TYCHONOFF HED-SPACES AND ZEMANIAN EXTENSIONS OF S4.3

G. BEZHANISHVILI, N. BEZHANISHVILI, J. LUCERO-BRYAN, J. VAN MILL

Abstract. We introduce the concept of a Zemanian logic above S4.3 and prove that an extension of S4.3 is the logic of a Tychonoff HED-space iff it is Zemanian.

1. Introduction

In topological semantics of modal logic, modal box is interpreted as topological interior and modal diamond as topological closure. Under this interpretation, Lewis's well-known modal system S4 is the logic of all topological spaces. McKinsey and Tarski [20] proved that S4 is the logic of any dense-in-itself separable metric space. This result was strengthened by Rasiowa and Sikorski [22, Sec. III.7 and III.8] where it was shown that S4 is the logic of any dense-in-itself metric space. Recently this result has been generalized in several directions. The McKinsey-Tarski completeness was generalized to strong completeness by Kremer [19], and the modal logic of an arbitrary metric space was axiomatized in [5].

The class of extremally disconnected spaces (ED-spaces) consists of mostly non-metrizable spaces. The only metrizable ED-spaces are discrete. The logic $\mathsf{S4.2} := \mathsf{S4} + \Diamond\Box p \to \Box\Diamond p$ is the logic of all ED-spaces (see, e.g., [1, pg. 253]). We point out that ED is not a hereditary property. The logic $\mathsf{S4.3} := \mathsf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$ is the logic of all hereditarily extremally disconnected spaces (HED-spaces); [2, Prop. 3.1].

ED-spaces play an important role in topology. Compact Hausdorff ED-spaces are exactly the projective objects in the category of compact Hausdorff spaces and continuous maps. Moreover, each compact Hausdorff space X has a projective cover $E(X)$, known as the Gleason cover. We recall that an *irreducible map* is an onto continuous map such that the image of a proper closed subset is proper. The *Gleason cover* $E(X)$ is the (unique up to homeomorphism) compact Hausdorff ED-space for which there exists an irreducible map $\pi : E(X) \to X$. The Gleason cover of X is realized as the Stone space of the complete Boolean algebra of regular open subsets of X, accompanied by the mapping $\pi(\nabla) = \bigcap\{\mathbf{c}_X(U) \mid U \in \nabla\}$; see [17]. By [6, Prop. 4.3], S4.2 is the logic of the Gleason cover $E(\mathbb{I})$ of the closed real unit interval $\mathbb{I} = [0, 1]$, and by [2, Thm. 3.6], S4.3 is the logic of a countable subspace of $E(\mathbb{I})$.

Tychonoff spaces are up to homeomorphism subspaces of compact Hausdorff spaces. In this note we characterize the logic of an arbitrary Tychonoff HED-space. We introduce the concept of a Zemanian logic above S4.3 and show that an extension of S4.3 is the logic of a Tychonoff HED-space iff it is Zemanian. We call these logics Zemanian because of their relationship to the Zeman logic $\mathsf{S4.Z} := \mathsf{S4} + \Box\Diamond\Box p \to (p \to \Box p)$ and its generalizations $\mathsf{S4.Z}_n$ introduced in [3].

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2. $\mathsf{S4.3}$ and its extensions

We assume the reader is familiar with the basic concepts and tools of modal logic (see, e.g., [10, 18, 7]). We will be mainly interested in the modal logic

$$\mathsf{S4.3} = \mathsf{S4} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$$

and its consistent extensions. By the Bull-Fine theorem ([9, 15]), there are countably many extensions of $\mathsf{S4.3}$, each is finitely axiomatizable, and has the finite model property (fmp). In fact, each $\mathsf{L} \supseteq \mathsf{S4.3}$ is a cofinal subframe logic (see, e.g., [10, Example 11.14]).

Rooted frames for $\mathsf{S4.3}$ are rooted $\mathsf{S4}$-frames $\mathfrak{F} = (W, R)$ such that wRv or vRw for each $w, v \in W$. They can be thought of as chains of clusters. We will refer to them as *quasi-chains*. By the Bull-Fine theorem, we will work only with finite quasi-chains. A finite quasi-chain $\mathfrak{F}$ is depicted in Figure 1, where $\mathsf{min}(\mathfrak{F})$ and $\mathsf{max}(\mathfrak{F})$ denote the minimum and maximum clusters of $\mathfrak{F}$, respectively.

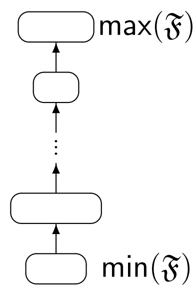


Figure 1. A finite quasi-chain $\mathfrak{F}$.

For a finite quasi-chain $\mathfrak{F}$, let $\chi_{\mathfrak{F}}$ denote the (negation of the) Jankov-Fine formula of $\mathfrak{F}$. By Fine's theorem [16, §2 Lem. 1], for any $\mathsf{S4.3}$-frame $\mathfrak{G}$,

$$\mathfrak{G} \vDash \chi_{\mathfrak{F}} \text{ iff } \mathfrak{F} \text{ is not a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Let $\mathcal{Q}$ be the set of all non-isomorphic finite quasi-chains. For $\mathfrak{F}, \mathfrak{G} \in \mathcal{Q}$, define $\mathfrak{F} \le \mathfrak{G}$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Then $\le$ is a partial ordering of $\mathcal{Q}$ and there are no infinite descending chains in $(\mathcal{Q}, \le)$. Thus, for any nonempty $S \subseteq \mathcal{Q}$, the set $\mathsf{min}(S)$ of minimal elements of S is nonempty, where

$$\mathsf{min}(S) = \{\mathfrak{F} \in S \mid \mathfrak{G} \le \mathfrak{F} \text{ and } \mathfrak{G} \in S \text{ imply } \mathfrak{G} = \mathfrak{F}\}.$$

For each extension L of $\mathsf{S4.3}$, let $\mathcal{F}_{\mathsf{L}}$ be the subset of $\mathcal{Q}$ consisting of L-frames. Then $\mathcal{F}_{\mathsf{L}}$ is a downset of $\mathcal{Q}$, and the assignment $\mathsf{L} \mapsto \mathcal{F}_{\mathsf{L}}$ is a dual isomorphism between the extensions of $\mathsf{S4.3}$ and the downsets of $\mathcal{Q}$. Moreover, each L is finitely axiomatizable by adding to $\mathsf{S4.3}$ the Jankov-Fine formulas $\chi_{\mathfrak{F}}$ where $\mathfrak{F} \in \mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$.

The following lemma, which shows that p-morphic images of a finite quasi-chain correspond to its cofinal subframes, is a version of Fine's result [15, §4 Lem. 6].

Lemma 2.1. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite quasi-chains. Then $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$ iff $\mathfrak{F}$ is isomorphic to a cofinal subframe of $\mathfrak{G}$.*

Proof. Let $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, S)$. Suppose there is a cofinal subframe $\mathfrak{H} = (U, S)$ of $\mathfrak{G}$ and an isomorphism f from $\mathfrak{H}$ to $\mathfrak{F}$. If $V = U$, then there is nothing to show. Suppose $V \neq U$. For $x \in V \setminus U$, since U is cofinal, $S[x] \cap U \neq \varnothing$. Therefore, $\mathsf{min}(S[x] \cap U) \neq \varnothing$ and is contained in a cluster of $\mathfrak{G}$. Pick $y_x \in \mathsf{min}(S[x] \cap U)$ and define $g : V \to W$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in U, \\ f(y_x) & \text{otherwise.} \end{cases}$$

That g is a well-defined onto map follows from the definition. To see that g is a p-morphism, suppose xSy. Then $S[y] \subseteq S[x]$. Therefore, $S[y] \cap U \subseteq S[x] \cap U$, and so for each $u \in \mathsf{min}(S[x] \cap U)$ and each $v \in \mathsf{min}(S[y] \cap U)$, we have uSv. Thus, $f(u)Rf(v)$, which yields $g(x)Rg(y)$. Next suppose $g(x)Rz$. Then there is $u \in U$ such that xSu and $f(u)Rz$. Since f is an isomorphism, there is $v \in U$ such that uSv and $f(v) = z$. Therefore, xSv and $g(v) = z$. Thus, g is an onto p-morphism, and hence $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G}$.

Conversely, suppose there is a p-morphism g from $\mathfrak{G}$ onto $\mathfrak{F}$. Since g is onto, $g^{-1}(w) \neq \varnothing$ for each $w \in W$. Thus, $\mathsf{max}(g^{-1}(w)) \neq \varnothing$. Pick $m_w \in \mathsf{max}(g^{-1}(w)) \neq \varnothing$ and let $U = \{m_w \mid w \in W\}$. Suppose $x \in V$. Then $xSm_{g(x)}$ and $m_{g(x)} \in U$. Therefore, U is cofinal in V. Let f be the restriction of g to U. Clearly f is a bijection between U and W. To see that f is an isomorphism, observe that wRv iff m_wSm_v. Thus, f is an isomorphism from a cofinal subframe of $\mathfrak{G}$ onto $\mathfrak{F}$. □

As an easy consequence of Lemma 2.1, we obtain:

Lemma 2.2. *A generated subframe of a finite quasi-chain* $\mathfrak{F}$ *is a p-morphic image of* $\mathfrak{F}$.

Proof. Since $\mathfrak{F}$ is a quasi-chain, a generated subframe of $\mathfrak{F}$ is a cofinal subframe of $\mathfrak{F}$. Now apply Lemma 2.1. □

As an immediate consequence of Lemmas 2.1 and 2.2, we obtain:

Lemma 2.3. *For finite quasi-chains* $\mathfrak{F}$ *and* $\mathfrak{G}$*, the following are equivalent:*

(1) $\mathfrak{F} \leq \mathfrak{G}$.
(2) $\mathfrak{F}$ *is a p-morphic image of* $\mathfrak{G}$.
(3) $\mathfrak{F}$ *is isomorphic to a cofinal subframe of* $\mathfrak{G}$.

