difference in Section 2.2. In particular, given our completeness theorem, all basic laws of substructural implication are derivable in the logic LCG. Consider the following formula where the equivalence symbol written in the middle is not a new operator in the calculus, but just a shorthand for writing a pair of implications, one in each direction:

[5] There are analogies here with the correspondence analysis of existential quantifiers in the antecedent of Sahlqvist-type axioms, [18, Ch. 3], which get replaced by prefix quantifiers over individual worlds.

$$(\psi \Rightarrow (\varphi \oplus \alpha)) \Leftrightarrow ((\psi \Rightarrow \varphi) \oplus \alpha)$$

From right to left, this principle is derivable in any standard basic implicational calculus. Start from a formula $\psi \oplus ((\psi \Rightarrow \varphi) \oplus \alpha)$, use associativity to rewrite to $(\psi \oplus (\psi \Rightarrow \varphi)) \oplus \alpha$, then use the Modus Ponens validity $\psi \oplus (\psi \Rightarrow \varphi) \Rightarrow \varphi$ to conclude $\varphi \oplus \alpha$. By Conditionalization, another basic principle in such calculi, we get the desired implication.

In the opposite direction, however, this principle is not valid on commutative groups. Consider the integers with addition (many other structures would do), and let φ denote the set $\{1, 2\}$, ψ the set $\{2, 3\}$, and α the set $\{1, 3\}$. Then $\varphi \Rightarrow \psi$ denotes $\{1\}$, $\psi \oplus \alpha$ denotes $\{3, 4, 5, 6\}$, $\varphi \Rightarrow (\psi \oplus \alpha)$ denotes $\{2, 3, 4\}$, while $(\varphi \Rightarrow \psi) \oplus \alpha$ denotes the set $\{2, 4\}$.[6]

But now consider the following special case of the above equivalence where the formula α has been replaced by he nominal n denoting one single object:

$$(\psi \Rightarrow (\varphi \oplus \mathsf{n})) \Leftrightarrow ((\psi \Rightarrow \varphi) \oplus \mathsf{n})$$

This mixed principle is valid on group models. We only show the non-trivial direction from left to right. Let point x satisfy the formula $\psi \Rightarrow (\varphi \oplus \mathsf{n})$, and write x as the sum $(x - n) + n$ where n is the denotation of n. Now, let y be any group element satisfying ψ: then $(x - n) + y = (x + y) - n$ where by the assumption, $x + y$ satisfies $\varphi \oplus \mathsf{n}$. But then $(x + y) - n$ and hence also $(x - n) + y$ satisfies φ, and therefore $x - n$ satisfies $\psi \Rightarrow \varphi$. By the decomposition $x = (x - n) + n$, object x also satisfies $(\psi \Rightarrow \varphi) \oplus \mathsf{n}$.

The preceding semantic argument can be reproduced formally in the proof system LCG by making use of the rules for nominals.[7]

4. Modal logic of subgroup closure

A basic notion in groups is that of *subgroups* closed under taking group sums and inverses. One can think of this as a sort of of dependence: everything in the closure of a set X is determined by the objects in X, and hence depends on X. This dependence perspective will be pursued in Section 5. For now, we just look at the modal logic content of closure.

4.1. **Language and definability.**

We add the following new modality to our language.

Definition 4.1. *$\mathfrak{M}, s \models \mathsf{C}\varphi$ iff s is obtained from the set $\{t \mid \mathfrak{M}, t \models \varphi\}$ by finitely many uses of the binary operation $+$, the unary operation $-$ and the nullary operation 0.*

Note that $\mathsf{C}\varphi$ is not an ordinary modality, since it does not distribute over conjunctions or disjunctions, as is clear from the behavior of closure in groups. Accordingly, our analysis in this section will use ideas from the broader *neighborhood semantics* for modal logic, [51], which deals with such generalized modalities.

[6]These calculations involve a generalized arithmetic with fixed and 'loose' numbers. We have specific numbers k, but also variables X whose value we only know as a set of numbers. Then our valid implication says that the standard arithmetical equality $(y+k)-x = (y-x)+k$ lifts to the valid equality $(Y+k)-X = (Y-X)+k$, but conversely, $(Y+Z)-X = (Y-X)+Z$ is no longer valid.

[7]Our example used the fact that our basic modal logic of groups can embed Minkowski-style or substructural categorial logics of product conjunction and implication. What we have not shown is whether this embedding is *faithful*. For instance, whether commutative linear logic is complete for vector space semantics, perhaps even for standard Euclidean vector spaces, is a long-standing open problem.

Remark 4.2 (Related definable notions)**.** The following notion of stepwise closure with n explicit arguments is already definable in our modal base language for each specific n:

$$\mathfrak{M}, v \models D^+(\varphi_1, \dots \varphi_n) \text{ iff } v \text{ is a sum of } n \text{ points } v_1, \dots v_n \text{ where } v_i \text{ satisfies } \varphi_i.$$

We can also introduce an abstract dependence notion $Dep_\varphi \psi$ as $\mathsf{U}(\psi \to \mathsf{C}\varphi)$, saying that each point satisfying ψ is in the subgroup generated by some finite set of points satisfying φ. This induces validities such as the following, derivable in our later proof system:

$$Dep_\varphi \psi \to Dep_{\langle - \rangle \varphi} \psi \qquad Dep_\varphi \psi \wedge Dep_\xi \eta \to Dep_{\varphi \vee \xi}(\psi \oplus \eta)$$

Perhaps the best way of understanding the closure modality involves a well-known modal fixed-point logic, viz. the modal μ-calculus, [20], over a language with binary modalities.

Fact 4.3. *$\mathsf{C}\varphi$ is definable in the μ-calculus extension of our basic modal language.*

Proof. The closure of a set denoted by a formula φ can be defined as the smallest fixed-point

$$\mu p. (\mathsf{0} \vee \varphi \vee \langle - \rangle p \vee (p \oplus p))$$

□

Remark 4.4. Given the invariance of μ-calculi for the same bisimulations as their underlying basic modal logics, it follows that the operator $\mathsf{C}\varphi$ is invariant for our earlier bisimulations of group models. However, we will not pursue this model-theoretic perspective here.

As for valid principles of reasoning with the closure modality, the preceding μ-calculus-style definition suggests the following two principles describing smallest fixed-points :

- $(\mathsf{0} \vee \varphi \vee \langle - \rangle \mathsf{C}\varphi \vee \mathsf{C}\varphi \oplus \mathsf{C}\varphi) \to \mathsf{C}\varphi$
- if $\mathsf{0} \to \alpha, \varphi \to \alpha, \langle - \rangle \alpha \to \alpha$ and $\alpha \oplus \alpha \to \alpha$ are valid, then so is $\mathsf{C}\varphi \to \alpha$.

As usual in modal logic, the latter implication between validities also holds when replacing validity by global truth in a model, yielding one valid axiom using the universal modality.

4.2. **Closure as a neighborhood modality.** One can view $\mathsf{C}\varphi$ as an upward monotonic neighborhood modality, describing points that are suitably related to the set of points $\{t \mid \mathfrak{M}, t \models \varphi\}$. One can then generalize the above intended models to frames with an abstract neighborhood relation NsX with the upward monotonic truth definition

$$\mathfrak{M}, s \models \mathsf{C}\varphi \quad \text{iff} \quad \exists X \ (NsX \ \& \ \forall x \in X : \mathfrak{M}, x \models \varphi)$$

This framework allows for an abstract modal frame correspondence analysis, [8, 18], of the content of natural principles governing the closure modality on group models. The following assertions rely on the earlier notion of frame truth for modal formulas.

Fact 4.5. *The following frame correspondences hold for the fixed-point principles governing the closure modality in abstract neighborhood frames enriched with a ternary relation R:*

- $p \to \mathsf{C}p$ *defines* $\forall x : Nx\{x\}$
- $\mathsf{0} \to \mathsf{C}p$ *defines* $\forall x : Nx\emptyset$ *iff* $x = 0$
- $\langle - \rangle \mathsf{C}p \to \mathsf{C}p$ *defines* $\forall x, Y : NxY$ *iff* $N_{-x}Y$
- $\mathsf{C}p \oplus \mathsf{C}p \to \mathsf{C}p$ *defines* $\forall x, y, z, U, V :$ *if* $Rxyz$ *and* NyU *and* NzV, *then* $NxU \cup V$
- *The fixed-point rule, with premises read as global truth in a model, enforces that the neighborhood relation is contained in the finite closure in the definition of* $\mathsf{C}\varphi$.

Proof. These facts follow by straightforward correspondence arguments. For instance, the LGC-axioms for inverse enforce in standard Sahlqvist style, [18, Ch. 3], the standard behavior of the inverse function in groups. Then, clearly, whatever valuation we choose for p on a given frame, $\langle - \rangle \mathsf{C}p \to \mathsf{C}p$ is going to be true when the stated condition holds, given our neighborhood-style truth definition for the modality C. Vice versa, if $\langle - \rangle \mathsf{C}p \to \mathsf{C}p$ is true for all valuations on a frame, and we have $N - xY$, then setting $V(p) = Y$ makes the antecedent $\langle - \rangle \mathsf{C}p$ true at x, whence we also have the consequent $\mathsf{C}p$ true at x, which implies that NxY. The sum axiom can be analyzed in the same Sahlqvist style, though here, the reader may find it easier to use the frame-equivalent version $\mathsf{C}p \oplus \mathsf{C}q \to \mathsf{C}(p \vee q)$ without repetitions of p in the antecedent. Finally, the proof for the last step is again a Sahlqvist-style argument, but in this case, without a first-order definition for the minimal valuation. This can be done as in the proof in [10] that the Induction Axiom of propositional dynamic logic PDL enforces that the accessibility relation for the modality $\Box^*$ is the reflexive transitive closure of the original accessibility relation R for the basic modality $\Box$. $\Box$

4.3. **Axiomatization and completeness.**

While the modality $\mathsf{C}\varphi$ does not distribute over conjunction or disjunction, it is obviously *upward monotonic*: $\mathsf{C}\varphi \to \mathsf{C}(\varphi \vee \psi)$ is valid. In combination with our earlier observations, this suggests the following proof system.

Definition 4.6. *The* proof system MCL *consists of the following components:*

(a) *The complete modal logic of abstract relational models, including the basic principles of hybrid logic for the universal modality and the nominals.*
(b) *The minimal monotonic neighborhood logic for the unary modality* C.
(c) *The following axiom and proof rule for smallest fixed-points:*
  - $(\mathsf{0} \vee \varphi \vee \langle - \rangle \mathsf{C}\varphi \vee \mathsf{C}\varphi \oplus \mathsf{C}\varphi) \to \mathsf{C}\varphi$
  - *if* $\vdash \mathsf{0} \to \alpha$, $\vdash \varphi \to \alpha$, $\vdash \langle - \rangle \alpha \to \alpha$ *and* $\vdash \alpha \oplus \alpha \to \alpha$, *then* $\vdash \mathsf{C}\varphi \to \alpha$.

Here are some provable formulas showing how the system works.

Example 4.7 (Formally derivable principles)**.**

- The formula $\mathsf{0} \leftrightarrow \mathsf{C}\bot$ is provable. From left to right, the fixed-point axiom gives $\mathsf{0} \to \mathsf{C}\varphi$ for any formula φ. From right to left, use a straightforward instance of the induction rule, noting that 0 satisfies all the premises.
- The formula $\mathsf{CC}\varphi \to \mathsf{C}\varphi$ expressing that the closure operator C is idempotent is valid in our semantics, and can be derived by taking both φ and α in the proof rule to be $\mathsf{C}\varphi$. For $\mathsf{C}\varphi \to \mathsf{CC}\varphi$, apply a distribution rule from the minimal modal logic to the consequence $\varphi \to \mathsf{C}\varphi$ of the first fixed-point principle.
- The formula $\mathsf{C}\varphi \oplus \mathsf{C}\psi \to \mathsf{C}(\varphi \vee \psi)$ is provable using upward monotonicity in both arguments to $\varphi \vee \psi$ and then applying the fixed-point axiom. In fact, the equivalence $\mathsf{C}(\varphi \vee \psi) \leftrightarrow (\mathsf{C}\varphi \vee \mathsf{C}\psi \vee \mathsf{C}\varphi \oplus \mathsf{C}\psi)$ is provable too, which we leave to the reader.