3. Zemanian logics

In this section we introduce the concept of a Zemanian logic above S4.3. We call $\mathfrak{F} \in \mathcal{Q}$ *uniquely rooted* if its root cluster is a singleton. Otherwise we call $\mathfrak{F}$ *non-uniquely rooted*. By $\mathfrak{C}_\kappa$ we denote a cluster of cardinality κ. Let $\mathfrak{F}^r$ be the *ordinal sum* $\mathfrak{C}_1 \oplus \mathfrak{F}$ which adds a 'new' unique root r beneath $\mathfrak{F}$ (see Figure 2). We view $\mathfrak{F}$ as a generated subframe of $\mathfrak{F}^r$.

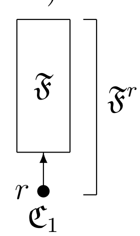


Figure 2. Adding a 'new' root to $\mathfrak{F}$.

Definition 3.1. Let L be a consistent logic above S4.3. We call L *Zemanian* provided for each non-uniquely rooted $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$, we have $\mathfrak{F}^r \in \mathcal{F}_{\mathsf{L}}$.

To motivate the name 'Zemanian logic' we recall that the *Zeman logic* S4.Z is obtained by adding to S4 the *Zeman axiom*

$$\mathsf{zem} = \Box\Diamond\Box p \to (p \to \Box p).$$

It is well known (see, e.g., [23]) that S4.Z is the logic of finite uniquely rooted S4-frames of depth 2. For $n \geq 1$, recall

$$\begin{aligned} \mathsf{bd}_1 &= \Diamond\Box p_1 \to p_1, \\ \mathsf{bd}_{n+1} &= \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1}. \end{aligned}$$

For transitive frames it is well known that $\mathfrak{F} \models \mathsf{bd}_n$ iff $\text{depth}(\mathfrak{F}) \leq n$, where $\text{depth}(\mathfrak{F})$ denotes the depth of $\mathfrak{F}$ (see, e.g., [10, Prop. 3.44]). In [3], the Zeman formula was generalized to n-Zeman formulas

$$\begin{aligned} \mathsf{zem}_0 &= p_1 \to \Box p_1, \\ \mathsf{zem}_n &= p_{n+1} \to \Box(\mathsf{bd}_n \vee p_{n+1}) \text{ for } n \geq 1, \end{aligned}$$

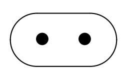
and the Zeman logic was generalized to n-Zeman logics $\mathsf{S4.Z}_n := \mathsf{S4} + \mathsf{zem}_n$ $(n \geq 0)$. By [3, Sec. 4], $\mathsf{S4.Z} = \mathsf{S4.Z}_1$ and each $\mathsf{S4.Z}_n$ is the logic of finite uniquely rooted $\mathsf{S4}$-frames of depth $n+1$.

Let $\mathsf{S4.3.Z}_n = \mathsf{S4.3} + \mathsf{zem}_n$. The next lemma shows that $\mathsf{S4.3.Z}_n$ is a Zemanian logic, hence Definition 3.1 generalizes the concept of n-Zeman logics for extensions of $\mathsf{S4.3}$.

Lemma 3.2. *If L is a Zemanian logic of finite depth, then $\mathsf{L} \vdash \mathsf{zem}_n$ for some $n \geq 0$.*

Proof. Suppose L is a Zemanian logic of finite depth. Since L is of finite depth, there is a least $n \geq 0$ such that $\mathsf{L} \vdash \mathsf{bd}_{n+1}$. Let $\mathfrak{F} \in \mathcal{F}_\mathsf{L}$. Then $\text{depth}(\mathfrak{F}) \leq n+1$. Suppose that $\mathfrak{F} \not\models \mathsf{zem}_n$. It follows from [3, Thm. 4.5] that $\text{depth}(\mathfrak{F}) = n+1$ and $\mathfrak{F}$ is non-uniquely rooted. Since L is Zemanian, $\mathfrak{F}^r \in \mathcal{F}_\mathsf{L}$. But $\text{depth}(\mathfrak{F}^r) = n+2$, yielding the contradiction $\mathfrak{F}^r \not\models \mathsf{bd}_{n+1}$. Thus, $\mathfrak{F} \models \mathsf{zem}_n$, and so $\mathsf{L} \vdash \mathsf{zem}_n$. $\square$

Remark 3.3. The converse of Lemma 3.2 is not true in general. To see this, let L be the logic of the two-point cluster $\mathfrak{C}_2$ shown in Figure 3. Then $\mathcal{F}_\mathsf{L} = \{\mathfrak{C}_1, \mathfrak{C}_2\}$. Since the depth of both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is $1 < 2$, we have that $\mathsf{L} \vdash \mathsf{zem}_1$. But L is not Zemanian because $\mathfrak{C}_2^r \notin \mathcal{F}_\mathsf{L}$.



$\mathfrak{C}_2$

FIGURE 3. The two-point cluster $\mathfrak{C}_2$.

Example 3.4.

1. It is clear that $\mathsf{S4.3}$ and $\mathsf{S4.3.Z}_n$ are Zemanian.
2. It is also obvious that $\mathsf{Grz.3}$ is Zemanian, and so is the logic of the cluster $\mathfrak{C}_1$.
3. On the other hand, neither $\mathsf{S5}$ nor $\mathsf{S4.3}_n = \mathsf{S4.3} + \mathsf{bd}_n$ is Zemanian. Neither is the logic of the cluster $\mathfrak{C}_n$ for $n \geq 2$.
4. If L is a consistent extension of $\mathsf{S4.3}$ such that $\mathsf{L} \not\subseteq \mathsf{S5}$, then $\mathsf{S5} \cap \mathsf{L}$ is not Zemanian. Indeed, since L is consistent and $\mathsf{L} \not\subseteq \mathsf{S5}$, there is $n \geq 2$ such that $\mathfrak{C}_n \notin \mathcal{F}_\mathsf{L}$. But then $\mathfrak{C}_n^r \notin \mathcal{F}_\mathsf{L}$. Therefore, $\mathfrak{C}_n \in \mathcal{F}_{\mathsf{S5}} \cup \mathcal{F}_\mathsf{L}$ but $\mathfrak{C}_n^r \notin \mathcal{F}_{\mathsf{S5}} \cup \mathcal{F}_\mathsf{L}$. Since $\mathcal{F}_{\mathsf{S5} \cap \mathsf{L}} = \mathcal{F}_{\mathsf{S5}} \cup \mathcal{F}_\mathsf{L}$, we see that $\mathsf{S5} \cap \mathsf{L}$ is not Zemanian. For example, $\mathsf{S5} \cap \mathsf{Grz.3}$ is not Zemanian.

We next describe all Zemanian logics above $\mathsf{S4.3.Z} = \mathsf{S4.3} + \mathsf{zem}$. It is clear that $\mathcal{F}_{\mathsf{S4.3.Z}} = \{\mathfrak{C}_n, \mathfrak{C}_n^r \mid n \geq 1\}$. A picture of $\mathcal{F}_{\mathsf{S4.3.Z}}$ with the partial order induced from $\mathcal{Q}$ is shown in Figure 4.

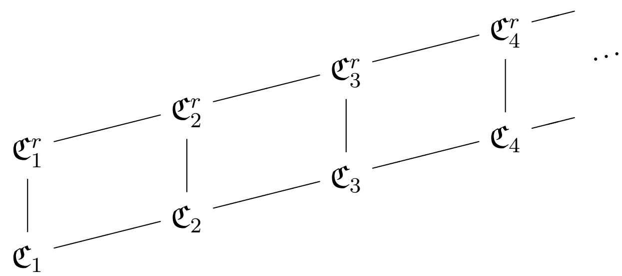


FIGURE 4. The poset $\mathcal{F}_{\mathbf{S4.3.Z}}$.

The lattice of extensions of $\mathsf{S4.3.Z}$ is dually isomorphic to the lattice of downsets of $\mathcal{F}_{\mathsf{S4.3.Z}}$. The lattice of consistent extensions of $\mathsf{S4.3.Z}$ is shown in Figure 5, where $\mathsf{Log}(\mathfrak{F})$ denotes the logic of $\mathfrak{F}$ and the Zemanian logics above $\mathsf{S4.3.Z}$ are denoted by the larger dots.

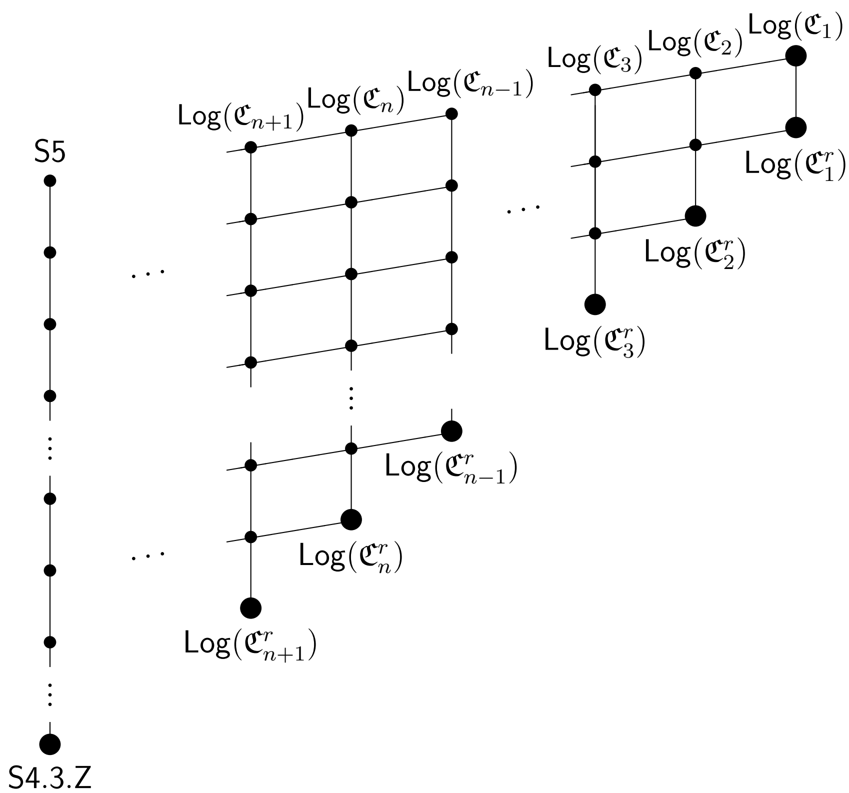


FIGURE 5. The lattice of consistent extensions of $\mathsf{S4.3.Z}$.