Soundness of the above proof system is immediate by the given intuitive explanations.

The completeness result to follow now is the best we have been able to obtain so far. It refers to the abstract relational models of Section 2 where, instead of a function $x = y + z$, we have a not necessarily functional ternary relation $Rxyz$. We include the result even so, since the proof presents several features of interest.

Theorem 4.8. *The above proof calculus* MCL *is complete for our modal base language extended with* $\mathsf{C}\varphi$ *over abstract relational models.*

Proof. We proceed as in the standard completeness proof for propositional dynamic logic, (cf. [18, Sec. 4.8] to which we refer for the notions employed in what follows), but thinking in the reverse direction for the values of the binary product function as in our basic semantics, and using a neighborhood-style semantic base for the closure modality. Our method is in essence the one developed for 'dynamic arrow logic' in [9].

Consider any formula φ that is consistent, where consistency is defined as usual with respect to the above proof system: we show that φ is satisfiable in an abstract model.

First fix a finite set F of relevant formulas in the same style as the Fisher-Ladner filtration for PDL: F contains φ, and is closed under (a) subformulas, (b) single negations, and (c) if $\mathsf{C}\varphi$ is in F, then so are $\langle - \rangle\varphi$, $\mathsf{C}\varphi \oplus \mathsf{C}\varphi$. That a set with these closure conditions stays finite follows by a standard argument. This will be our language in what follows.

Now we construct the *finite Henkin model* $\mathfrak{M}$ whose points are all maximally consistent sets of formulas in the finite set F, endowed with the following structure. The 0 point is the unique maximally consistent set containing the nominal $\mathsf{0}$. The valuation for a proposition letter p just selects the set of all maximally consistent sets containing p. Group inverses are treated as follows, using the axioms for the inverse modality in our proof system to show that we are mapping maximally consistent sets to maximally consistent sets.

Fact 4.9. *The stipulation* $inv(\Sigma) = \{\varphi \mid \langle - \rangle\varphi \in \Sigma\}$ *defines a function.*

Next, the ternary relation $\mathrm{R}\Gamma\Delta\Sigma$ is defined as follows:

$$conj(\Gamma) \wedge (conj(\Delta) \oplus conj(\Sigma)) \text{ is consistent}$$

where $conj(\Gamma)$ denotes the conjunction of all formulas in the finite set Γ, and so on.

Next, we show a Truth Lemma for maximally consistent sets and relevant formulas:

Fact 4.10. *A formula* φ *belongs to a set* Γ *iff* φ *is true at* Γ *in the finite Henkin model* $\mathfrak{M}$.

Proof. The inductive argument is straightforward for atomic formulas, Boolean combinations, and modalities $\langle - \rangle\varphi$, $\varphi \oplus \psi$, using the standard modal completeness argument in a filtration format, [13], adapted to deal with binary modalities. For instance, if $\varphi \oplus \psi \in \Gamma$, then maximally consistent sets Δ, Σ as above can be found by extending $\{\varphi\}, \{\psi\}$, respectively, with ever new formulas while keeping the above consistency statement true for Γ and the conjunctions of the two sets obtained. The crucial inductive step is the one for $\mathsf{C}\varphi$.

(a) Suppose that $\mathsf{C}\varphi$ belongs to the set Γ. Consider the set $\mathfrak{D}$ that consists of all maximally consistent sets containing φ, the unique maximally consistent set for the nominal denoting zero, plus all maximally consistent sets that can be obtained from these, iteratively, by closing under the results of unary inverse and binary relational combination as defined above. The family $\mathfrak{D}$ is finite, given that our Henkin model is finite, and hence it can be described by one formula δ, the disjunction of all conjunctions for its member sets. By its construction, this formula δ satisfies the conditions of the Fixed-Point Rule.

For instance, consider $\delta \oplus \delta \to \delta$. If this formula were not provable, then $\delta \oplus \delta \wedge \neg\delta$ would be consistent. But given the disjunctive definition of δ and basic theorems of the minimal modal logic like distribution of $\oplus$ over $\vee$, this implies that some formula $\delta_1 \oplus \delta_2 \wedge \neg\delta$ is consistent, where δ_1 is the conjunction of some set $\Delta_1 \in \mathfrak{D}$, and likewise δ_2 w.r.t. some Δ_2. But then, extending step by step with formulas in the initial set F, we can construct a set Γ whose conjunction γ is consistent with $\delta_1 \oplus \delta_2 \wedge \neg\delta$, and we have a contradiction: the first conjunct $\delta_1 \oplus \delta_2$ implies that $R\Gamma\Delta_1\Delta_2$ while the second conjunct implies that $\Gamma \notin \mathfrak{D}$.

Applying the fixed-point proof rule, we then have $\mathsf{C}\varphi \to \delta$ provable, and hence Γ is consistent with δ, and therefore, Γ must be in the family $\mathfrak{D}$ described. Finally, given the inductive hypothesis, membership and truth for φ coincide for all maximally consistent sets, and it follows that $\mathsf{C}\varphi$ is true at Γ in the finite Henkin model $\mathfrak{M}$.

(b) Conversely, let Γ make $\mathsf{C}\varphi$ true in $\mathfrak{M}$. Then Γ belongs to the finite family $\mathfrak{D}$ described above, where we again use the inductive hypothesis for φ to identify sets where φ is true with sets containing this formula. Now, we prove by induction on the depth of the construction for Γ from maximally consistent sets containing φ that $\varphi \in \Gamma$, by a repeated appeal to the fixed-point axiom plus properties of our closure. The base cases are obvious, we just consider the case of the sum modality. Suppose that $R\Gamma\Gamma_1\Gamma_2$ in $\mathfrak{D}$ with Γ_1, Γ_2 having decompositions of lower depth. By the inductive hypothesis, $\mathsf{C}\varphi$ belongs to both Γ_1, Γ_2. Now $\mathsf{C}\varphi \oplus \mathsf{C}\varphi$ belongs to the initial set F by one of our closure conditions, and using the definition of the ternary sum modality, it follows that $\mathsf{C}\varphi \oplus \mathsf{C}\varphi$ is consistent with the maximally consistent Γ and thus belongs to it. But then by the Fixed-Point Axiom, the formula $\mathsf{C}\varphi$ is consistent with Γ too, and hence it belongs to the set Γ. $\square$

Incorporating the global existential modality and nominals does not pose a problem in the preceding argument. In particular, a maximally consistent set containing a relevant nominal in the finite language will be unique with this property. $\square$

Remark 4.11 (Limitations)**.** Our completeness proof does not enforce *functionality* of the ternary relation behind the group operation. Moreover, our proof does not guarantee validity for the group axioms of LCG. While filtration does preserve some axioms expressing universal frame properties, it does not guarantee validity of axioms that express *existential requirements* such as Associativity. We leave these extensions as open problems.

We conclude with a perhaps not well-known observation on the proof system for standard propositional dynamic logic, triggered by the above role of neighborhood models.

Remark 4.12 (Distribution in propositional dynamic logic)**.** $\mathsf{C}\varphi$ is not a normal modality, since it only satisfies monotonicity. Given the strong analogies of the above proof with the completeness proof for the normal iteration modality $\Box^*\varphi$ in PDL, the lack of appeal to a distribution axiom in our completeness proof may seem surprising. But as it happens, monotonicity is the only active principle for $\Box^*$ in the completeness proof for PDL too: distribution over conjunction is already *derivable*. For better comparison with the existential flavor of $\mathsf{C}\varphi$, we derive distribution over disjunctions for the existential iteration modality $\Diamond^*\varphi$: it turns out that distribution for the base modality $\Diamond$ suffices plus the fixed-point principles for the inductive definition $\mu p.\,(\varphi \vee \Diamond p)$.

(a) We have directly from the PDL fixed-point axiom that $(\varphi \vee \psi) \to (\Diamond^*\varphi \vee \Diamond^*\psi)$.

(b) Also, $\Diamond(\Diamond^*\varphi \vee \Diamond^*\psi)$ implies $\Diamond\Diamond^*\varphi \vee \Diamond\Diamond^*\psi$ by distribution for the base modality, and this again implies $\Diamond^*\varphi \vee \Diamond^*\psi$ by the fixed-point axiom.

(c) Applying the PDL fixed-point rule to (a) and (b) yields $\Diamond^*(\varphi \vee \psi) \to (\Diamond^*\varphi \vee \Diamond^*\psi)$.

This example provides an interesting instance of how adding mathematical principles such as induction can boost the propositional modal base logic one started from.

A major question left open by the preceding result is completeness of modal closure logic for genuine group models where $+$ is a *function*. We have not been able to adjust our filtration method to achieve this feature, and must leave this as an open problem.

5. Dependence and independence in vector spaces

We now make a first step from groups to the richer structure of vector spaces. In such spaces, linear dependence is a basic notion, making it an obvious target for qualitative analysis in our style. At the same time, dependence is a topic of growing importance in the broader logical literature [56]. In particular, the recent paper [7] initiates a modal approach to dependence and explored some first connections with Linear Algebra. In this section, we take this modal line further, bringing in independence as well. However, toward the end we identify an important conceptual difference between our logics and those in the cited literature. Our treatment will focus on dependence and independence per se – an integration with our earlier complete logics is possible, and indeed desirable for expressing important properties such as dimension, but will not be undertaken in this paper.

5.1. **Linear dependence of vectors and Steinitz Exchange.**

The subgroup closure of Section 4 is related to dependence in vector spaces: a vector x depends on a set of vectors X if x is in the linear span of X. But these linear combinations of vectors can use scalars from a field that is part of the vector space. The difference shows up in new principles.

Example 5.1. The law of *Steinitz Exchange* says that, for finite sets of vectors X,

If vector z depends on $X \cup \{y\}$, then either z depends on X or y depends on $X \cup \{z\}$.

Steinitz Exchange is the crucial property underlying the elementary theory of dependence and dimension in vector spaces, [22]. Its validity, though simple to see, depends essentially on the field properties in vector spaces. If z is a linear combination of the form $c_1 \cdot x_1 + \cdots + c_k \cdot x_k + d \cdot y$, then there are two cases. Either $d = 0$ and z depends on X alone, or $d \neq 0$, and we can divide the equality by d to obtain, after suitably transposing terms, that y depends on $X \cup \{z\}$.

Note. In what follows, all notations $D_X y$ used will refer to finite sets of vectors X.

Next, let us specialize the modal-style models introduced in Section 2 from arbitrary groups to vector spaces as the underlying structure. More precise definitions of such 'vector models' $\mathfrak{M}$ will be given in Section 6, but the following can be understood without these details. In such structures, the dependence relation in vector spaces will be denoted by

$$D_X y : \quad y \text{ is a linear combination of vectors in the set } X.$$

We will also use the notation $D_X Y$ for $\forall y \in Y : D_X y$.

Fact 5.2. *Dependence in vector spaces satisfies the following properties:*

(a) $D_X x$ *for* $x \in X$ — *Reflexivity*
(b) $D_X y$ *and* $X \subseteq Z$ *imply* $D_Z y$ — *Monotonicity*
(c) $D_X Y, D_Y Z$ *imply* $D_X Z$ — *Transitivity*
(d) $D_{X,y} z$ *implies* $D_X z$ *or* $D_{X,z} y$ — *Steinitz Exchange*

To bring such facts into a logical syntax, we introduce a formal modality $\mathsf{D}\varphi$ for vector dependence with the following semantic interpretation in vector models :

$$\mathfrak{M}, x \models \mathsf{D}\varphi \text{ iff } x \text{ is a linear combination of vectors satisfying } \varphi.$$

Then the preceding four abstract properties express the following valid modal principles, of which the first three also hold for the earlier closure modality C of Section 4:

(a) $\varphi \to \mathsf{D}\varphi$,
(b) $\mathsf{D}\varphi \to \mathsf{D}(\varphi \vee \psi)$,
(c) $(\mathsf{D}\varphi \wedge \mathsf{U}(\varphi \to \mathsf{D}\psi) \to \mathsf{D}\psi$, or equivalently: $\mathsf{DD}\varphi \to \mathsf{D}\varphi$[8]
(d) Steinitz Exchange can be defined using one general formula variable plus nominals:

$$\mathsf{m} \wedge \mathsf{D}(\varphi \vee \mathsf{n}) \to (\mathsf{D}\varphi \vee \mathsf{E}(\mathsf{n} \wedge \mathsf{D}(\varphi \vee \mathsf{m})))$$

Notice that this principle does not hold for our earlier closure operation in groups.

Example 5.3 (Steinitz fails in commutative groups.)**.** Consider the additive group $(\mathbb{Z}\times\mathbb{Z}, +)$, where we have the pair $(5,3)$ in the set generated by $\{(1,1),(2,1)\}$. Clearly, $(5,3)$ is not in the set generated by $\{(1,1)\}$ alone. But also $(2,1)$ is not in the set generated by $\{(1,1),(5,3)\}$. To see this, write equations $2 = x + 5y$, $1 = x + 3y$ and try to solve for integer coefficients x, y (negative numbers record the use of inverses). Eliminating y, we obtain $1 = -2x$ which cannot be true in integers. Thus, for Steinitz we really need the presence of division.

Our final observation concerns an abstract perspective on linear dependence.

Remark 5.4. Like the earlier closure modality $\mathsf{C}\varphi$, the new $\mathsf{D}\varphi$ can be seen as a *neighborhood modality* interpreted over abstract relational models, in the format

$$\mathfrak{M}, s \models \mathsf{D}\varphi \quad \text{iff} \quad \exists X\, (NsX \;\&\; \forall x \in X : \mathfrak{M}, x \models \varphi)$$

Similar points apply to those made in Section 4. In particular, a frame correspondence analysis applies, and the preceding formula expressing Steinitz Exchange then enforces an abstract version of this property for the neighborhood relation.

5.2. **A modal logic of dependence for vectors.** A complete proof system for our modal logic of linear dependence should contain (a) all the principles of our earlier complete logic LCG for group models plus (b) suitable axioms and rules for the new dependence modality D adapting those found for our earlier system MCL. In particular, in vector spaces, the earlier smallest fixed-point analysis for subgroup closure simplifies to

$$\mu p.(\varphi \vee \mathfrak{m}p \vee p \oplus p)$$

Here we folded the disjuncts used in MCL for the zero-vector and additive inverses under a new modality $\mathfrak{m}$ for taking linear multiples by a scalar from the field of the vector space.

Let us focus on the new ingredient which comes to light here, as a simple way of bringing out the contribution of the field of the vector space in its own terms. We add the new modal operator to our language, describing the behavior of 'multiples' in vector models $\mathfrak{M}$ consisting of a vector space with a valuation for proposition letters:

[8]We omit the simple argument establishing this equivalence.

Definition 5.5. *The modality $\mathfrak{m}\varphi$ is true for a vector x in a model $\mathfrak{M}$ if $x = ay$ for some vector y with $\mathfrak{M}, y \models \varphi$, where a is an element of the field.*

What is the complete logic of this new modality for field multiples which describes collections of rays? We add one nominal for the zero vector plus the existential modality for ease in stating basic properties. One could also easily add the modalities for inverse and sum in tis setting, but we focus on the basics here.

Fact 5.6. *The following principles hold for $\mathfrak{m}$:*

(1) $\mathfrak{m}(\varphi \vee \psi) \leftrightarrow \mathfrak{m}\varphi \vee \mathfrak{m}\psi$
(2) $\varphi \to \mathfrak{m}\varphi$
(3) $\mathfrak{m}\mathfrak{m}\varphi \to \mathfrak{m}\varphi$
(4) $(\varphi \wedge \neg \mathsf{0} \wedge \mathfrak{m}(\neg \mathsf{0} \wedge \psi)) \to \mathfrak{m}(\psi \wedge \mathfrak{m}\varphi)$
(5) $\mathfrak{m}(\mathsf{0} \wedge \varphi) \to \varphi$
(6) $\mathsf{E}\varphi \to \mathsf{E}(\mathsf{0} \wedge \mathfrak{m}\varphi)$
(7) $\mathfrak{m}\varphi \to \mathsf{E}\varphi$

Proof. The pattern described here is that of a vector space as a collection of independent one-dimensional subspaces, whose intersection is just the zero vector. From an abstract modal perspective, the above list is close to the axioms for the standard normal logics $\mathsf{S5}$ or $\mathsf{KD45}$, in a language with one special nominal added describing a distinguished point. The first principle is a standard modal distribution law. The second and third principles express reflexivity and transitivity, while the fourth expresses symmetry among non-zero vectors in a one-dimensional subspace. Here is some detail for this case. Suppose that vector $x \neq 0$ satisfies φ, while also $x = ay$ for some field element a, where $y \neq 0$ satisfies ψ. We have $a \neq 0$ since x is not the zero vector. But then $y = a^{-1}x$ satisfies $\mathfrak{m}\varphi$, and the consequent follows. The fifth principle says that multiples of 0 are unique, and the sixth that the zero vector is a multiple of every point. Finally, the seventh principle is just the usual connection between modalities of special interest and the existential modality. □

Taking all this together as a modal logic and reading $\mathfrak{m}\varphi$ as an existential modality over a binary accessibility relation R and E as the true existential modality, it is easy to describe the binary relational frames for this logic by standard frame correspondences. They consist of a family of equivalence classes for R plus one unique point 0 outside of these which has every point accessible, but is only accessible from 0 itself. The number of disjoint equivalence classes can be large, being an abstract reflection of the dimension of a vector space.

Theorem 5.7. *The complete modal logic of $\mathfrak{m}\varphi$ is axiomatized by the above six principles.*

Proof. By standard arguments, [18, 23, 24], using the Sahlqvist form of all six principles, and taking a submodel for the relation R_{E} defined as in our completeness proof for LCG, any non-provable formula φ in the above axiom system can be falsified at either 0 or at some point in some equivalence class in the models just described. The valuation only has to assign truth values at points for the finitely many proposition letters occurring in φ, yielding a finite partition of 'state descriptions' in each equivalence class. Still without loss of generality, as in a standard completeness proof for $\mathsf{S5}$, we can even assume that each partition element occurs only once in the equivalence class. Moreover, we need only finitely many equivalence classes, by a familiar analysis of the 'branching width' needed to verify a modal formula. The resulting model $\mathfrak{M}$ can be easily visualized, as depicted in Figure 5.

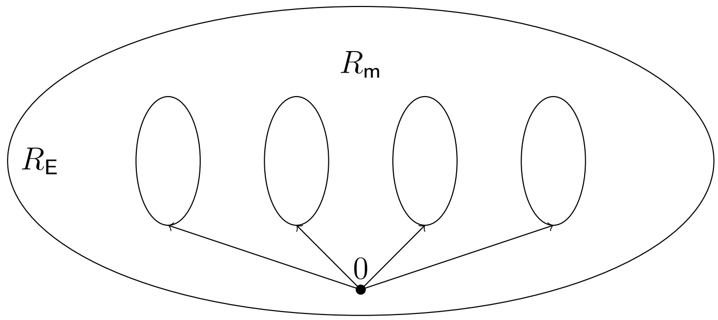

FIGURE 5. The finite cluster model $\mathfrak{M}$

Now it is straightforward to reproduce the model $\mathfrak{M}$ thus described with its valuation pattern up to bisimulation on, say, an n-dimensional vector space $\mathbb{R}^n$ with n equal to the number of equivalence classes. We choose some one-to-one match between n standard base vectors s and equivalence classes X, and make sure that each state description occurring in X occurs somewhere in the one-dimensional subspace (the 'ray') generated by s, while only state descriptions of this sort occur for vectors in that subspace. We make all proposition letters false in all other vectors in $\mathbb{R}^n$. This construction yields an obvious relation between points in $\mathfrak{M}$ and vectors in the n rays that we have used. It is straightforward to check that this relation is a bisimulation in the standard modal sense. Thus, our initial modal formula has also been falsified with the true 'multiple relation' in some vector model. $\square$

Remark 5.8 (A connection with a known modal logic)**.** The logic of $\mathsf{m}\varphi$ identified here in vector spaces resembles a well-known system in the modal literature. Disregarding the role of the nominal $\mathsf{0}$ and the existential modality E, its special case with one equivalence class (one single ray in a vector space), is related to the 'pin logic' of [27, 48], a largest non-tabular logic below $\mathsf{S5}$ which does have the Finite Model Property. If the root of the pin is irreflexive, one obtains the well-known logic $\mathsf{KD45}$, [18]. The more general logic of 'multiple pins' (the structures in our proof above but with irreflexive roots) has been studied in [5] under the name of $\mathsf{wKD45}$ ('weak $\mathsf{KD45}$"). Known issues and results then transfer to our modal logic of vector multiples. Conversely, our setting suggests that adding new vocabulary to the existing modal languages to bring out the special nature of the frames might pose some interesting new questions in the modal literature on pre-tabular logics.

An axiom system for the full language with the linear dependence modality D would combine this base logic for m with the principles used in the MCL-style axiomatization of subgroup closure. However, we may also need additional devices representing the true force of Steinitz Exchange, and we leave this completeness issue as an open problem here.

5.3. **From dependence to independence, the abstract way.** In Linear Algebra, *independence* of sets of vectors is arguably just as important as dependence, since it underlies the fundamental notions of basis and dimension. Independence can be seen as just the negation of the earlier dependence in the following sense: $\neg D_X y$ says that y cannot be written as a linear combination of vectors in Y.

However, a more common and useful related notion is that of an *independent set* of vectors, which we will view as follows in an abstract format.

Definition 5.9. *An* independence predicate I *satisfies the following conditions:*

(a) $I(X)$ *and* $Y \subseteq X$ *imply* $I(Y)$ *Downward Monotonicity*

(b) *if* $I(X)$ *and* $I(Y)$ *and* $|X| < |Y|$, *then there is some* $y \in Y - X$ *s.t.* $I(X \cup \{y\})$.

In the finite case, there is a unique maximal size of independent sets, the 'dimension'.

Note. In this subsection and the next, we consider abstract notions of independence and dependence on arbitrary underlying sets, not necessarily vector spaces.

Remark 5.10 (Matroid Theory)**.** Definition 5.9 states the key properties of a *matroid*, a well-known abstraction out of linearly independent sets in vector spaces, [58]. A matroid is a finite family $\mathbf{F}$ of finite sets (i) containing the empty set, (ii) closed under subsets, and satisfying an abstract analogue of Steinitz Exchange (iii) if A, B are in $\mathbf{F}$, and $|B| > |A|$, then there exists a $b \in B$ s.t. $A \cup \{b\} \in \mathbf{F}$. Matroids have both vector and graph interpretations, and it is a non-trivial issue just when a representation in vector spaces is possible, [62]. The restriction to the finite case has been lifted recently, and we will return to this later on.

Before turning to a modal perspective on independence, we note that the two abstract approaches: via dependence relations and independence predicates are equivalent. We state some relevant results, that can also be found in Matroid Theory, [22], with set-theoretic proofs that might be a bit more easily accessible to logicians placed in an Appendix.

Definition 5.11. *The* induced predicate I^D *of a dependence relation* D *holds precisely for all sets* X *s.t. for no* $x \in X, D_{X-\{x\}}\, x$.

Fact 5.12. *The* I^D *are independence predicates.*

There is also a converse to this result.

Definition 5.13. *The* induced relation D^I *of a predicate* I *is defined as follows:* $D^I_X y$ *iff either (a)* $y \in X$, *or (b) there exists some* $X' \subseteq X : I(X') \wedge \neg I(X' \cup \{y\}$.

Fact 5.14. *The induced relation* D^I *of an independence predicate is a dependence relation.*

Of course, we can also compose the two constructions I^D and D^I, but we will not continue at this level of generality in this paper. Abstract analyses like this can be seen as a sort of 'proto-logic': we now turn to a richer modal language which includes Boolean structure.

Remark 5.15 (Matroid theory revisited)**.** The above results heavily depend on the use of finite sets. However, recently, [22], a natural notion of infinite matroid has been introduced. It extends the conditions for the finite case with this second-order statement:

> Take any subset X of the total domain of objects. Any independent set I contained in X can be extended to a maximally independent set *among the subsets of X.*

This states well-foundedness of the inclusion order among independent sets satisfying some property encoded by the family X. Well-founded partial orders are well-known in modal logic $\mathsf{S4.Grz}$ [25, Ch. 3], and given suitable abstract models with vectors and sets of these, there might be an angle for modal analysis of infinite matroids as well as finite ones.