The remainder of this section is dedicated to establishing some basic facts about Zemanian logics. For $\mathsf{L} \supseteq \mathsf{S4.3}$, let $\mathcal{U}_{\mathsf{L}} = \{\mathfrak{F} \in \mathcal{F}_{\mathsf{L}} \mid \mathfrak{F} \text{ is uniquely rooted}\}$.

Lemma 3.5. *Let* $\mathsf{L} \supseteq \mathsf{S4.3}$ *be consistent. Then* L *is Zemanian iff* $\mathcal{U}_{\mathsf{L}}$ *is cofinal in* $\mathcal{F}_{\mathsf{L}}$.

Proof. Suppose L is Zemanian and let $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$. If $\mathfrak{F} \in \mathcal{U}_{\mathsf{L}}$, then there is nothing to show. So let $\mathfrak{F} \notin \mathcal{U}_{\mathsf{L}}$. Then $\mathfrak{F}$ is non-uniquely rooted. Since L is Zemanian, $\mathfrak{F}^r \in \mathcal{F}_{\mathsf{L}}$. Clearly $\mathfrak{F}^r$ is uniquely rooted and $\mathfrak{F} \leq \mathfrak{F}^r$. Thus, $\mathcal{U}_{\mathsf{L}}$ is cofinal in $\mathcal{F}_{\mathsf{L}}$.

Conversely, suppose $\mathcal{U}_{\mathsf{L}}$ is cofinal in $\mathcal{F}_{\mathsf{L}}$. Let $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$ be non-uniquely rooted. Then there is $\mathfrak{G} \in \mathcal{U}_{\mathsf{L}}$ such that $\mathfrak{F} \leq \mathfrak{G}$. By Lemma 2.3, up to isomorphism, $\mathfrak{F}$ is a cofinal subframe of $\mathfrak{G}$. Since $\mathfrak{G}$ is uniquely rooted and $\mathfrak{F}$ is non-uniquely rooted, the root of $\mathfrak{G}$ is not in $\mathfrak{F}$. Thus, we may identify the root of $\mathfrak{F}^r$ with the root of $\mathfrak{G}$, yielding that $\mathfrak{F}^r$ is isomorphic to a cofinal subframe of $\mathfrak{G}$. Consequently, $\mathfrak{F}^r \leq \mathfrak{G}$. Since $\mathcal{F}_{\mathsf{L}}$ is a downset of $\mathcal{Q}$ and $\mathfrak{G} \in \mathcal{F}_{\mathsf{L}}$, we see that $\mathfrak{F}^r \in \mathcal{F}_{\mathsf{L}}$. Thus, L is a Zemanian logic. □

For a class of frames $\mathcal{K}$, let $\mathsf{Log}(\mathcal{K})$ denote the logic of $\mathcal{K}$.

Lemma 3.6. *A Zemanian logic is the logic of its finite uniquely rooted quasi-chains.*

Proof. Because L has the fmp, we have that $\mathsf{L} = \mathsf{Log}(\mathcal{F}_{\mathsf{L}}) \subseteq \mathsf{Log}(\mathcal{U}_{\mathsf{L}})$. Suppose that $\mathsf{L} \not\vdash \varphi$. Then there is $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$ such that $\mathfrak{F} \not\models \varphi$. If $\mathfrak{F} \in \mathcal{U}_{\mathsf{L}}$, then there is nothing to show. Suppose $\mathfrak{F} \notin \mathcal{U}_{\mathsf{L}}$. Then $\mathfrak{F}$ is non-uniquely rooted. Since L is Zemanian, $\mathfrak{F}^r \in \mathcal{U}_{\mathsf{L}}$. As $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}^r$, from $\mathfrak{F} \not\models \varphi$ it follows that $\mathfrak{F}^r \not\models \varphi$. Thus, $\mathsf{L} = \mathsf{Log}(\mathcal{U}_{\mathsf{L}})$. □

We finish the section by axiomatizing Zemanian logics by means of Jankov-Fine formulas. For $\mathfrak{F} \in \mathcal{Q}$, let $\mathfrak{F}^a$ be the ordinal sum $\mathfrak{C}_2 \oplus (\mathfrak{F} \setminus \mathsf{min}(\mathfrak{F}))$ shown in Figure 6. Intuitively, $\mathfrak{F}^a$ is obtained by replacing the root cluster of $\mathfrak{F}$ by the two-point cluster. When $\mathfrak{F}$ is uniquely rooted, this amounts to adding a second root.

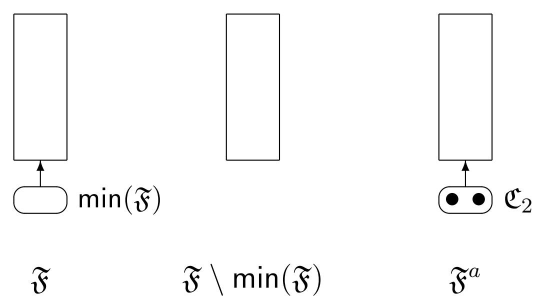


FIGURE 6. The frame $\mathfrak{F}^a$.

Theorem 3.7. *Let* $\mathsf{L} \supseteq \mathsf{S4.3}$ *be consistent. Then* L *is Zemanian iff for each* $\mathfrak{G} \in \mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$*, either* $\mathfrak{G}$ *is non-uniquely rooted or* $\mathfrak{G} \setminus \{r\}$ *is uniquely rooted and* $(\mathfrak{G} \setminus \{r\})^a \notin \mathcal{F}_{\mathsf{L}}$*.*

Proof. For the right to left direction, suppose for each $\mathfrak{G} \in \mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$, either $\mathfrak{G}$ is non-uniquely rooted or $\mathfrak{G} \setminus \{r\}$ is uniquely rooted and $(\mathfrak{G} \setminus \{r\})^a \notin \mathcal{F}_{\mathsf{L}}$. Let $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$ be non-uniquely rooted. If $\mathfrak{F}^r \notin \mathcal{F}_{\mathsf{L}}$, then there is $\mathfrak{G} \in \mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$ such that $\mathfrak{G} \leq \mathfrak{F}^r$. Therefore, up to isomorphism, $\mathfrak{G}$ is a cofinal subframe of $\mathfrak{F}^r$. Since $\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}}$ is an upset of $\mathcal{Q}$ and $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$, we have that $\mathfrak{G} \not\leq \mathfrak{F}$, so $\mathfrak{G}$ is not isomorphic to any cofinal subframe of $\mathfrak{F}$. Thus, $\varnothing \neq \mathfrak{G} \setminus \mathfrak{F} \subseteq \mathfrak{F}^r \setminus \mathfrak{F} = \{r\}$, and hence $\mathfrak{G}$ is uniquely rooted. By assumption, this yields that $\mathfrak{G} \setminus \{r\}$ is uniquely rooted and $(\mathfrak{G} \setminus \{r\})^a \notin \mathcal{F}_{\mathsf{L}}$. Because $\mathfrak{G}$ is cofinal in $\mathfrak{F}^r$, it follows that $\mathfrak{G} \setminus \{r\}$ is cofinal in $\mathfrak{F}^r \setminus \{r\} = \mathfrak{F}$.

Let t be the root of $\mathfrak{G} \setminus \{r\}$. We show that without loss of generality we may assume that $t \in \mathsf{min}(\mathfrak{F})$. Clearly, either $\mathfrak{G} \setminus \{r\} = \{t\}$ or $\mathfrak{G} \setminus \{r\} \neq \{t\}$. If $\mathfrak{G} \setminus \{r\} = \{t\}$, then since $\mathfrak{G}$ is not isomorphic to any cofinal subframe of $\mathfrak{F}$, we have that $\mathfrak{F}$ consists of a single cluster, and hence $\mathsf{max}(\mathfrak{F}) = \mathsf{min}(\mathfrak{F})$. Since $\mathfrak{G}$ is cofinal in $\mathfrak{F}^r$, we have that

$$t \in \mathsf{max}(\mathfrak{G}) \subseteq \mathsf{max}(\mathfrak{F}^r) = \mathsf{max}(\mathfrak{F}) = \mathsf{min}(\mathfrak{F}).$$

If $\mathfrak{G} \setminus \{r\} \neq \{t\}$, then $t \notin \mathsf{max}(\mathfrak{G})$. Since $\mathfrak{G}$ is cofinal in $\mathfrak{F}^r$, we obtain that $t \notin \mathsf{max}(\mathfrak{F}^r)$, and hence without loss of generality we may assume that $t \in \mathsf{min}(\mathfrak{F})$.

Since $\mathfrak{F}$ is non-uniquely rooted, we have that $(\mathfrak{G} \setminus \{r\})^a$ is isomorphic to a cofinal subframe of $\mathfrak{F}$. Therefore, $(\mathfrak{G} \setminus \{r\})^a \leq \mathfrak{F}$. As $\mathcal{F}_{\mathsf{L}}$ is a downset, we obtain that $(\mathfrak{G} \setminus \{r\})^a \in \mathcal{F}_{\mathsf{L}}$. The obtained contradiction proves that $\mathfrak{F}^r \in \mathcal{F}_{\mathsf{L}}$, and hence L is Zemanian.