5.4. **A modal logic of independence.** In terms of logical syntax, the notion of independence in Linear Algebra is not a modal operator on formulas, but a *predicate* of formulas denoting sets of vectors. This predicate view, too, is of logical interest. It makes I a relative of global modalities like E, U which express simpler, permutation-invariant, properties of sets such as non-emptiness. We introduce the following new syntactic construction:[9]

$$\mathsf{I}\varphi : \varphi \text{ defines an independent set in the current model.}$$

This seems a new notion compared to what we had before. We believe that the earlier set-theoretic definition of independence predicates in terms of dependence relations does not have a modal counterpart, and conjecture that the above notion I is not definable in the modal language with the earlier D-modality. However, pursuing this would require a deeper bisimulation analysis whose details do not seem relevant to our main interest here. Even so, some of the above set-theoretic conditions have reflections in the modal language:

- The fact that I is a property of sets is reflected by $(\neg)\mathsf{I}\varphi \to U(\neg)\mathsf{I}\varphi$
- Downward monotonicity is expressed by $I\varphi \to \mathsf{I}(\varphi \wedge \psi)$
- In addition, we have In for all nominals n except $\mathsf{0}$.

It may be of interest to axiomatize the logic of I by itself, but perhaps the more interesting logical language also includes our modality for dependence.

Example 5.16. Here are two modally expressible directions of the above-mentioned transformations between the abstract dependence predicates D and I:

- $(\mathsf{I}\varphi \wedge \neg\mathsf{I}(\varphi \vee \mathsf{n})) \to \mathsf{E}(\mathsf{n} \wedge \mathsf{D}\varphi)$
- $(\mathsf{n} \wedge \mathsf{D}\varphi \wedge \mathsf{I}(\varphi \vee \mathsf{n})) \to \mathsf{E}(\mathsf{n} \wedge \varphi)$

The first implication says that, if adding an object n to the independent set X defined by φ results in a non-independent set, then the added n depends on the set X. The second implication says that, if the object n depends on the set X, and adding it leaves $X \cup \{n\}$ independent, this can only be because n was already a member of X.

We leave a complete axiomatization of the modal logic of I, D as an open problem.

5.5. **Excursion: Connections with logics of dependence and independence.** The modal logics of dependence and independence introduced here invite comparison with current dependence logics, [56, 7]. These logics work on sets of assignments (or more abstract states) where not all possible maps from variables to values in their ranges need to be available in the model. In this setting, functional dependence is 'determination': fixing the values of X fixes the value of y uniquely on the available assignments. This semantic determination matches functional definability: it holds iff there is a map F from values to values s.t. for all states s in the model, $s(y) = F(\mathbf{s}(\mathbf{X}))$. This, of course, seems close to our notion of dependence as definability in a vector space.[10] What, in particular, the recent dependence logic LFD shares with the present paper is a modal-style approach taking a local perspective (formulas are always interpreted at a given assignment), and yielding modal-style axiomatizations.

Even so, there are also striking differences in the notion of *independence* of y from X, whose definition in LFD says that fixing the local values of X at a state s still leaves y free to take on all its possible values in the available assignments of the model. This is much stronger than just the negation of local semantic dependence. What is going on?

[9]For another general logical analysis of notions of independence, we refer to [35].

[10]The additional requirement that the function be linear amounts to an even more demanding notion of semantic determination, which might be captured by imposing further structural invariance conditions.

We propose a distinction here. Dependence in an algebraic *value setting* means definability by some term in the similarity type of the algebra. Independence is then just the negation of this. This is how the notions are connected in our group and vector models. But LFD and other dependence logics *lift* this value setting to a *function setting*. Rephrasing our earlier formulation, the variables of LFD are really functions from 'states' into a value algebra, and semantic dependence of y on X is equivalent to saying that there exists a term T for a partial function in that algebra such that, for all states $s, y(s) = T[X(s)]$. In these richer function spaces, naturally, more complex notions become available, and in particular, independence now means something else from the value setting, namely, that the functions take their values independently. Even so, our modality $I\varphi$ might make sense in the setting of LFD as an alternative to the undecidable logic $\mathsf{LFD} + \mathsf{I}$, [7].

In all, however, the precise connections between our logics of dependence and independence and those developed in the function tradition remain to be clarified.

6. Fields and vector spaces

With our basic modal approach in place, and having made some forays into linear dependence, we now turn to the modal logic of vector spaces in the full sense of our introduction. Vector spaces include a field whose members can be seen as actions, in fact, linear transformations, on the additive group of vectors. Algebraic notations for these objects serve as scalars in linear terms defining linear transformations. In this section, we extend our earlier modal logic of commutative groups (now thought of as the additive group of vectors) to include this field structure. The idea behind the formal language to follow is like that for propositional dynamic logic PDL: formulas describe properties of states, but there are also terms for actions from the field that denote transition relations between states, in our case: functions from vectors to vectors. One technical issue in this approach concerns field inverses and avoiding division by zero, which requires a treatment of undefined terms. This complication does not arise with modules (vector spaces with a ring instead of a field), and all results in this section apply directly, in simplified form, to that broader class of structures.[11]

6.1. **Language and semantics.**

Definition 6.1. *The* terms *of the* modal language of vector spaces MVL *are given by the following schema, starting with some set of variables x, while 0, 1 are individual constants:*

$$t := x \mid 0 \mid 1 \mid t + t \mid -t \mid t.t \mid t^{-1}$$

This definition also includes the term 0^{-1} or $(x + -x)^{-1}$ which do not denote objects in fields, and as a result, our semantics must deal with terms lacking a denotation.

Next we define formulas extending the earlier modal language of groups. Here, we use the new non-standard notation $\mathsf{0}$ to denote the zero vector, in order to distinguish this from the zero element of the field of the vector space.

Definition 6.2. Formulas *are defined as follows, starting with proposition letters p and nominals i from given sets, and using t to denote an arbitrary term as defined just now:*

$$\varphi := p \mid \mathsf{i} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\varphi \mid \mathsf{0} \mid \langle - \rangle\varphi \mid \varphi \oplus \varphi \mid \langle t \rangle\varphi$$

[11]Our style of analysis also seems to apply to the algebraically specifiable 'meadows' of [15].

Clearly, this modal language extends the earlier one of the group logic LCG. The new modality $\langle t\rangle\varphi$ describes the crucial product of vectors and field scalars linking the two components of a vector space. Note also that the modal notation distinguishes some notions that mathematical notation for vector spaces collapses with a benign ambiguity: the zero of the field and the zero vector, product between field elements and between field elements and vectors. This prising apart of notions in our logical syntax is not just pedantry, since it affords us a finer look at the underlying reasoning principles.

Remark 6.3 (More redical modalization)**.** Our language does not treat vectors and field elements in exactly the same style. A more thorough modalization would have two sorts of objects: vectors and field elements, and instead of terms, it would have formulas that can be true or false of field elements. For an analogy, cf. the 'arrow logic' version of PDL in [9].

Now, we set up our semantic structures.

Definition 6.4. *A* vector model *over a field is a structure $\mathfrak{M} = (S, F, V, h)$ with S a commutative group as described in Section 2, F a field, and V a valuation for proposition letters on S. Next, the* assignment map *h sends basic variable terms to objects in the field F. This map extends uniquely to a partial map from the whole set of terms to objects in F, also denoted by h. Here the convention is that (a) complex terms with undefined components do not get a value, (b) if a term t has value 0, then t^{-1} does not get a value.*

As an illustration, assignments h are undefined on the terms 0^{-1}, $(x + (-x))^{-1}$.

The semantics for our language is the same as in earlier sections, with the following addition. Note here that terms t denote the same object throughout a vector model, while the modality $\langle t\rangle\varphi$ describes their effect as actions on vectors:

$$\begin{array}{l}(S, V, h), v \models \langle t\rangle\varphi \quad \text{iff} \quad \text{there exists a vector } w \text{ such that}\\ \text{(i) } (S, V, h), w \models \varphi, \text{ (ii) } h(t) \bullet w \text{ is defined and equals } v\end{array}$$

Remark 6.5 (Two directions)**.** This choice of semantics needs explanation. We treat $\langle t\rangle\varphi$ in the same 'reverse' style here as the earlier group addition modality $\varphi \oplus \psi$ of LCG. Instead, one might have expected the 'forward' formulation that $h(t) \bullet v \models \varphi$ for the current point v, in analogy with the semantics of program modalities in PDL – but we found that this hinders the statement of perspicuous valid principles. Even so, the two transition relations involved here are converses of each other. In line with this, modulo some care with undefined terms, it should also be possible to view the syntax of our modal logic as including a two-directional *tense logic*, since the modalities $\langle t\rangle$ and $\langle t^{-1}\rangle$ function largely as converses.

6.2. **Bisimulation and expressive power.** As in Section 2.3, the extended modal language MVL comes with a notion of bsisimulation, where more structure now needs to be preserved. Again, we could define this between relational models generalizing vector models, but the following case will suffice for an illustration.

Definition 6.6. *A* field bisimulation *between two vector models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ over possibly different sets of vectors and fields is a relation Z between vectors in the two models satisfying all the conditions of Definition 2.12 for bisimulations between group models plus the following extra item for each term t of the language and each Z-related pair v, w:*

- *If $v = h_1(t) \bullet v'$, then there exists a w' s.t. $v'\mathsf{Z}w'$ and $w = h_2(t) \bullet w'$*
- *Likewise in the opposite direction.*

Remark 6.7 (Operations that are safe for bisimulation)**.** In propositional dynamic logic PDL, one only needs a bisimulation between atomic actions, and this is then automatically a bisimulation for the transition relations of complex programs by the *safety* of the program operations, [18]. Is there a similar extension property here? Indeed, it is easy to see by spelling out the preceding definition that, if Z is a field bisimulation for terms t and t', then it is also a field bisimulation for the term $t \cdot t'$. But there are two obstacles. First, consider safety for inverse terms s^{-1}. Here reducing $v = s^{-1} \cdot w$ to $w = s \cdot v$ requires our bisimulation to also work in the converse direction. This might be solved by making field bisimulations two-sided, on the earlier-noted analogy of our modal vector logic with a tense-logical one. However, here is one more problem. If Z is a field bisimulation for terms t and t', it is not clear that Z is also a bisimulation for $t+t'$, since the assumption that $v = h_1(t+t') \bullet v'$ means that $v = h_1(t) \bullet v' + h_1(t') \bullet v'$ and the double occurrence of v' requires Z-links to the same point in the opposite model, which bisimulation typically does not guarantee. We feel that more can be said on the topic of safety, but must leave this for further investigation.

Fact 6.8. *Field-bisimilar points in two vector models satisfy the same* MVL*-formulas.*

Example 6.9. The bisimulation in Example 2.20 between $\mathbb{Q} \times \mathbb{Q}$ and $\mathbb{Q}$ extends immediately to a field bisimulation where the values $h(t)$ are the same in both models.

Example 6.10. Some bisimulations between group models as defined in Section 2 do not extend to field bisimulations in the preceding sense. Consider the vector space $\mathbb{R}$ over the field $\mathbb{R}$. Let $\mathbb{R}'$ be an isomorphic copy. The binary relation $v\mathsf{Z}u'$ defined by

- $v = 0$ *and* $u' = 0'$,
- $v \in \mathbb{Z} - \{0\}$ *and* $u' \in \mathbb{Z}' - \{0'\}$,
- $v \in \mathbb{R} - \mathbb{Z}$ *and* $u' \in \mathbb{R}' - \mathbb{Z}'$

is a bisimulation between the two structures $\mathbb{R}$ and $\mathbb{R}'$ viewed as group models. To see this, note first that the additive zero-element is mapped uniquely. Next, the clause for additive inverses is satisfied since the three areas $\{0\}, \mathbb{Z} - \{0\}, \mathbb{R} - \mathbb{Z}$ are closed under applying the operation $-x$. Finally, consider the back and forth clauses for addition.