For the left to right direction, we proceed by contraposition. Suppose there is $\mathfrak{G} \in \mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$ such that $\mathfrak{G}$ is uniquely rooted, and either $\mathfrak{G} \setminus \{r\}$ is non-uniquely rooted or $(\mathfrak{G} \setminus \{r\})^a \in$

$\mathcal{F}_{\mathsf{L}}$. Since $\mathfrak{G}$ is uniquely rooted, $\mathfrak{G} = (\mathfrak{G} \setminus \{r\})^r$. First suppose $\mathfrak{G} \setminus \{r\}$ is non-uniquely rooted. The minimality of $\mathfrak{G}$ in $\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}}$ yields that $\mathfrak{G} \setminus \{r\} \in \mathcal{F}_{\mathsf{L}}$. Therefore, L is not Zemanian because $\mathfrak{G} \setminus \{r\} \in \mathcal{F}_{\mathsf{L}}$ is non-uniquely rooted and $(\mathfrak{G} \setminus \{r\})^r = \mathfrak{G} \notin \mathcal{F}_{\mathsf{L}}$. Next suppose $\mathfrak{G} \setminus \{r\}$ is uniquely rooted. Then $(\mathfrak{G} \setminus \{r\})^a \in \mathcal{F}_{\mathsf{L}}$. By construction, $(\mathfrak{G} \setminus \{r\})^a$ is non-uniquely rooted. Because $\mathfrak{G} \setminus \{r\}$ is uniquely rooted, $\mathfrak{G} \setminus \{r\}$ is isomorphic to a cofinal subframe of $(\mathfrak{G} \setminus \{r\})^a$, so $\mathfrak{G} \setminus \{r\} \leq (\mathfrak{G} \setminus \{r\})^a$. Since $\mathfrak{G}$ is uniquely rooted and $\mathfrak{G} \setminus \{r\} \leq (\mathfrak{G} \setminus \{r\})^a$, it follows that $\mathfrak{G} = (\mathfrak{G} \setminus \{r\})^r$ is isomorphic to a cofinal subframe of $((\mathfrak{G} \setminus \{r\})^a)^r$, hence $\mathfrak{G} \leq ((\mathfrak{G} \setminus \{r\})^a)^r$. As $\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}}$ is an upset in $\mathcal{Q}$ containing $\mathfrak{G}$, we have that $((\mathfrak{G} \setminus \{r\})^a)^r \notin \mathcal{F}_{\mathsf{L}}$. Thus, $(\mathfrak{G} \setminus \{r\})^a \in \mathcal{F}_{\mathsf{L}}$ but $((\mathfrak{G} \setminus \{r\})^a)^r \notin \mathcal{F}_{\mathsf{L}}$, and so L is not Zemanian. □

Corollary 3.8. *Let* $\mathsf{L} \supseteq \mathsf{S4.3}$. *If* $\mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}}) = \{\mathfrak{G}\}$, *then* L *is Zemanian iff* $\mathfrak{G}$ *is non-uniquely rooted.*

Proof. Suppose that $\mathfrak{G}$ is non-uniquely rooted. Then every quasi-chain in $\mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$ is non-uniquely rooted, so L is Zemanian by Theorem 3.7. Conversely, suppose that L is Zemanian. Then Theorem 3.7 yields that either $\mathfrak{G}$ is non-uniquely rooted or $\mathfrak{G} \setminus \{r\}$ is uniquely rooted and $(\mathfrak{G} \setminus \{r\})^a \notin \mathcal{F}_{\mathsf{L}}$. We show that the latter condition is never satisfied when $\mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{L}})$ is a singleton. Suppose that both $\mathfrak{G}$ and $\mathfrak{G} \setminus \{r\}$ are uniquely rooted. Since the depth of $\mathfrak{G}$ is greater than the depth of $(\mathfrak{G} \setminus \{r\})^a$, we have that $\mathfrak{G}$ is not isomorphic to any subframe of $(\mathfrak{G} \setminus \{r\})^a$. Therefore, $\mathfrak{G} \not\leq (\mathfrak{G} \setminus \{r\})^a$, and so $(\mathfrak{G} \setminus \{r\})^a \in \mathcal{F}_{\mathsf{L}}$. □

4. S4.3 and HED-spaces

We assume the reader is familiar with basic topological concepts (see, e.g., [14]). For a topological space X, we use $\mathbf{c}_X$ and $\mathbf{i}_X$ for closure and interior in X, respectively. We recall that a topological space X is *extremally disconnected* (ED) if the closure of any open set is open, and X is *hereditarily extremally disconnected* (HED) if every subspace of X is ED. While HED is clearly a stronger concept than ED, it is of note that every countable Hausdorff ED-space is HED (see, e.g., [8, pg. 86]). As we pointed out in the introduction, if we interpret $\Box$ as topological interior and $\Diamond$ as topological closure, then $\mathsf{S4.2}$ is the logic of all ED-spaces, and $\mathsf{S4.3}$ is the logic of all HED-spaces.

Since $\mathsf{S4}$-frames can be viewed as special topological spaces, called *Alexandroff spaces*, in which each point has a least open neighborhood (namely the set of points that are R-accessible from it), relational completeness of logics above $\mathsf{S4}$ clearly implies their topological completeness. However, Alexandroff spaces do not satisfy higher separation axioms. In fact, an Alexandroff space is T_1 iff it is discrete. Therefore, obtaining completeness with respect to "good" topological spaces, such as Tychonoff spaces, requires additional work.

As we pointed out in the introduction, $\mathsf{S4.2}$ is the logic of the Gleason cover $E(\mathbb{I})$ of the real unit interval $\mathbb{I} = [0, 1]$, and $\mathsf{S4.3}$ is the logic of a countable subspace of $E(\mathbb{I})$. Our goal is to build on this and show that an extension of $\mathsf{S4.3}$ is the logic of a Tychonoff HED-space iff it is a Zemanian logic. The key technique is to associate a Tychonoff HED-space $X_{\mathfrak{F}}$ with each uniquely rooted finite quasi-chain $\mathfrak{F}$ of depth > 1 so that the logic $\mathsf{Log}(X_{\mathfrak{F}})$ of the space $X_{\mathfrak{F}}$ is equal to $\mathsf{Log}(\mathfrak{F})$. For this we require some tools.

The *Cantor cube*, $2^{\mathfrak{c}}$, is the topological product of continuum many copies of the two-point discrete space 2. We will consider the Gleason cover $E(2^{\mathfrak{c}})$ of the Cantor cube $2^{\mathfrak{c}}$.

A space X is *resolvable* provided there is a dense subset D of X such that $X \setminus D$ is dense in X. If X is not resolvable, then X is *irresolvable.* If every subspace of X is irresolvable, then X is *hereditarily irresolvable*, and X is *open-hereditarily irresolvable* if every open subspace of X is irresolvable. A space X is *nodec* provided every nowhere dense subset is closed (equivalently, closed and discrete).

Definition 4.1. [11, §2] Suppose X is a topological space.

(1) For a subspace Y of X, we define the set $\mathscr{N}(Y)$ of *near-points* of Y by

$$\mathscr{N}(Y) = \bigcup\{\mathbf{c}_X(D) \mid D \text{ is a countable discrete subspace of } Y\}.$$

(2) The subspaces Y and Z of X are *far* if $\mathscr{N}(Y) \cap \mathscr{N}(Z) = \varnothing$.

A topological space is *dense-in-itself* or *crowded* if it has no isolated points.

Theorem 4.2. [11, §4] *There is a countable pairwise disjoint family $\mathcal{A}$ of countable crowded dense subsets of $E(2^{\mathfrak{c}})$ such that*

(1) *each element of $\mathcal{A}$ is a nodec open-hereditarily irresolvable* ED*-space;*
(2) *distinct elements of $\mathcal{A}$ are far.*

Remark 4.3. As follows from [11, §4], each element of $\mathcal{A}$ is not only nodec and open-hereditarily irresolvable, but also maximal, hence submaximal, and hence also hereditarily irresolvable.

A *dense partition* of a topological space X is a pairwise disjoint collection $\mathcal{P}$ of dense subsets of X such that $X = \bigcup \mathcal{P}$. Call X *n-resolvable* provided there is a dense partition of X consisting of n elements; otherwise X is called *n-irresolvable*.

Let $\mathcal{A}$ be as in Theorem 4.2. Enumerate $\mathcal{A} = \{A_1, \dots, A_n, \dots\}$ and set $X_n = A_1 \cup \cdots \cup A_n$.

Lemma 4.4.

(1) *X_n is nodec.*
(2) *If $k > n$ and N is nowhere dense in X_n, then $\mathscr{N}(A_k) \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(N) = \varnothing$.*
(3) *A nonempty open subspace U of X_n is n-resolvable and $(n+1)$-irresolvable.*

Proof. (1). Suppose N is nowhere dense in X_n. We show that $N_i := N \cap A_i$ is nowhere dense in the subspace A_i. Let U be an open subset of A_i such that $U \subseteq \mathbf{c}_{X_n}(N_i)$. Then there is an open subset V of X_n such that $U = V \cap A_i$. Since A_i is dense in X_n, we have $V \subseteq \mathbf{c}_{X_n}(U)$. Therefore, $V \subseteq \mathbf{c}_{X_n}(N_i) \subseteq \mathbf{c}_{X_n}(N)$. Because N is nowhere dense in X_n, we have $V = \varnothing$. Thus, $U = \varnothing$, and so N_i is nowhere dense in A_i.

Since A_i is nodec, N_i is closed and discrete. If $i \neq j$, then A_i and A_j are far. Therefore, as N_i is countable,

$$\mathbf{c}_{E(2^{\mathfrak{c}})}(N_i) \cap A_j \subseteq \mathscr{N}(A_i) \cap \mathscr{N}(A_j) = \varnothing.$$

Because $N = N \cap X_n = N \cap \bigcup_{i=1}^n A_i = \bigcup_{i=1}^n (N \cap A_i) = \bigcup_{i=1}^n N_i$, we have that

$$\begin{aligned}
\mathbf{c}_{X_n}(N) &= \mathbf{c}_{X_n}\left(\bigcup_{i=1}^n N_i\right) = \bigcup_{i=1}^n \mathbf{c}_{X_n}(N_i) = \bigcup_{i=1}^n \left[\mathbf{c}_{E(2^{\mathfrak{c}})}(N_i) \cap X_n\right] \\
&= \bigcup_{i=1}^n \left[\mathbf{c}_{E(2^{\mathfrak{c}})}(N_i) \cap \bigcup_{j=1}^n A_j\right] = \bigcup_{i=1}^n \bigcup_{j=1}^n \left[\mathbf{c}_{E(2^{\mathfrak{c}})}(N_i) \cap A_j\right] \\
&= \bigcup_{i=1}^n \left[\mathbf{c}_{E(2^{\mathfrak{c}})}(N_i) \cap A_i\right] = \bigcup_{i=1}^n \mathbf{c}_{A_i}(N_i) = \bigcup_{i=1}^n N_i = N.
\end{aligned}$$

So N is closed in X_n. This yields that X_n is a nodec space.

(2). Suppose $k > n$. Then A_i and A_k are far for each $i \le n$. Since N_i is a countable discrete subset of A_i, we have

$$\begin{aligned}\mathscr{N}(A_k) \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(N) &= \mathscr{N}(A_k) \cap \bigcup_{i=1}^{n} \mathbf{c}_{E(2^{\mathfrak{c}})}(N_i) = \bigcup_{i=1}^{n} [\mathscr{N}(A_k) \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(N_i)] \\ &\subseteq \bigcup_{i=1}^{n} [\mathscr{N}(A_k) \cap \mathscr{N}(A_i)] = \varnothing.\end{aligned}$$

(3). Let U be a nonempty open subspace of X_n. Note that X_n is n-resolvable since $\{A_1, \dots, A_n\}$ is a dense partition of X_n. Therefore, U is n-resolvable by [12, Prop. 1.1(c)]. Since A_i is dense, $U \cap A_i$ is a nonempty open subset of A_i, and hence a crowded open-hereditarily irresolvable space. Because $U = \bigcup_{i=1}^{n} (U \cap A_i)$, it follows from [12, Lem. 3.2(a)] that U is $(n+1)$-irresolvable. □

For $m > 1$ and a finite uniquely rooted quasi-chain $\mathfrak{F}$ of depth m, we construct $X_{\mathfrak{F}}$ by recursion on m. Suppose $\mathsf{max}(\mathfrak{F})$ consists of n elements.

Base case: For $m = 2$, set $X_{\mathfrak{F}} = \bigcup_{i=1}^{n} A_i$. Then $X_{\mathfrak{F}}$ is a countable dense subspace of $E(2^{\mathfrak{c}})$, and hence $X_{\mathfrak{F}}$ is a countable crowded ED-space.

Recursive step: Suppose $m > 2$, $\mathfrak{G} := \mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})$, and $Y := X_{\mathfrak{G}}$ is already built. So Y is a countable crowded ED-space constructed from the finite uniquely rooted quasi-chain $\mathfrak{G}$. Let $Z = \bigcup_{i=1}^{n} A_i$. Since A_{n+1} is crowded, it is easy to construct a countable family $\{U_i \mid i \in \omega\}$ of open sets in A_{n+1} such that their closures in A_{n+1} are pairwise disjoint. Picking a point from each U_i then yields a countably infinite closed discrete subset D of A_{n+1}. Let $\beta\omega$ denote the Čech-Stone compactification of the discrete space ω. By [24, Prop. 1.48], $\mathbf{c}_{E(2^{\mathfrak{c}})}(D)$ is homeomorphic to $\beta\omega$ since countable sets in an ED-space are C^*-embedded (see, e.g., [24, Prop. 1.64]). Also, $\mathbf{c}_{E(2^{\mathfrak{c}})}(D) \cap Z = \varnothing$ since A_i and A_{n+1} are far for all $i \le n$.

By Efimov's theorem [13] (see also [21, Thm. 1.4.7]), each compact Hausdorff ED-space of weight $\le \mathfrak{c}$ can be embedded in $\beta\omega$. Therefore, βY and hence Y is embedded in $\beta\omega$, which is homeomorphic to $\mathbf{c}_{E(2^{\mathfrak{c}})}(D)$. Since Y is crowded, we may assume that Y is a subspace of $\mathbf{c}_{E(2^{\mathfrak{c}})}(D) \setminus D$. We set $X_{\mathfrak{F}}$ to be the subspace $Y \cup Z$ of $E(2^{\mathfrak{c}})$; see Figure 7.

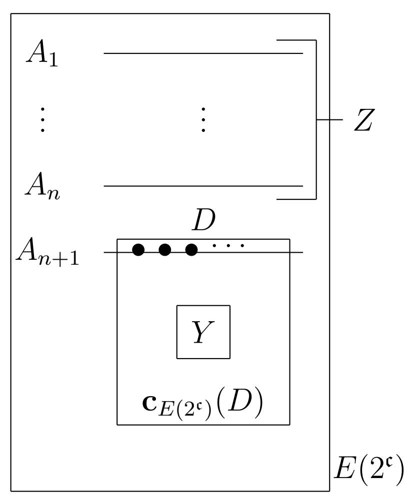


FIGURE 7. Recursive step defining $X_{\mathfrak{F}} = Y \cup Z$.

5. Properties of $X_{\mathfrak{F}}$

It follows from the construction that $X_{\mathfrak{F}}$ is a countable crowded Tychonoff ED-space, and hence an HED-space. Moreover, Z is open and dense in $X_{\mathfrak{F}}$ and Y is closed and nowhere dense in $X_{\mathfrak{F}}$. To see this, $Y \subseteq \mathbf{c}_{E(2^{\mathfrak{c}})}(D)$ gives $Y \cap Z = \varnothing$, so $Y = X_{\mathfrak{F}} \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(D)$ is closed in $X_{\mathfrak{F}}$, and so $Z = X_{\mathfrak{F}} \setminus Y$ is open in $X_{\mathfrak{F}}$. Since each A_i is dense in $E(2^{\mathfrak{c}})$, it follows that Z is dense in $X_{\mathfrak{F}}$. As Z is open and dense in $X_{\mathfrak{F}}$, we see that $Y = X_{\mathfrak{F}} \setminus Z$ is nowhere dense.

Let $\mathfrak{F} = (W, R)$ be a finite quasi-chain. Call $U \subseteq W$ an *R-upset* provided $w \in U$ and wRv imply $v \in U$ (*R-downsets* are defined dually). Recall that the opens in the Alexandroff topology on W are the R-upsets, and the closure in the Alexandroff topology is given by $R^{-1}(A) := \{w \in W \mid \exists v \in A \text{ with } wRv\}$.

We recall that a map $f : X \to Y$ between topological spaces is *interior* provided f is continuous and open. If f is an onto interior map, then we call Y an *interior image* of X. Our next goal is to show that $\mathfrak{F}$, viewed as an Alexandroff space, is an interior image of $X_{\mathfrak{F}}$. To prove Lemma 5.2, we utilize the following two straightforward facts, which we gather together in a lemma for easy reference.

Lemma 5.1.
(1) *Let X, Y be topological spaces and $f : X \to Y$ an onto interior map. Suppose $C \subseteq Y$ and $D = f^{-1}(C)$. Then the restriction of f to D is an interior mapping onto C.*
(2) *A dense subspace of a crowded T_1-space is crowded.*

Lemma 5.2. *Let X be a T_1-space and $\mathfrak{F}$ a non-uniquely rooted finite quasi-chain. Then $\mathfrak{F}$ is an interior image of X iff $\mathfrak{F}^r$ is an interior image of X.*

Proof. First suppose there is an onto interior mapping $f : X \to \mathfrak{F}^r$. As $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}^r$, by Lemma 2.2, there is an onto p-morphism $g : \mathfrak{F}^r \to \mathfrak{F}$. Since p-morphisms correspond to interior maps between Alexandroff spaces, the composition $g \circ f : X \to \mathfrak{F}$ is an onto interior map, showing that $\mathfrak{F}$ is an interior image of X.

Next suppose there is an onto interior mapping $f : X \to \mathfrak{F}$. For each $w \in \mathsf{min}(\mathfrak{F})$, let $A_w = f^{-1}(w)$. Then $D := f^{-1}(\mathsf{min}(\mathfrak{F}))$ is partitioned into $\{A_w \mid w \in \mathsf{min}(\mathfrak{F})\}$. By Lemma 5.1(1), the restriction of f is an interior mapping of D onto $\mathsf{min}(\mathfrak{F})$. Therefore, since $R^{-1}(w) = \mathsf{min}(\mathfrak{F})$, each A_w is dense in D. Because $\mathsf{min}(\mathfrak{F})$ contains more than one point, D is crowded. By Lemma 5.1(2), each A_w is crowded, hence infinite.

Choose $x_0 \in D$ and define $g : X \to \mathfrak{F}^r$ by

$$g(x) = \begin{cases} r & \text{if } x = x_0, \\ f(x) & \text{if } x \neq x_0. \end{cases}$$

Clearly g is a well-defined map, and g is onto since $g(x_0) = r$ and $D \setminus \{x_0\} \neq \varnothing$. For $w \in \mathfrak{F}^r$, observe that

$$g^{-1}(R[w]) = \begin{cases} X & \text{if } w = r, \\ X \setminus \{x_0\} & \text{if } w \in \mathsf{min}(\mathfrak{F}), \\ f^{-1}(R[w]) & \text{otherwise.} \end{cases}$$

Therefore, g is continuous since X is T_1 and f is continuous. For a nonempty open subset U of X, observe that

$$g(U) = \begin{cases} f(U) & \text{if } x_0 \notin U, \\ \mathfrak{F}^r & \text{if } x_0 \in U. \end{cases}$$

Thus, g is open since f is open and $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}^r$. Consequently, $\mathfrak{F}^r$ is an interior image of X. $\square$

We are ready to prove that $\mathfrak{F}$ is an interior image of $X_{\mathfrak{F}}$.

Theorem 5.3. *Each finite uniquely rooted quasi-chain $\mathfrak{F}$ of depth $m > 1$ is an interior image of $X_{\mathfrak{F}}$.*

Proof. Suppose $\mathsf{max}(\mathfrak{F})$ consists of n elements. Let $\mathfrak{G} = \mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})$. We proceed by induction on $m \geq 2$. First suppose $m = 2$. By Lemma 4.4(3), $X_{\mathfrak{F}}$ is n-resolvable. By [3, Lem. 5.9], $\mathsf{max}(\mathfrak{F})$ is an interior image of $X_{\mathfrak{F}}$. Therefore, since $\mathfrak{F} = \mathsf{max}(\mathfrak{F})^r$, Lemma 5.2 yields that $\mathfrak{F}$ is an interior image of $X_{\mathfrak{F}}$.