Given the isomorphism between $\mathbb{R}$ and $\mathbb{R}'$, it suffices to analyze one direction. The following facts are easy to see: (a) Every $a \in \mathbb{R}$ can be written as a sum $a = u + v$ with $u, v \in \mathbb{R} - \mathbb{Z}$ – moreover, if $a \neq 0$, we can take $u, v \neq 0$, (b) If $a \in \mathbb{R} - \mathbb{Z}$, then $a = u + v$ for $u \in \mathbb{Z} - \{0\}$ and $v \in \mathbb{R} - \mathbb{Z}$ [take $u = 1$ and $v = a - 1$], and (c) If $a \in \mathbb{Z}$, then $a = u + v$ with $u, v \in \mathbb{Z} - \{0\}$ [take $a - n$, $a + n$ for some natural number n such that neither $a - n$ nor $a + n$ equals 0]. Now let $x \in \mathbb{R}$, $u' \in \mathbb{R}'$ and $x\mathsf{Z}u'$. Suppose that $x = y + z$. We distinguish a few cases. If $y = 0$, then we take $v' = 0'$ and $w' = u'$ for which $u' = v' + w'$, $y\mathsf{Z}v'$ and $z\mathsf{Z}w'$. If $y, z \in \mathbb{R} - \mathbb{Z}$, by (a) we find $v', w' \in \mathbb{R}' - \mathbb{Z}'$ with $u' = v' + w'$. So we have $y\mathsf{Z}v'$ and $z\mathsf{Z}w'$. If $y, z \in \mathbb{Z} - \{0\}$, then $x \in \mathbb{Z}$ and so $u' \in \mathbb{Z}'$. If $x = 0$, then $u' = 0'$ and we take $v' = y'$ and $w' = z'$. If $x \neq 0$, we apply (c). Finally, if $y \in \mathbb{Z} - \{0\}$ and $z \in \mathbb{R} - \mathbb{Z}$, then $x = y + z$ belongs to $\mathbb{R} - \mathbb{Z}$. Therefore, $u' \in \mathbb{R}' - \mathbb{Z}'$ and we apply (b).

However, the relation Z is not a field bisimulation with respect to the natural assignments where $h_1(t) = 2$ and $h_2(t) = 2'$ for some term t. For, $6 \ \mathsf{Z} \ 7'$ and $6 = 2 \cdot 3$, but there is no p' with $3 \ \mathsf{Z} \ p'$, where p' must be in $\mathbb{Z}' - \{0'\}$ by the definition of Z, such that $2' \cdot p' = 7'$.

We will not continue with model-theoretic aspects of vector models here. Our focus in the rest of this section will be on valid reasoning principles.

6.3. **Exploring the valid principles of modal vector logic.** Before stating any formal results, we will first discuss some crucial valid principles in the system MVL just defined. This first round of discussion will be in a semi-formal style, so as to motivate the later formal proof system where things will be made more precise.

For a start, since we have included the earlier system LCG for vector addition, everything we have seen about its deductive power remains available here.

Next, we consider *definedness of terms.* The formula

$$\mathsf{E}\langle s\rangle\top$$

holds at a point (in fact, at any point) if some vector x equals $s \bullet y$ for some vector y. Given our semantics, definedness of a term holds globally, so our formula expresses that the value of the term s is defined. But then we can express all our stipulations for definedness of terms as principles in our modal language. For instance, the formula

$$\mathsf{E}\langle s+t\rangle\top \leftrightarrow (\mathsf{E}\langle s\rangle\top \wedge \mathsf{E}\langle t\rangle\top)$$

says that sum terms are defined iff their summands are. The most complex case here was that of terms s^{-1} for multiplicative inverse. This time, s does not just need to be defined, but its value must also not be the additive zero-element of the field. To ensure this, we can use the following formula:

$$\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top)$$

This says that the value of the term s is defined, but also that some vector can be multiplied by s to produce a non-zero vector. This can only happen if the value of s differs from the additive zero of the field, given the definition of vector spaces.

Next, consider the field structure. In our style of semantics, the structure of field elements is represented indirectly through the structure of the associated linear transformations on vectors.[12] Accordingly, we must first discuss *linear transformations* $s \bullet x$. First, we can make sure that the linear transformation associated with a field element s is a *function*

$$(\langle t\rangle\mathsf{i} \wedge \varphi) \to \mathsf{U}(\langle t\rangle\mathsf{i} \to \varphi)$$

Next, we consider the two basic distributive laws of vector spaces that regulate the modality $\langle s\rangle$. First, the algebraic identity $a \bullet (x+y) = a \bullet x + a \bullet y$ underlies the following fact:

Lemma 6.11. *The formula* $\langle t\rangle(\varphi \oplus \psi) \leftrightarrow (\langle t\rangle\varphi \oplus \langle t\rangle\psi)$ *is valid on every vector space.*

Proof. From left to right, if $v = h(t) \bullet w$ with $w \models \varphi\oplus\psi$, then $w = x+y$ for some $y \models \varphi, z \models \psi$. So, $v = h(t) \bullet x + h(t) \bullet y$ and the right hand side follows. From right to left, if $v = x+y$ with $x \models \langle t\rangle\varphi$ and $y \models \langle t\rangle\psi$ then $x = a \bullet x'$ for some $x' \models \varphi$, and $y = a \bullet y'$ for some $y' \models \psi$, and using the algebraic equality once more, $v = a \bullet (x' + y')$ and the left-hand side follows. □

Next consider the second distributive law $(a + b) \bullet x = a \bullet x + b \bullet x$. The modal formula $\langle s+t\rangle\varphi \to (\langle s\rangle\varphi \oplus \langle t\rangle\varphi)$ is valid. If $v = (s+t) \bullet w$ with $w \models \varphi$, then $v = s \bullet w + t \bullet w$, which implies the right-hand side. But the converse $(\langle s\rangle\varphi \oplus \langle t\rangle\varphi) \to \langle s+t\rangle\varphi$ is not valid. For a counter-example, take the rationals, set $a = 0.5, b = 1$ and let φ be true just at $1, 2$. The

[12] Readers familiar with propositional dynamic logic may find the analogy helpful with the way program expressions denote binary transition relations between states.

vectors 0.5 and 2 satisfy $\langle a\rangle\varphi$ and $\langle b\rangle\varphi$, respectively, but their sum 2.5 is not a 1.5 multiple of any point satisfying φ.

However, it is easy to see the following fact, replacing the earlier formula φ which can hold at more than one point by a nominal denoting a single point, thereby staying closer to the distribution law for vector spaces as usually stated:

$$\langle s+t\rangle\mathsf{i} \leftrightarrow (\langle s\rangle\mathsf{i} \oplus \langle t\rangle\mathsf{i}) \text{ is valid}$$

In our later axiom system, we will use this principle strengthened by a rule of substitution for 'point formulas' denoting unique vectors, which are our modal analogue of algebraic terms.

What remains is to see which valid principles express the definition of a *field*. For the *additive structure* of the field, to some extent, the earlier distribution laws help us enlist the additive structure of the vectors. Here are some illustrations.

Associativity for nominals can be demonstrated as follows. The formula $\langle s+(t+u)\rangle\mathsf{i}$ is equivalent, using the last distribution principle mentioned above twice, to $\langle s\rangle\mathsf{i} \oplus (\langle t\rangle\mathsf{i} \oplus \langle u\rangle\mathsf{i})$ and now we can use the associativity of $\oplus$ in LCG (and hence DVL) and then use the distribution equivalence backwards. *Commutativity* can be derived in a similar manner. As we shall see later, our informal discussion here will be reflected in formal derivations in our proof system, where we can even replace the nominal i by arbitrary formulas φ.

This leaves the zero element of the field, for which we have this valid equivalence:

$$\langle 0\rangle\varphi \leftrightarrow (\mathsf{0} \wedge \mathsf{E}\varphi)$$

With this in place, the modal form of the valid identity $t+(-t)=0$ will follow directly from the properties of vector inverse by the following clearly valid formula

$$\langle -t\rangle\varphi \leftrightarrow \langle -\rangle\langle t\rangle\varphi$$

It remains to deal with the *multiplicative structure* of the field. Some properties of the product operation are again immediate. In particular, *associativity* follows merely though the syntactic properties of our modal language. We have these provable equivalences:

$$\langle s\cdot(t\cdot u)\rangle\varphi \leftrightarrow \langle s\rangle\langle t\cdot u\rangle\varphi \leftrightarrow \langle s\rangle\langle t\rangle\langle u\rangle\varphi \text{ which recombines to } \langle s\cdot t\rangle\langle u\rangle\varphi$$

In addition, the following modal principle is obviously valid for the unit element:

$$\langle 1\rangle\varphi \leftrightarrow \varphi.$$

However, in general, algebraic properties of the field product cannot be read off from the additive structure of the vectors. For instance, there is no easy derivation for *commutativity*. We will just postulate the following valid equivalences as principles in our system:

$$\langle s\rangle\langle t\rangle\varphi \leftrightarrow \langle t\rangle\langle s\rangle\varphi^{13}$$

The hardest task remaining arises with division in a field, i.e., the inverse of the product operation. We need a modal equivalent for the algebraic equality $t\,.\,t^{-1}=1$. Here a first observation is that we can partially define the modality for term inverse in line with our truth definition. Precise details of the relevant principles are found in our proof system defined

[13] Commutativity may be a momentous feature affecting the complexity of our system, since multimodal logics with the universal modality plus commutation axioms are usually undecidable, [49].

later, but here is one useful validity, where the second conjunct to the right is the above definition for the term s being defined and having a non-zero value:

$$(\mathsf{i} \wedge \langle s^{-1} \rangle \mathsf{j}) \leftrightarrow (\mathsf{E}(\mathsf{j} \wedge \langle s \rangle \mathsf{i}) \wedge \mathsf{E}(\neg \mathsf{0} \wedge \langle s \rangle \top)).$$

For the law of inverses, using the preceding principle or by direct inspection, we then get

$$\mathsf{E}(\neg \mathsf{0} \wedge \langle s \rangle \top) \to (\langle s \rangle \langle s^{-1} \rangle \varphi \leftrightarrow \varphi).[14]$$

Remark 6.12. The preceding observations can be summed up in the following modal frame correspondence result. Interpreted in terms of their truth on general relational frames, the preceding principles express exactly that an abstract bimodal model with a product map is a vector space. We do not provide details for this result, since the proof of our completeness theorem to follow is more informative about the content of our modal system.

6.4. **A complete proof system.** With the preceding preliminary tour of principles in place, we can now introduce a formal proof system. But first, we define a useful auxiliary notion of *point formulas* as follows:

(i) nominals are point formulas,
(ii) if P is a point formula, then so is $\langle s \rangle P$, for every term s.

In our semantics, when defined, point formulas clearly just denote one vector, so they function in our modal language as counterparts to algebraic terms over the field.

Definition 6.13. *The proof calculus of* dynamic vector logic DVL *consists of*

- *All axioms and rules of the proof system* LCG *from Section 3, now for the present extended language.*
- *The following additional modules:*
  (a) *Axioms for definedness of terms:*
    (a1) $\mathsf{E}\langle s+t \rangle \top \leftrightarrow \mathsf{E}\langle s \rangle \top \wedge \mathsf{E}\langle t \rangle \top$
    (a2) $\mathsf{E}\langle s \cdot t \rangle \top \leftrightarrow \mathsf{E}\langle s \rangle \top \wedge \mathsf{E}\langle t \rangle \top$
    (a3) $\mathsf{E}\langle -s \rangle \top \leftrightarrow \mathsf{E}\langle s \rangle \top$
    (a4) $\mathsf{E}\langle s^{-1} \rangle \top \leftrightarrow \mathsf{E}(\neg \mathsf{0} \wedge \langle s \rangle \top)$.
  (b) *Axioms for scalar-vector product:*
    (b1) $\langle s \rangle (\varphi \vee \psi) \leftrightarrow \langle s \rangle \varphi \vee \langle s \rangle \psi$
    (b2) $\langle s \rangle \varphi \to \mathsf{E}\varphi$
    (b3) $\mathsf{E}(\neg \mathsf{0} \wedge \langle s \rangle \top) \to (\langle s \rangle \neg \varphi \leftrightarrow (\langle s \rangle \top \wedge \neg \langle s \rangle \varphi))$
    (b4) $\langle s \cdot t \rangle \varphi \leftrightarrow \langle s \rangle \langle t \rangle \varphi$
    (b5) $\langle t \rangle (\varphi \oplus \psi) \leftrightarrow (\langle t \rangle \varphi \oplus \langle t \rangle \psi)$
    (b6) $\langle s+t \rangle \mathsf{i} \leftrightarrow \langle s \rangle \mathsf{i} \oplus \langle t \rangle \mathsf{i}$
  (c) *Further laws for field addition and multiplication:*
    (c1) $\langle 0 \rangle \varphi \leftrightarrow (\mathsf{0} \wedge \mathsf{E}\varphi)$

[14] The left to right direction in the final equivalence does not need the antecedent condition.