Next suppose $m > 2$. By construction, $X_{\mathfrak{F}} = Y \cup Z$, where $Y = X_{\mathfrak{G}}$ and $Z = \bigcup_{i=1}^{n} A_i$. By the inductive hypothesis, there is an onto interior map $g : Y \to \mathfrak{G}$. By Lemma 4.4(3), the open subspace Z of $X_{\mathfrak{F}}$ is n-resolvable. Therefore, by [3, Lem. 5.9], there is an onto interior map $h : Z \to \mathsf{max}(\mathfrak{F})$. Define $f : X_{\mathfrak{F}} \to \mathfrak{F}$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in Y, \\ h(x) & \text{if } x \in Z. \end{cases}$$

Since Y and Z are complements in $X_{\mathfrak{F}}$, the map f is well-defined. It is onto since g is onto $\mathfrak{G}$ and h is onto $\mathsf{max}(\mathfrak{F})$. Moreover,

$$f^{-1}(R^{-1}(w)) = \begin{cases} X_{\mathfrak{F}} & \text{if } w \in \mathsf{max}(\mathfrak{F}), \\ g^{-1}(R^{-1}(w)) & \text{if } w \in \mathfrak{G}. \end{cases}$$

Notice that $f^{-1}(R^{-1}(w))$ is closed in $X_{\mathfrak{F}}$ whenever $w \in \mathfrak{G}$ since g is continuous and Y is closed in $X_{\mathfrak{F}}$. Therefore, f is continuous. To see that f is open, let U be a nonempty open subset of $X_{\mathfrak{F}}$. Since A_i is dense in Z and hence in $X_{\mathfrak{F}}$, we have $U \cap A_i \neq \varnothing$ for all $i \leq n$. So

$$f(U) = f(U \cap Z) \cup f(U \cap Y) = h(U \cap Z) \cup g(U \cap Y) = \mathsf{max}(\mathfrak{F}) \cup g(U \cap Y).$$

Because g is open and $U \cap Y$ is open in Y, we have $g(U \cap Y)$ is an R-upset of $\mathfrak{G}$. Therefore, $f(U)$ is an R-upset of $\mathfrak{F}$. Thus, f is open, so f is an onto interior map, and hence $\mathfrak{F}$ is an interior image of $X_{\mathfrak{F}}$. □

We next recall the definition of the *modal Krull dimension* $\mathrm{mdim}(X)$ of a topological space X from [3]:

$$\begin{array}{ll} \mathrm{mdim}(X) = -1 & \text{if } X = \varnothing, \\ \mathrm{mdim}(X) \leq n & \text{if } \mathrm{mdim}(D) \leq n-1 \text{ for every nowhere dense subset } D \text{ of } X, \\ \mathrm{mdim}(X) = n & \text{if } \mathrm{mdim}(X) \leq n \text{ and } \mathrm{mdim}(X) \not\leq n-1, \\ \mathrm{mdim}(X) = \infty & \text{if } \mathrm{mdim}(X) \not\leq n \text{ for any } n = -1, 0, 1, 2, \ldots. \end{array}$$

As follows from [3, Rem. 4.8 & Thm. 4.9], for a T_1-space X, we have $\mathrm{mdim}(X) \leq n$ iff $X \vDash \mathsf{zem}_n$; in particular, X is nodec iff $\mathrm{mdim}(X) \leq 1$.

Theorem 5.4. *For a finite uniquely rooted quasi-chain $\mathfrak{F}$ of depth $m > 1$, the modal Krull dimension of $X_{\mathfrak{F}}$ is $m - 1$.*

Proof. The proof is by induction on $m \geq 2$. First suppose $m = 2$. Then $X_{\mathfrak{F}}$ is nodec by Lemma 4.4(1). Since $X_{\mathfrak{F}}$ is a crowded T_1-space, it follows from [3, Rem. 4.8 & Thm. 4.9] that $\mathrm{mdim}(X_{\mathfrak{F}}) = 1$.

Next suppose $m > 2$. Let $\mathsf{max}(\mathfrak{F})$ consist of n elements and $\mathfrak{G} = \mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})$. By construction, $X_{\mathfrak{F}} = Y \cup Z$, where $Y = X_{\mathfrak{G}}$, $Y \subseteq \mathbf{c}_{E(2^{\mathfrak{c}})}(D) \subseteq \mathscr{N}(A_{n+1})$, and $Z = \bigcup_{i=1}^{n} A_i$. By the inductive hypothesis, $\mathrm{mdim}(Y) = m - 2$. Let N be a nowhere dense subset of $X_{\mathfrak{F}}$. Since Z is open in $X_{\mathfrak{F}}$, we see that $N \cap Z$ is nowhere dense in Z. By Lemma 4.4(2),

$$Y \cap \mathbf{c}_N(N \cap Z) \subseteq \mathscr{N}(A_{n+1}) \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(N \cap Z) = \varnothing.$$

Therefore, $\mathbf{c}_N(N \cap Z) \subseteq N \setminus Y = N \cap Z$, showing that $N \cap Z$ is closed in N. Clearly $N \cap Z$ is open in N since Z is open in $X_{\mathfrak{F}}$. Thus, $N \cap Z$ is clopen in N. It follows that N is the topological sum of $N \cap Z$ and $N \cap Y$. By Lemma 4.4(1), Z is nodec. So by [3, Lem. 3.3],

$\mathrm{mdim}(N \cap Z) \leq \mathrm{mdim}(Z) \leq 1 \leq m-2$ and $\mathrm{mdim}(N \cap Y) \leq \mathrm{mdim}(Y) = m-2$. Therefore, [3, Lem. 5.6] yields $\mathrm{mdim}(N) \leq m-2$. Thus, by definition, $\mathrm{mdim}(X_{\mathfrak{F}}) \leq m-1$. But since Y is a nowhere dense subspace of $X_{\mathfrak{F}}$ with $\mathrm{mdim}(Y) = m-2$, we see that $\mathrm{mdim}(X_{\mathfrak{F}}) \not\leq m-2$. Consequently, $\mathrm{mdim}(X_{\mathfrak{F}}) = m-1$. □

Lemma 5.5. *Suppose a finite quasi-chain $\mathfrak{F}$ is an interior image of X. If X has an isolated point, then $\mathsf{max}(\mathfrak{F})$ is a singleton.*

Proof. Let $f : X \to \mathfrak{F}$ be an onto interior mapping. If $x \in X$ is an isolated point, then since f is interior, $\{f(x)\}$ is an R-upset of $\mathfrak{F}$. But the least nonempty R-upset of $\mathfrak{F}$ is $\mathsf{max}(\mathfrak{F})$. Thus, $\mathsf{max}(\mathfrak{F}) = \{f(x)\}$ is a singleton. □

Lemma 5.6. *Suppose X is a nodec space and $\mathfrak{F}$ is a finite quasi-chain. If $f : X \to \mathfrak{F}$ is an onto interior mapping, then $\mathfrak{F} = \mathsf{max}(\mathfrak{F})$ or $\mathfrak{F} = \mathsf{max}(\mathfrak{F})^r$.*

Proof. It is shown in [4, Prop. 3.8] that $\mathsf{S4.Z}$ defines the class of nodec spaces. Therefore, an interior image of a nodec space is a nodec space. It is a consequence of [4, Prop. 4.1] that a finite quasi-chain, viewed as an Alexandroff space, is a nodec space iff $\mathfrak{F}$ is a cluster or $\mathfrak{F} = \mathsf{max}(\mathfrak{F})^r$. The result follows. □

Lemma 5.7. *If C is a nonempty closed subset of a nodec ED-space X, then C is a disjoint union of a clopen set and a closed discrete set.*

Proof. Let $E = \mathbf{c}_X\mathbf{i}_X(C)$. Then $C \supseteq E$ and E is clopen since X is ED. Also $F := C \setminus E$ is a closed nowhere dense subset of X. Therefore, F is discrete since X is nodec. Clearly E, F are disjoint and $C = E \cup F$. □

The next lemma is the main technical result of the section.

Lemma 5.8. *If a finite quasi-chain $\mathfrak{G} = (V, R)$ is an interior image of a closed subspace C of $X_{\mathfrak{F}}$, then $\mathfrak{G}$ is isomorphic to a subframe of $\mathfrak{F}$. Moreover, if the interior of C is nonempty, then $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$.*

Proof. Suppose that $g : C \to \mathfrak{G}$ is an onto interior mapping, $\mathrm{depth}(\mathfrak{F}) = m$, $\mathsf{max}(\mathfrak{F})$ consists of n elements, and $\mathsf{max}(\mathfrak{G})$ consists of k elements. By [3, Lem. 3.3] and Theorem 5.4, $\mathrm{mdim}(C) \leq \mathrm{mdim}(X_{\mathfrak{F}}) = m-1$. Therefore, by [3, Thm. 3.6], $C \vDash \mathsf{bd}_m$. Since $\mathfrak{G}$ is an interior image of C, we have $\mathfrak{G} \vDash \mathsf{bd}_m$, and hence $\mathrm{depth}(\mathfrak{G}) \leq m$. If $\mathrm{depth}(\mathfrak{G}) = m$ and $\mathfrak{G}$ is non-uniquely rooted, then Lemma 5.2 yields that $\mathfrak{G}^r$ is an interior image of C. This is a contradiction since $\mathfrak{G}^r \not\vDash \mathsf{bd}_m$. Thus, if $\mathrm{depth}(\mathfrak{G}) = m$, then $\mathfrak{G}$ is uniquely rooted. We prove that $\mathfrak{G}$ is isomorphic to a subframe of $\mathfrak{F}$ by induction on $m \geq 2$.

Base case: Suppose $m = 2$. Then $\mathfrak{G} = \mathsf{max}(\mathfrak{G})$ or $\mathfrak{G} = \mathsf{max}(\mathfrak{G})^r$. We show that $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$. For this it is sufficient to show that $\mathsf{max}(\mathfrak{G})$ consists of no more than n elements. Since $m = 2$, we have that $X_{\mathfrak{F}}$ is a nodec ED-space, so Lemma 5.7 gives that $C = E \cup F$, where E and F are disjoint, E is clopen in $X_{\mathfrak{F}}$, and F is closed and discrete in $X_{\mathfrak{F}}$. If $F \neq \varnothing$, then since F is discrete, every point in F is isolated in C. Therefore, C has an isolated point. Thus, by Lemma 5.5, $\mathsf{max}(\mathfrak{G})$ is a singleton, and hence $\mathsf{max}(\mathfrak{G})$ consists of no more than n elements. If $F = \varnothing$, then $C = E$ is open in $X_{\mathfrak{F}}$, so $g^{-1}(\mathsf{max}(\mathfrak{G}))$ is open in $X_{\mathfrak{F}}$. By Lemma 4.4(3), $g^{-1}(\mathsf{max}(\mathfrak{G}))$ is $(n+1)$-irresolvable. Therefore, by [3, Lem. 5.9], $\mathsf{max}(\mathfrak{G})$ consists of no more than n elements. Thus, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$.

Inductive step: Suppose $m > 2$. By construction, $X_{\mathfrak{F}} = Y \cup Z$, where $Y := X_{\mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})}$ is closed and nowhere dense in $X_{\mathfrak{F}}$ and $Z = \bigcup_{i=1}^{n} A_i$ is open and dense in $X_{\mathfrak{F}}$. If $C \subseteq Y$, then by the inductive hypothesis, $\mathfrak{G}$ is isomorphic to a subframe of $\mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})$, and hence $\mathfrak{G}$ is isomorphic to a subframe of $\mathfrak{F}$.

Suppose $C \nsubseteq Y$, so $C \cap Z \neq \varnothing$. We first show that $\mathsf{max}(\mathfrak{G})$ has no more than n elements. Since $C \cap Z$ is open in C, it follows that $g|_{C\cap Z}$ is an interior mapping of $C \cap Z$ onto $g(C \cap Z)$, which is a generated subframe of $\mathfrak{G}$, and hence contains $\mathsf{max}(\mathfrak{G})$. Also $C \cap Z$ is closed in Z. By Lemma 4.4(1), Z is nodec, so by Lemma 5.7, there are disjoint subsets E and F of Z such that E is clopen in Z, F is closed and discrete in Z, and $C \cap Z = E \cup F$. If $F \neq \varnothing$, then $C \cap Z$ has an isolated point, and so $\mathsf{max}(\mathfrak{G}) = \mathsf{max}(g(C \cap Z))$ is a singleton by Lemma 5.5. So we may assume that $F = \varnothing$. But then $C \cap Z = E$ is open in Z, and so $(g|_{C\cap Z})^{-1}(\mathsf{max}(\mathfrak{G}))$ is open in Z. By Lemma 4.4(3), $(g|_{C\cap Z})^{-1}(\mathsf{max}(\mathfrak{G}))$ is $(n+1)$-irresolvable, so it follows from [3, Lem. 5.9] that $\mathsf{max}(\mathfrak{G})$ contains no more than n elements.

We next show that $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$. If $\mathrm{depth}(\mathfrak{G}) = 1$, then $\mathfrak{G} = \mathsf{max}(\mathfrak{G})$. Since $\mathsf{max}(\mathfrak{G})$ has no more than n elements and $\mathsf{max}(\mathfrak{F})$ has n elements, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$. Suppose $\mathrm{depth}(\mathfrak{G}) > 1$. The set $N := g^{-1}(\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G}))$ is a closed nowhere dense subset of C. Since the restriction $g|_{C\cap Z}$ is interior, we have $N \cap Z = (g|_{C\cap Z})^{-1}(\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G}))$ is a closed nowhere dense subset of Z. By Lemma 4.4(2),

$$Y \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(N \cap Z) \subseteq \mathscr{N}(A_{n+1}) \cap \bigcup_{i=1}^{n} \mathscr{N}(A_i) = \varnothing.$$

Therefore,

$$\begin{aligned}
\mathbf{c}_{X_{\mathfrak{F}}}(N \cap Z) &= \mathbf{c}_{X_{\mathfrak{F}}}(N \cap Z) \cap (Y \cup Z) \\
&= \big(\mathbf{c}_{X_{\mathfrak{F}}}(N \cap Z) \cap Y\big) \cup \big(\mathbf{c}_{X_{\mathfrak{F}}}(N \cap Z) \cap Z\big) \\
&= \big(X_{\mathfrak{F}} \cap \mathbf{c}_{E(2^{\mathfrak{c}})}(N \cap Z) \cap Y\big) \cup \mathbf{c}_{Z}(N \cap Z) \\
&= \varnothing \cup (N \cap Z) = N \cap Z.
\end{aligned}$$

Thus, $N \cap Z$ is closed in $X_{\mathfrak{F}}$. Clearly $N \cap Z$ is open in N since Z is open in $X_{\mathfrak{F}}$. Because $N \cap Z$ is closed in $X_{\mathfrak{F}}$, we have that $N \cap Z$ is clopen in N. Consequently, $N \cap Y = N \setminus Z$ is also clopen in N. We proceed by cases.

First suppose $N \subseteq Z$. Then $N = N \cap Z$, so N is closed in $X_{\mathfrak{F}}$, and hence N is closed in $C \cap Z$. Therefore, $(C \cap Z) \setminus N$ is open in $C \cap Z$. The restriction $g|_{C\cap Z} : C \cap Z \to \mathfrak{G}$ is interior and onto $\mathfrak{G}$ since

$$\begin{aligned}
g|_{C\cap Z}(C \cap Z) &= g((C \cap Z) \setminus N) \cup g((C \cap Z) \cap N) \\
&\supseteq \mathsf{max}(\mathfrak{G}) \cup g(N) = \mathsf{max}(\mathfrak{G}) \cup (\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})) = \mathfrak{G}.
\end{aligned}$$

Because Z is nodec and $C \cap Z$ is a (closed) subspace of Z, we see that $C \cap Z$ is nodec. Since $\mathrm{depth}(\mathfrak{G}) > 1$, Lemma 5.6 yields that $\mathrm{depth}(\mathfrak{G}) = 2$ and $\mathfrak{G}$ is uniquely rooted. As $\mathrm{depth}(\mathfrak{F}) = m > 2$, $\mathsf{max}(\mathfrak{F})$ consists of n elements, and $\mathsf{max}(\mathfrak{G})$ has no more than n elements, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$.

Next suppose $N \subseteq Y$. It follows from Lemma 5.1(1) that the restriction $g|_N : N \to \mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$ is an onto interior map. Moreover, N is closed in C, which is closed in $X_{\mathfrak{F}}$, so N is closed in $X_{\mathfrak{F}}$. Therefore, N is also closed in Y. By the inductive hypothesis, $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$ is isomorphic to a subframe of $\mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})$. Thus, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$ since $\mathsf{max}(\mathfrak{F})$ consists of n elements and $\mathsf{max}(\mathfrak{G})$ has no more than n elements.

Finally, suppose $N \cap Z \neq \varnothing$ and $N \cap Y \neq \varnothing$. By Lemma 5.1(1), $g|_N : N \to \mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$ is an onto interior map. Let r denote a root of $\mathfrak{G}$ and hence a root of $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$. Since $N \cap Z$ and $N \cap Y$ are clopen in N, both $g|_N(N \cap Z)$ and $g|_N(N \cap Y)$ are R-upsets in $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$. Either $r \in g|_N(N \cap Z)$ or $r \in g|_N(N \cap Y)$.

If $r \in g|_N(N \cap Z)$, then $g|_N(N \cap Z) = \mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$, so $g|_{N\cap Z}$ is an interior mapping onto $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$. Since $N \cap Z$ is nowhere dense in the nodec space Z, we have that $N \cap Z$ is discrete, so $\mathrm{mdim}(N \cap Z) = 0$, and hence $\mathrm{depth}(\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})) = 1$ by [3, Thm. 3.6]. Since discrete spaces are irresolvable, $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$ is a singleton by [3, Lem. 5.9]. Thus, $\text{depth}(\mathfrak{G}) = 2$ and $\mathfrak{G} = \mathsf{max}(\mathfrak{G})^r$. Because $\text{depth}(\mathfrak{F}) = m > 2$, $\mathsf{max}(\mathfrak{F})$ consists of n elements, $\text{depth}(\mathfrak{G}) = 2$, and $\mathsf{max}(\mathfrak{G})$ has no more than n elements, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$.

If $r \in g|_N(N \cap Y)$, then $g|_N(N \cap Y) = \mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$, so $g|_{N\cap Y}$ is an interior mapping onto $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$. Since C is closed in $X_\mathfrak{F}$ and N is closed in C, N is closed in $X_\mathfrak{F}$. But Y is also closed in $X_\mathfrak{F}$, giving that $N \cap Y$ is closed in $X_\mathfrak{F}$, and so $N \cap Y$ is closed in Y. By the inductive hypothesis, $\mathfrak{G} \setminus \mathsf{max}(\mathfrak{G})$ is isomorphic to a subframe of $\mathfrak{F} \setminus \mathsf{max}(\mathfrak{F})$. Therefore, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$ since $\mathsf{max}(\mathfrak{F})$ consists of n elements and $\mathsf{max}(\mathfrak{G})$ has no more than n elements.

Consequently, we have shown that $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$ whenever $C \not\subseteq Y$. If the interior of C is nonempty, then $C \not\subseteq Y$ since Y is nowhere dense in $X_\mathfrak{F}$. Thus, $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$ and the proof is complete. □

We conclude this section by the following consequence of Lemma 5.8, which will be utilized in the last section.