(c2) $\langle 1\rangle\varphi \leftrightarrow \varphi$
(c3) $\langle -s\rangle\varphi \leftrightarrow \langle -\rangle\langle s\rangle\varphi$
(c4) $\langle s\cdot t\rangle\varphi \leftrightarrow \langle t\cdot s\rangle\varphi$
(c5) $(\mathsf{i} \wedge \langle s^{-1}\rangle \mathsf{j}) \leftrightarrow (\mathsf{E}(\neg \mathsf{0} \wedge \langle s\rangle\top) \wedge \mathsf{E}(\mathsf{j} \wedge \langle s\rangle \mathsf{i}))$
(c6) $(\mathsf{E}(\neg \mathsf{0} \wedge \langle s\rangle\top) \wedge \mathsf{i}) \to \langle s^{-1}\rangle\langle s\rangle \mathsf{i}$

- *The additional rules of inference for* DVL *over* LCG *are as follows.*

  Necessitation Rules: $\frac{\neg\varphi}{\neg\langle s\rangle\varphi}$ *for each term s*

  Extra Witness Rule: $\frac{\langle s\rangle(\mathsf{j}\wedge\varphi)\to\theta}{\langle s\rangle\varphi\to\theta}$, *where the nominal* j *does not occur in* φ *or* θ.

  Substitution Rule: Nominals in provable formulas can be replaced by point formulas.

The reader may have missed some obvious counterparts of field axioms in the preceding list, but these turn out to be derivable in the system as presented here. Before giving examples and details, we state two facts that will be used throughout.

Lemma 6.14.
- $\vdash \mathsf{U}(\varphi \leftrightarrow \psi) \to \mathsf{U}(\chi \leftrightarrow \chi[\varphi/\psi])$.
- *If* $\vdash \langle s\rangle \mathsf{i} \leftrightarrow \langle t\rangle \mathsf{i}$, *then* $\vdash \langle s\rangle\varphi \leftrightarrow \langle t\rangle\varphi$

Proof. The first fact is standard for modal logics with global modalities. The second fact may be seen as follows. Let j be a nominal not occurring in φ. By the substitution rule, we have $\vdash \langle s\rangle \mathsf{j} \leftrightarrow \langle t\rangle \mathsf{j}$. Therefore, $\langle s\rangle(\mathsf{j}\wedge\varphi)$ implies $\langle t\rangle \mathsf{j}$. We also have the hybrid law $\mathsf{E}(\mathsf{j}\wedge\varphi) \to \mathsf{U}(\mathsf{j}\to\varphi)$. Therefore we get $\langle t\rangle\varphi$. Now apply the Witness Rule for modalities $\langle s\rangle$ to get the desired conclusion. □

Example 6.15. Here are some derivable formulas in the logic DVL:
(1) $\langle (s+t)+u\rangle\varphi \leftrightarrow \langle s+(t+u)\rangle\varphi$
(2) $\langle s+t\rangle\varphi \leftrightarrow \langle t+s\rangle\varphi$
(3) $\langle s+0\rangle\varphi \leftrightarrow \langle s\rangle\varphi$
(4) $\langle s+(-s)\rangle\varphi \leftrightarrow \langle 0\rangle\varphi$
(5) $\langle (s\cdot t)\cdot u\rangle\varphi \leftrightarrow \langle s\cdot(t\cdot u)\rangle\varphi$
(6) $\langle s\cdot 1\rangle\varphi \leftrightarrow \langle s\rangle\varphi$
(7) $\langle s\cdot(t+u)\rangle\varphi \leftrightarrow \langle s\cdot t + s\cdot u\rangle\varphi$
(8) $\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \to (\langle s^{-1}\cdot s\rangle\varphi \leftrightarrow \langle 1\rangle\varphi)$.

For all equivalences here, by Lemma 6.14, it suffices to show the case where φ is a nominal i. (i) E.g, for principle (2), $\langle s+t\rangle \mathsf{i}$ is provably equivalent to $\langle s\rangle \mathsf{i} \oplus \langle t\rangle \mathsf{i}$ by Axiom (b6), which is equivalent to $\langle t\rangle \mathsf{i} \oplus \langle s\rangle \mathsf{i}$ by the commutativity of $\oplus$ in the proof system LCG, and using Axiom (b6) once more, we derive $\langle t+s\rangle \mathsf{i}$. (ii) $\langle (s\cdot t)\cdot u\rangle \mathsf{i} \leftrightarrow \langle s\cdot(t\cdot u)\rangle \mathsf{i}$ is proved by applying Axiom (b4) a number of times and recombining in an obvious way. (iii) $\langle s+(-s)\rangle \mathsf{i} \leftrightarrow \langle 0\rangle \mathsf{i}$ is proved as follows. Using Axiom (c1) we can replace $\langle 0\rangle \mathsf{i}$ by $\mathsf{0} \wedge \mathsf{E}\mathsf{i}$, where the right-hand conjunct is provable, whence we get the provable equivalent $\mathsf{0}$. The left-hand side $\langle s+(-s)\rangle \mathsf{i}$ is provably equivalent to $\langle s\rangle \mathsf{i} \oplus \langle -s\rangle \mathsf{i}$ and so, by Axiom (c3), to $\langle s\rangle \mathsf{i} \oplus \langle -\rangle\langle s\rangle \mathsf{i}$ which is provably equivalent to $\mathsf{0}$ already in the initial proof system LCG. (iv) Principle

(8) simplifies up to provable equivalence to $\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \to (\langle s^{-1}\rangle\langle s\rangle\varphi \leftrightarrow \varphi)$, and by a reasoning similar to that establishing Lemma 6.14.(ii) it suffices to derive the nominal form $\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \to (\langle s^{-1}\rangle\langle s\rangle\mathsf{i} \leftrightarrow \mathsf{i})$. From left to right, we assume $\mathsf{j} \wedge \langle s^{-1}\rangle\langle s\rangle\mathsf{i}$, and use Axiom (c5), substituting the point formula $\langle s\rangle\mathsf{i}$ for i, to derive $\mathsf{j} \wedge \mathsf{E}(\langle s\rangle\mathsf{i} \wedge \langle s\rangle\mathsf{j})$, which implies in hybrid logic that $\mathsf{E}(\mathsf{i} \wedge \mathsf{j})$, which together with j implies i. Then we drop the assumption j by the Naming Rule. For the direction from right to left, we appeal to Axiom (c6).[15]

Remark 6.16. In fact a more general statement is true which can be proved by induction on terms. Let $s = t$ be any identity of terms that is derivable from our stated axioms by principles of equational logic. Then for each φ, the formula $\langle s\rangle\varphi \leftrightarrow \langle t\rangle\varphi$ is derivable in DVL.

Incidentally, we have not aimed for a minimal set of principles for inverse and other notions in the proof system DVL, and simplifications may be possible.

Theorem 6.17. *The proof calculus* DVL *is sound and complete for validity in vector models.*

Proof. Soundness is obvious from the earlier semantic explanation of the axioms.

The proof of completeness follows the pattern used for Theorem 3.8 about the logic LCG of commutative groups, so we suppress details of the proof that work analogously.

We start with some consistent set Σ and create a family of maximally consistent sets, forming one equivalence class of the modal accessibility relation for the global modalities, that are each named by a nominal, and where additional nominals witness all the modalities employed in our system.. In this setting, we then select a designated *guidebook*: one maximally consistent set $\Sigma^\bullet$ containing Σ which contains all the information that is needed to create a vector model as desired.

As before, we first use the guidebook to define a commutative additive group on the maximally consistent sets, viewed as standing for vectors. Next, we need to define a field.

We first define an equivalence relation $s \sim t$ on terms of our language by setting

$$s \sim t \quad \text{iff} \quad \mathsf{U}(\langle s\rangle P \leftrightarrow \langle t\rangle P) \in \Sigma^\bullet \text{ for every point formula } P$$

Recall that point formulas were introduced at the beginning of this section as a sort of modal formula counterparts to algebraic terms for specific objects. They were then used in the definition of our proof system DVL.

The elements of our field will be the equivalence classes $[t]^\sim$ of this relation for *defined terms* t in the sense of having the formula $\mathsf{E}\langle t\rangle\top$ present in $\Sigma^\bullet$. To develop this further, we first state a useful auxiliary fact.

Lemma 6.18. *Let s and t be terms such that $s \sim t$. Then*

(1) *s, t are both defined or undefined according to the guidebook. That is,*
$$\mathsf{E}\langle s\rangle\top \in \Sigma^\bullet \;\textit{iff}\; \mathsf{E}\langle t\rangle\top \in \Sigma^\bullet.$$
(2) *s, t are both defined and unequal to zero according to the guidebook. That is,*
$$\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \in \Sigma^\bullet \;\textit{iff}\; \mathsf{E}(\neg\mathsf{0} \wedge \langle t\rangle\top) \in \Sigma^\bullet.$$
(3) $\mathsf{U}(\mathsf{E}\langle s\rangle\top \leftrightarrow \mathsf{E}\langle t\rangle\top) \in \Sigma^\bullet$ *and* $\mathsf{U}(\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \leftrightarrow \mathsf{E}(\neg\mathsf{0} \wedge \langle t\rangle\top)) \in \Sigma^\bullet$

[15] The reader may also want to prove $\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \to \mathsf{U}(\langle s^{-1}\rangle\top \wedge \langle s\rangle\top)$.

Proof. (1) Suppose that $\mathsf{E}\langle s\rangle\top \in \Sigma^\bullet$. Then by our witness construction, some maximally consistent set Γ exists named by a unique nominal j s.t. $\langle s\rangle\top \in \Gamma$, and this again implies the existence of a maximally consistent set Δ named by some k such that $\mathsf{E}(\mathsf{j} \wedge \langle s\rangle\mathsf{k}) \in \Sigma^\bullet$. Now by the definition of $s \sim t$, $\mathsf{U}(\langle s\rangle\mathsf{k} \leftrightarrow \langle t\rangle\mathsf{k}) \in \Sigma^\bullet$. But then by the properties of derivability in DVL, $\mathsf{E}(\mathsf{j} \wedge \langle t\rangle\mathsf{k}) \in \Sigma^\bullet$, which in its turn implies $\mathsf{E}\langle t\rangle\top \in \Sigma^\bullet$. (2) is proved analogously.

Finally, (3) is a stronger statement implying the first two. It can be proved by showing that the argument for Cases (1), (2) will work for any maximally consistent set Γ, using the fact that its behavior lies encoded precisely in the guidebook $\Sigma^\bullet$. □

Next, we must define the field operations on these equivalence classes. This can be done by setting $[s]^\sim + [t]^\sim$ equal to $[s+t]^\sim$, and so on for all other functions in the signature. However, to see that these stipulations are well-defined, we must show several facts:

Lemma 6.19. *For all terms s, s', t, t' we have:*

(1) *If $s \sim s'$ and $t \sim t'$, then $s + t \sim s' + t'$,*
(2) *If $s \sim s'$ and $t \sim t'$, then $s \cdot t \sim s' \cdot t'$,*
(3) *If $s \sim t$, then $-s \sim -t$,*
(4) *If $s \sim t$, then $s^{-1} \sim t^{-1}$.*

Proof. We will rely heavily on theorems in DVL which either belong automatically to $\Sigma^\bullet$, or justify closure properties of the guidebook.

For (1), let $s \sim s'$, $t \sim t'$. Then we have $\mathsf{U}(\langle s\rangle P \leftrightarrow \langle s'\rangle P) \in \Sigma^\bullet$ and $\mathsf{U}(\langle t\rangle P \leftrightarrow \langle t'\rangle P) \in \Sigma^\bullet$, for every point formula P. But the formula $\mathsf{U}(\langle s\rangle P \leftrightarrow \langle s'\rangle P) \wedge \mathsf{U}(\langle t\rangle P \leftrightarrow \langle t'\rangle P)) \to \mathsf{U}(\langle s+t\rangle P \leftrightarrow \langle s'+t'\rangle P$ is a theorem of DVL. Therefore, $\mathsf{U}(\langle s+t\rangle P \leftrightarrow \langle s'+t'\rangle P) \in \Sigma^\bullet$, and so $s+t \sim s'+t'$.

For (2), let $s \sim s'$, $t \sim t'$. Then we have $\mathsf{U}(\langle s\rangle P \leftrightarrow \langle s'\rangle P) \in \Sigma^\bullet$ and $\mathsf{U}(\langle t\rangle P \leftrightarrow \langle t'\rangle P) \in \Sigma^\bullet$, for every point formula P. We also have $\mathsf{U}(\langle s\cdot t\rangle P \leftrightarrow \langle s\rangle\langle t\rangle P) \in \Sigma^\bullet$. Then $\mathsf{U}(\langle s\rangle\langle t\rangle P \leftrightarrow \langle s\rangle\langle t'\rangle P) \in \Sigma^\bullet$ and thus $\mathsf{U}(\langle s\rangle\langle t\rangle P \leftrightarrow \langle s'\rangle\langle t'\rangle P) \in \Sigma^\bullet$.[16] So $\mathsf{U}(\langle s\cdot t\rangle P \leftrightarrow \langle s'\rangle\langle t\rangle P) \in \Sigma^\bullet$, and hence $s \cdot t \sim s' \cdot t'$.

For (3), let $s \sim t$. Then $\mathsf{U}(\langle s\rangle P \leftrightarrow \langle t\rangle P) \in \Sigma^\bullet$, and hence also $\mathsf{U}(\langle -\rangle\langle s\rangle P \leftrightarrow \langle -\rangle\langle t\rangle P) \in \Sigma^\bullet$. But we also have $\mathsf{U}(\langle -s\rangle P \leftrightarrow \langle -\rangle\langle s\rangle P) \in \Sigma^\bullet$, $\mathsf{U}(\langle -t\rangle P \leftrightarrow \langle -\rangle\langle t\rangle P) \in \Sigma^\bullet$ as instances of theorems. Therefore, $\mathsf{U}(\langle -s\rangle P \leftrightarrow \langle -t\rangle P) \in \Sigma^\bullet$ and $-s \sim -t$.

For (4), let $s \sim t$. We have to show that $\mathsf{U}(\langle s^{-1}\rangle P \leftrightarrow \langle t^{-1}\rangle P) \in \Sigma^\bullet$. Axiom (c5) plus the Substitution Rule for the point term P and Necessitation for U gives us the theorem $\mathsf{U}((\mathsf{i} \wedge \langle s^{-1}\rangle P) \leftrightarrow (\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \wedge \mathsf{E}(P \wedge \langle s\rangle\mathsf{i}))$, which will then be in $\Sigma^\bullet$. Now we have $\mathsf{U}(\langle s\rangle P \leftrightarrow \langle t\rangle P) \in \Sigma^\bullet$, and therefore also the formula $\mathsf{U}((\mathsf{i} \wedge \langle s^{-1}\rangle P) \leftrightarrow (\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \wedge \mathsf{E}(P \wedge \langle t\rangle\mathsf{i}))$. Moreover, by Lemma 6.18(3), we have $\mathsf{U}(\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top \leftrightarrow \mathsf{E}(\neg\mathsf{0} \wedge \langle t\rangle\top) \in \Sigma^\bullet$, and we immediately also get $\mathsf{U}((\mathsf{i} \wedge \langle s^{-1}\rangle P) \leftrightarrow (\mathsf{i} \wedge \langle t^{-1}\rangle P)) \in \Sigma^\bullet$. It remains to remove the nominal i. We show this in one direction only [the other is symmetric] by appealing to the following closure property:.

Claim 6.20. *For each formula φ, if $\mathsf{U}(\mathsf{i} \to \varphi) \in \Sigma^\bullet$ for all nominals i that do not occur in φ, then $\mathsf{U}\varphi \in \Sigma^\bullet$.*

Proof. We can assume, thanks to the Witness Rules in DVL that, if $\mathsf{E}\varphi \in \Sigma^\bullet$, then there is a maximally consistent set Δ containing φ which is named by some nominal k that does not

[16]Here we make crucial use of the fact that, if P is a point formula, then so is $\langle t'\rangle P$.

occur in φ. In other worlds, $\mathsf{E}(\mathsf{k} \wedge \varphi) \in \Sigma^\bullet$. Now we show that $\mathsf{U}\varphi \in \Sigma^\bullet$. Let $\mathsf{U}\varphi \notin \Sigma^\bullet$. Then by maximal consistency, $\mathsf{E}\neg\varphi \in \Sigma^\bullet$. But the preceding observation about witnessing nominals in the set $\Sigma^\bullet$ then gives a formula $\mathsf{E}(\mathsf{k} \wedge \neg\varphi) \in \Sigma^\bullet$ with k not occurring in $\neg\varphi$, and we arrive at a contradiction. □

This completes the proof of Lemma 6.19. □

Given the preceding well-defined operations $+, -, \cdot$ on equivalences classes of terms, our next task is to show that these classes form a field.

Lemma 6.21. *The set* $\mathbb{F} = \{[t]^\sim : \mathsf{E}\langle t\rangle\top \in \Sigma^\bullet\}$ *forms a field.*

Proof. We must check that all axioms of a field hold according to the content of the guidebook $\Sigma^\bullet$. In principle, this is taken care of by either explicit axioms in DVL or derivable theorems in the list of Example 6.14. We only display two examples. (a) For associativity, consider the set $([s]^\sim + [t]^\sim) + [u]^\sim$. By the above definition, this equals $[(s+t)+u]^\sim$. We need to show that this set coincides with $[s+(t+u)]^\sim$. That is, we have to show that $(s+t)+u \sim s+(t+u)$. This means that $\mathsf{U}(\langle(s+t)+u\rangle P \leftrightarrow \langle s+(t+u)\rangle P) \in \Sigma^\bullet$, for every point formula P. But this follows immediately from the derivable associativity theorem about field addition listed as Example 6.14.(1). (b) Likewise, to verify that $[s]^\sim \neq 0$ implies $[s]^\sim \cdot ([s]^\sim)^{-1} = [1]^\sim$, we use Axiom (c6) and item (8) in the list of derivabilities in Example 6.15. □

Given this definition of the field, the *assignment function* h of our model is now straightforward: for all terms t with $\mathsf{E}\langle t\rangle\top \in \Sigma^\bullet$, we set $h(t) = [t]^\sim$.

Lemma 6.22. *The assignment function* h *is well-defined.*

Proof. We need to show that h is defined in the sense of Definition 6.4 for every term t such that $\mathsf{E}\langle t\rangle\top \in \Sigma^\bullet$. This easily done by induction on terms starting with basic variables by using the axioms of DVL that govern definedness of terms. □

Our next task is to define the *scalar-vector product* of field elements and vectors in the sense of our syntactic model. This requires a little twist given our semantics for the modality $\langle t\rangle\varphi$. Let us assume that the set Δ is named by the nominal i. Then we set:

$$[t]^\sim \bullet \Delta = \{\alpha \mid \mathsf{E}(\langle t\rangle\mathsf{i} \wedge \alpha) \in \Sigma^\bullet\}$$

Given our Functionality axiom for the field modalities $\langle t\rangle$, it is easy to see that this stipulation defines a unique maximally consistent set.

The product thus defined satisfies the distribution axioms for vector spaces.

Lemma 6.23. *The set of all named maximally consistent sets* Δ *such that* $\mathsf{E}\langle t\rangle\mathsf{i} \in \Sigma^\bullet$*, where* i *is the nominal naming* Δ *forms a vector space over* $\mathbb{F}$*.*

Proof. Again we rely on the proof principles available in the calculus DVL. For an illustration, we spell out the case of $[t]^\sim \bullet (\Delta + \Gamma) = [t]^\sim \bullet \Delta + [t]^\sim\Gamma$. The argument that follows here looks complex, but it consists essentially in just spelling out definitions.

Suppose that the name of Δ is j and the name of Γ is k. As in the completeness proof for LCG, there is a unique maximally consistent set forming the vector sum $\mathsf{j} \oplus \mathsf{k}$, named by m say, where we have $\mathsf{E}(\mathsf{m} \wedge \mathsf{j} \oplus \mathsf{k})$ in the guidebook $\Sigma^\bullet$. Now by the definition of scalar-vector product, the function value on the left-hand side is $\{\alpha \mid \mathsf{E}(\langle t\rangle\mathsf{m} \wedge \alpha) \in \Sigma^\bullet\}$. Using some basic hybrid principles for nominals and global modalities, we can also view this set of formulas as $\{\alpha \mid \mathsf{E}(\langle t\rangle(\mathsf{j} \oplus \mathsf{k}) \wedge \alpha) \in \Sigma^\bullet\}$.

Next consider the right-hand side of our equality. Let n_1 be the name for the maximally consistent set $[t]^\sim \bullet \Delta$, and n_2 for $[t]^\sim\Gamma$. Again by the presence of unique sums of vectors in our model, there is a maximally consistent set m' s.t. the following three formulas are in the guidebook $\Sigma^\bullet$: $\mathsf{E}(\mathsf{m}' \wedge \mathsf{n}_1 \oplus \mathsf{n}_2, \mathsf{E}(\mathsf{n}_1 \wedge \langle t\rangle\mathsf{j}), \mathsf{E}(\mathsf{n}_2 \wedge \langle t\rangle\mathsf{k})$. With some obvious manipulations in the logic, we can view the set associated with m' as $\{\alpha \mid \mathsf{E}(\langle t\rangle\mathsf{j} \oplus \langle t\rangle\mathsf{k} \wedge \alpha) \in \Sigma^\bullet\}$.

With this description, the only difference is the presence of the formula $\langle t\rangle(\mathsf{j} \oplus \mathsf{k})$ on the left, and $\langle t\rangle\mathsf{j} \oplus \langle t\rangle\mathsf{k}$ on the right. But these are provably equivalent by Axiom (b5).

The case of the other distribution law is similar. $\square$

It remains to prove a *Truth Lemma* for DVL in the same format as the one for the system LCG in Section 3, now with respect to the vector model $\mathfrak{M}$ defined by combining all the above definitions for its various components.

Lemma 6.24 (Truth lemma). *Let Δ be a maximally consistent set named by nominal* i. *Then*

$$\mathfrak{M}, \Delta \models \varphi \textit{ iff } \mathsf{E}(\mathsf{i} \wedge \varphi) \in \Sigma^\bullet \quad \textit{for all formulas } \varphi$$

Proof. As before, a straightforward induction on formulas φ is all that is required.

We will only demonstrate the case $\varphi = \langle t\rangle\psi$. Let $\mathfrak{M}, \Delta \models \langle t\rangle\psi$. Then $[t]^\sim$ is defined and there is a maximally consistent set Γ named by some nominal j such that $[t]^\sim \bullet \Gamma = \Delta$ and $\mathfrak{M}, \Gamma \models \psi$. By the inductive hypothesis we have that $\mathsf{E}(\mathsf{j} \wedge \psi) \in \Sigma^\bullet$. The definition of the scalar product $\bullet$ then yields $\mathsf{E}(\mathsf{i} \wedge \langle t\rangle\mathsf{j}) \in \Sigma^\bullet$. Using the principles for nominals and the existential modality in the proof system DVL, we get that $\mathsf{E}(\mathsf{i} \wedge \langle t\rangle\psi) \in \Sigma^\bullet$.

Conversely, suppose $\mathsf{E}(\mathsf{i} \wedge \langle t\rangle\psi) \in \Sigma^\bullet$. Then by the Witnessing properties of our model (analogous to the Witnessing properties used in the proof of Theorem 3.8), there is a nominal j such that $\mathsf{E}(\mathsf{i} \wedge \langle t\rangle\mathsf{j}), \mathsf{E}(\mathsf{j} \wedge \psi) \in \Sigma^\bullet$. Let Γ be the maximally consistent set named by j. Then $[t]^\sim \bullet \Gamma = \Delta$ and by the inductive hypothesis, $\mathfrak{M}, \Gamma \models \psi$. Thus, $\mathfrak{M}, \Delta \models \langle t\rangle\psi$. $\square$

The completeness theorem now follows from the Truth Lemma and Lemma 6.23. $\square$

6.5. **Discussion: concretizations and extensions.** We end by discussing some variations on the style of analysis presented here. First, it may be possible to simplify our axiom system in the spirit of the discussion of hybrid logic and links with complex algebra in Section 3.3. We will not pursue this proof-theoretic line here. Instead, we mention a few other topics.

Concretizations. A concrete version of our system arises when we *fix the field F* we are working on, say, the rationals or the reals, drop the algebraic terms, and just add names for the elements of our field to our language, making the modalities $\langle a\rangle\varphi$ refer to concrete elements $a \in F$. We will still have completeness now, and in fact the above construction may become easier since we have the actual field represented by terms of the language and concrete assertions can be formulated about the structure of the field.

Alternatively, one may consider concrete *classes of fields*, e.g., those with a finite characteristic. There are some issues how to formulate this. We could express, say, $1 + 1 = 0$ in terms of nominals as $\mathsf{i} \oplus \mathsf{i} \leftrightarrow \mathsf{0}$, but we can also use the more modal-style axiom $\mathsf{E}\varphi \to \mathsf{E}((\varphi \oplus \varphi) \wedge \mathsf{0})$. We do not know yet how such properties can be built into our completeness proof.

A perhaps more ambitious concrete result would be an analogue of the Tarski-McKinsey Theorem for modal topology, where the reals, or any metric space without isolated points, are complete for the logic. Do invalid principles in our logic always have counterexamples in a restricted class of vector spaces, maybe only countable ones?

Language extensions. The simple hybrid modal language of DVL stays close to the similarity type and algebraic laws for vector spaces. An urgent issue is clearly how to add the earlier *dependence* and *independence* modalities of Section 5 to our language. This is a natural companion to the explicit field scalrs in DVL, and adds expressive power about undefinability. We have not yet found the right way of achieving this combination.

7. Further Directions

Richer languages for more linear algebra structures Our modal languages only describe the basic structure of vector spaces. They do not talk about such important notions as bases, dimension, angles, orthogonality, inner product, outer product, eigenvectors, and other staples of linear algebra. While some of these notions have been studied by logicians, cf. [33] on modal logics of abstract orthogonality, and [45] for uses of angles between vectors, an incorporation into our framework remains to be undertaken. A further direction of analysis would be to determine what minimal logical formalisms are needed to represent and derive basic benchmark results in linear algebra such as the Dimension Theorem for linear maps [39], the sufficiency of orthonormal bases, or other highlights of its elementary theory.

Bringing linear transformations and group actions into the logic One extension seems particularly natural, given our logic for vector spaces where elements a of a field are linear transformations on the set of vectors. Structure-preserving maps are internalized in dynamic topological logic, [42], which adds an abstract modality $\langle F \rangle$ referring to a continuous function on the state space which interacts with the modality for topological interior. In the same style, we could analyze logics of maps F satisfying commutation axioms such as

$$\langle F \rangle(\varphi \oplus \psi) \leftrightarrow (\langle F \rangle\varphi \oplus \langle F \rangle\psi)$$

Also on this road toward abstraction, in Section 6, we stressed analogies between our logic for vector spaces with propositional dynamic logic of actions on a state space. This analogy suggests an interest to logics of arbitrary groups of actions over any class of models.

However, one can also be more concrete in extending the languages of Section 6. Given a basis, linear transformations are given by *matrices*, so we can put matrices inside our modalities, either concrete or in symbolic form, and state axioms such as the following:

$$\langle \mathbf{M_1} \rangle \langle \mathbf{M_2} \rangle \varphi \ \leftrightarrow \ \langle \mathbf{M_1} \times \mathbf{M_2} \rangle \varphi$$

One could view the resulting systems as combining general logical reasoning about patterns in vector spaces with concrete computations in linear algebra involving matrices and vectors. A further development of this perspective must be left to a future occasion.

8. Conclusion

This paper is a first exploration of modal structures in commutative groups and vector spaces. We have shown how notions and techniques from modal model theory and proof systems apply with some natural adjustments. Also, we found suggestive connections to existing substructural implicational logics and dependence logics, as well as new modal logics of subgroup closure and field multiples that seem interesting in their own right.

Our treatment uncovered many open technical questions, formulated throughout the paper. We also did not pursue the model theory of group models and vector spaces in their own terms. Perhaps most importantly, we have not investigated matters of decidability and computational complexity for our systems, one of the usual considerations in favor of using modal formalisms. It may be the case that the 'grid structure' of vector spaces poses serious obstacles here, [18, Ch. 6], but all this needs careful scrutiny.

But perhaps the most caveat concerns the point of adding a modal approach to the existing literature. So far, we have captured only very little of vector spaces, up to dependence, independence and matrices. We did not formalize further powerful notions such as inner products, orthogonality, or eigenvectors, and our logics do not yet represent basic results at the heart of Linear Algebra. One can think about this distance to the target in several ways. If a logical approach faithfully copies an existing mathematical theory, it may just offer restatement in another syntax. While this may still be useful for practical purposes such as automated deduction, it need not yield novel insights. Moreover, distance can also be a beneficial source of new abstract structures and perspectives, as in our discussion of bisimulations and generalized relational models. We see this as analogous to the way in which non-vector models for Matroid Theory are considered a benefit rather than a drawback – though we admit that the best abstractions are still to be found, and may well lie somewhere in between our bare relational models and vector spaces.

Finally, another way of continuing the program of this paper might stick with the simple modal apparatus that we have introduced, not logicize more of the structure of vector spaces, and then *combine* these logics with a computational component from Linear Algebra, in the cooperative style of 'qualitative plus quantitative' advocated in [14] where qualitative logic and quantitative mathematics each do what they do best.

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9. Appendix. Proofs for some results in Section 5.3

We first formulate an auxiliary observation on finite-dimensional vector spaces.

Remark 9.1 (Generalized Steinitz)**.** Dependence relations in vector spaces satisfy the following generalized version of Steinitz Exchange:

$$D_{X,Y}z \text{ implies } D_X z \text{ or, for some } y \in Y : D_{X,Y-y+z}\, y \quad (GS)$$

This can be proved by induction on the size of Y. The base case is ordinary Steinitz. Now suppose the assertion holds for sets Y of size n, we consider the case $n+1$: or in other words, $Y+u$ with $|Y| = n$. By Steinitz, $D_{X,Y,u}\, z$ implies $D_{X,Y}\, z$ (i) or $D_{X,Y,z}\, u$ (ii). Case (ii) satisfies the second disjunct of our statement, as u ands z have been interchanged. In Case (i), by the inductive hypothesis, we have either $D_X z$ (iia) or there is some $y \in Y$ s.t. $D_{X,Y-y+z}\, y$ (iib). Case (iia) is the first disjunct of our statement, while, by monotonicity, (iib) implies the second disjunct $D_{X,Y-y+z+u}\, y$.

Proof of Fact 5.12

Proof. Downward Monotonicity is obvious from the universally quantified definition of I^D, using also the Upward Monotonicity of the relation D in its X-argument. However, the second condition requires more work.

Suppose that the independent sets X, Y satisfy $|X| < |Y|$, but for no $y \in Y - X$, $I(X \cup \{y\})$. Then $D_X y$ for all $y \in Y - X$. Here is why. Since $\neg I(X \cup \{y\})$, either $D_X y$, and we are done, or for some $x \in X$, $D_{X-x+y}\, x$. By Steinitz, the latter yields either (i) $D_{X-x}\, x$, contradicting $I(X)$, or (ii) $D_X\, y$ again. In all then, $D_X y$ since points in the $X \cap Y$ depend on X by Reflexivity. It follows that $D_X Y$: every member of Y depends on X.

We now show that there is no larger number of independent points in Y than the cardinality of X. In general, X, Y may overlap: set $Z = X \cap Y$. Clearly, $|X - Y| < |Y - X|$. Now pick any object $y \in Y - X$. We have that $D_X y$, i.e., $D_{Z,X-Y}\, y$. By Generalized Steinitz, we have either $D_Z y$, which is impossible since $I(Y)$ holds, or for some $s \in X - Y : D_{Z,(X-Y)-s+y}, s$, i.e., just $D_{Z,(X-Y)-s}, s$ since $y \in X - Y$. But this implies $D_{Z,(X-Y)-s} X$ by Reflexivity, since all $x \in X$ different from s are in the set $Z \cup (X - Y) - \{s\}$. Finally, by the given Transitivity of D applied to $D_X Y$, we get that $D_{Z,(X-Y)-s} Y$.What this means is that we have now exchanged one element in $X - Y$ with one in $Y - X$ while still having all of Y dependent on the new set after the exchange. Repeating this process, we exhaust $X - Y$, and get a proper subset of Y [here we use the assumption that $|X - Y| < |Y - X|$] on which each object in Y is dependent. But this contradicts $I(Y)$. $\square$

Proof of Fact 5.14:

Proof. Reflexivity and Monotonicity are straightforward. Next, we prove *Steinitz Exchange.* Suppose that $D_{X,y}\, z$, that is either (a) $z \in X \cup \{y\}$, or (b) for some $X' \subseteq X \cup \{y\}, I(X') \wedge \neg I(X' \cup \{z\})$. In Case (a), we either have $z \in X$, and so by definition $D_X z$, or we have $z = y$, and then $y \in X \cup \{y\}$, so again by definition $D_{X,y}, y$ and therefore $D_{X,z} y$. We are done in both cases. In Case (b), there are two subcases. Case (b1.) The set X' does not contain y. Then by definition, $D_X z$. Case (b2): No set as in case (b1) exists, and so we must have $y \in X'$. This implies in particular that $I(X' - y + z)$. But we also had $\neg I(X' \cup \{z\})$, and $X' \cup \{z\} = (X' - y + z) + y$. By definition then $D_{X+z}\, y$.

Next we prove *Transitivity* of the defined relation D. Suppose that $D_X\, y$ for all $y \in Y$ and $D_Y z$: we must show that $D_X z$. Here are the main steps. For a start, there are some special cases to consider first, such as $z \in Y$ or $z \in X$. In both these cases, the conclusion follows immediately by the definition of D. So we can assume that z is a new object.

Some comments before the argument to come. Independence is an absolute property of sets. But *maximal independence* inside the subsets of a given set X depends essentially on X, and may change truth value when we change that background set. We will use the latter changes repeatedly in combination with the next standard observation. The Extension Property in the definition of the predicate I can be used easily to show that *any two maximally independent sets inside a given set have the same cardinality.* Here is one more useful fact:

Claim Let X, Y be two maximally independent subsets of a set U: then X, Y have the same dependent objects z outside of U in the sense of the above definition.

Proof. Let $D_X z$. Since $z \notin U$ and $z \notin X$, the second clause of the definition applies, and we have $\neg I(X \cup \{z\})$ with X independent. This means that X is also maximally independent in the extended set $U \cup \{z\}$, and hence X has the largest size of an independent set there. But then, the independent set Y of the same size is maximally independent in $U \cup \{z\}$, too, and therefore, adding the further object z to it results in a non-independent set: $\neg I(Y \cup \{z\})$. $\square$

Now consider the general situation for Transitivity. We have two sets X, Y with all $y \in Y$ dependent [in our defined sense] on X, and z dependent on Y. Since $z \notin Y$, this means that there is some independent $Y' \subseteq Y$ with $\neg I(Y' \cup \{z\})$. Here also $D_X Y'$, by the definition of the relation D for set arguments Y. Thus, it suffices to consider Y' instead of Y, or, without loss of generality, we can take the above set Y to be independent.

Consider the elements of $Y - X$. Case 1: There are no such elements, and $Y \subseteq X$. Then we are done, since Y is then an independent subset of X which loses independence when adding z, and this implies that $D_X z$ holds in our defined sense. Case 2: there are such elements. For each such u, there exists an independent subset X' of X with $\neg I(X' \cup \{u\})$. Obviously, we can extend X' to a maximally independent set inside X, and we still have $\neg I(X' \cup \{u\})$ by the downward monotonicity of the independence predicate I. Moreover, for all these u, we can take the same maximally independent subset by the above observation: call this set $X^\bullet$. We show that this is the required independent subset of X with $\neg I(X^\bullet \cup \{z\})$.

First note that $X^\bullet$ is maximally independent in the set $X \cup Y$. For, the only objects one can add are either (i) in X: but this is impossible since $X^\bullet$ was maximally independent in X, or (ii) in $Y - X$: but this is impossible since by the assumption $D_X Y$ as analyzed above, adding such objects to $X^\bullet$ makes the resulting set non-independent. Next, consider the independent set Y and extend it to a maximally independent set Y^+ in $X \cup Y$. We still have that $D_{Y^+} z$, by our definition of dependence and the downward monotonicity of independence. Now we use our earlier observation that two maximally independent sets inside a given set have the same dependent objects outside of that set to conclude that $D_{X^\bullet} z$ since z was a fresh object outside of $X \cup Y$. $\square$