Theorem 5.9. *If a finite quasi-chain $\mathfrak{G}$ is an interior image of an open subspace of $X_\mathfrak{F}$, then $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$.*

Proof. Suppose that there exist an open subspace U of $X_\mathfrak{F}$ and an onto interior mapping $g : U \to \mathfrak{G}$. Since g is onto, for each $v \in \mathfrak{G}$, there is $x_v \in g^{-1}(v)$. As $X_\mathfrak{F}$ is a Tychonoff ED-space, $X_\mathfrak{F}$ is zero-dimensional by [14, Thm. 6.2.25]. Therefore, for each $v \in \mathfrak{G}$, there is a clopen subset U_v of $X_\mathfrak{F}$ such that $x_v \in U_v \subseteq U$. Let $C = \bigcup_{v\in\mathfrak{G}} U_v$. Since $\mathfrak{G}$ is finite, C is a clopen subset of $X_\mathfrak{F}$ contained in U. Because C is open in U, $g|_C$ is an interior mapping of C onto $\mathfrak{G}$. Since C is closed in $X_\mathfrak{F}$ and has nonempty interior, it follows from Lemma 5.8 that $\mathfrak{G}$ is isomorphic to a cofinal subframe of $\mathfrak{F}$. Thus, $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$ by Lemma 2.1. □

6. Main results

In this final section we will prove the main results of the paper. Our first result determines the logic of $X_\mathfrak{F}$. The proof utilizes a topological version of Fine's theorem: for a finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$ and a topological space X, we have $X \vDash \chi_\mathfrak{F}$ iff $\mathfrak{F}$ is not an interior image of an open subspace of X [3, Lem. 3.5].

Theorem 6.1. $\mathsf{Log}(X_\mathfrak{F}) = \mathsf{Log}(\mathfrak{F})$.

Proof. By Theorem 5.3, $\mathfrak{F}$ is an interior image of $X_\mathfrak{F}$. Therefore, since interior images preserve validity, $\mathsf{Log}(X_\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F})$. For the reverse inclusion, let $\mathfrak{G}$ be a finite quasi-chain. By Fine's theorem [16, §2 Lem. 1], Lemma 2.3, Theorem 5.9, and [3, Lem. 3.5],

$$\begin{aligned} \mathfrak{F} \vDash \chi_\mathfrak{G} \quad &\text{iff} \quad \mathfrak{G} \text{ is not a p-morphic image of a generated subframe of } \mathfrak{F} \\ &\text{iff} \quad \mathfrak{G} \text{ is not a p-morphic image of } \mathfrak{F} \\ &\text{iff} \quad \mathfrak{G} \text{ is not an interior image of an open subspace of } X_\mathfrak{F} \\ &\text{iff} \quad X_\mathfrak{F} \vDash \chi_\mathfrak{G}. \end{aligned}$$

Since $\mathsf{Log}(\mathfrak{F}) = \mathsf{S4.3} + \{\chi_{\mathfrak{G}_1}, \dots, \chi_{\mathfrak{G}_n}\}$, where $\mathsf{min}(\mathcal{Q} \setminus \mathcal{F}_{\mathsf{Log}(\mathfrak{F})}) = \{\mathfrak{G}_1, \dots, \mathfrak{G}_n\}$, we have $\mathfrak{F} \vDash \chi_{\mathfrak{G}_i}$ for each i. Therefore, $X_\mathfrak{F} \vDash \chi_{\mathfrak{G}_i}$ for each i. Thus, $\mathsf{Log}(X_\mathfrak{F}) \vdash \chi_{\mathfrak{G}_i}$ for each i, and so $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(X_\mathfrak{F})$. □

Lemma 6.2. *Let X be a nonempty topological space and $\mathfrak{F}$ be a finite rooted $\mathsf{S4}$-frame. If $\mathfrak{F} \vDash \mathsf{Log}(X)$, then $\mathfrak{F}$ is an interior image of an open subspace of X.*

Proof. Suppose that $\mathfrak{F}$ is not an interior image of an open subspace of X. By [3, Lem. 3.5], $X \vDash \chi_{\mathfrak{F}}$, so $\mathsf{Log}(X) \vdash \chi_{\mathfrak{F}}$. Therefore, since $\mathfrak{F} \vDash \mathsf{Log}(X)$, we have $\mathfrak{F} \vDash \chi_{\mathfrak{F}}$. The obtained contradiction proves that $\mathfrak{F}$ is an interior image of an open subspace of X. □

Theorem 6.3. [Main Theorem] *Let* $\mathsf{L} \supseteq \mathsf{S4.3}$ *be consistent. Then* L *is the logic of a Tychonoff* HED*-space iff* L *is Zemanian.*

Proof. First suppose that L is the logic of a Tychonoff HED-space X. Let $\mathfrak{F} \in \mathcal{F}_{\mathsf{L}}$ be non-uniquely rooted. By Lemma 6.2, $\mathfrak{F}$ is an interior image of an open subspace U of X. Since X is Tychonoff, U is T_1. Therefore, by Lemma 5.2, $\mathfrak{F}^r$ is an interior image of U. Because open subspaces and interior images preserve validity, $\mathfrak{F}^r \in \mathcal{F}_{\mathsf{L}}$. Thus, L is Zemanian.

Conversely, suppose L is Zemanian. If $\mathsf{L} \vdash \mathsf{zem}_0$, then L is the logic of a singleton space X, and hence the logic of a Tychonoff HED-space. Suppose $\mathsf{L} \nvdash \mathsf{zem}_0$. Then $\mathcal{F}_{\mathsf{L}}$ contains a quasi-chain consisting of more than a single point. Therefore, since L is Zemanian, there is $\mathfrak{F} \in \mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\}$. By Lemma 3.6, $\mathsf{L} = \mathsf{Log}(\mathcal{U}_{\mathsf{L}}) \subseteq \mathsf{Log}(\mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\})$. Because $\mathfrak{C}_1$ is a p-morphic image of $\mathfrak{F}$, we have that $\mathfrak{F}$ can refute any formula refuted on $\mathfrak{C}_1$, and hence $\mathsf{Log}(\mathcal{U}_{\mathsf{L}}) \supseteq \mathsf{Log}(\mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\})$. Let X be the topological sum of the $X_{\mathfrak{F}}$ where $\mathfrak{F} \in \mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\}$. Since the logic of a topological sum is the intersection of the logics of the summands, by Theorem 6.1,

$$\begin{aligned} \mathsf{Log}(X) &= \bigcap \{\mathsf{Log}(X_{\mathfrak{F}}) \mid \mathfrak{F} \in \mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\}\} \\ &= \bigcap \{\mathsf{Log}(\mathfrak{F}) \mid \mathfrak{F} \in \mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\}\} = \mathsf{Log}(\mathcal{U}_{\mathsf{L}} \setminus \{\mathfrak{C}_1\}) = \mathsf{Log}(\mathcal{U}_{\mathsf{L}}) = \mathsf{L}. \end{aligned}$$

As each $X_{\mathfrak{F}}$ is a Tychonoff HED-space, X is a Tychonoff HED-space. Thus, L is the logic of a Tychonoff HED-space. □

Remark 6.4.

(1) The Tychonoff HED-space X built in the proof of Theorem 6.3 is countable because in the case when $\mathsf{L} \vdash \mathsf{zem}_0$, X is a singleton; and in the case when $\mathsf{L} \nvdash \mathsf{zem}_0$, since $\mathcal{U}_{\mathsf{L}}$ is countable, X is a countable topological sum of countable spaces, hence X is countable. On the other hand, since a countable Tychonoff ED-space is HED, the only logics above S4.2 that have the countable model property with respect to Tychonoff spaces are Zemanian extensions of S4.3.

(2) Since S4.3 is Zemanian, by Theorem 6.3, S4.3 is the logic of a countable crowded Tychonoff HED-space X. A different construction of such an X was given in [2], where X was constructed as a subspace of the Gleason cover $E(\mathbb{I})$ of the real unit interval $\mathbb{I} = [0, 1]$. The recursive process of [2] constructing X is based on nesting ω copies of $E(\mathbb{I})$ within itself by first selecting a countable ω-resolvable dense subspace X_1 of $E(\mathbb{I})$ such that a homeomorphic copy E_1 of $E(\mathbb{I})$ is contained in $E(\mathbb{I}) \setminus X_1$, then repeating the base step in each E_n giving X_{n+1} and $E_{n+1} \subseteq E_n \setminus X_{n+1}$, and finally setting $X = \bigcup_{n=1}^{\infty} X_n$. Comparing [2] to this paper, we note that the current construction builds 'upwards from the bottom' whereas the previous construction builds 'downwards from the top'. Also, the current construction provides control over the resolvability at each stage, while the previous one does not. On the other hand, the previous construction does not require topological sums.

(3) Instead of nesting ω copies of $E(\mathbb{I})$ within itself we can nest ω copies of $\beta\omega$ within itself as follows. Observe that there is a subspace of $\beta\omega \setminus \omega$ homeomorphic to $\beta\omega$. Let β_n be homeomorphic to $\beta\omega$ and D_n be the isolated points of β_n for $n \geq 1$. Embed β_{n+1} in $\beta_n \setminus D_n$ and set $X = \bigcup_{n=1}^{\infty} D_n$. Then X a countable scattered Tychonoff HED-space, and hence $\mathsf{Log}(X) = \mathsf{Grz.3}$. If we nest only $n+1$ copies of $\beta\omega$ within itself, then the logic of the so obtained X is $\mathsf{Grz.3.Z}_n := \mathsf{Grz.3} + \mathsf{zem}_n$ (note that $\mathsf{Grz.3.Z}_n = \mathsf{Grz.3} + \mathsf{bd}_{n+1}$).

(4) In contrast to (3), the Tychonoff HED-space X built in the proof of Theorem 6.3 for the case when $\mathsf{L} \nvdash \mathsf{zem}_0$ is crowded since $X_{\mathfrak{F}}$ is crowded for each $\mathfrak{F} \in \mathcal{U}_{\mathsf{L}}$ of depth > 1. If the uniquely rooted $\mathfrak{F}$ is such that it has a unique maximal point (and depth$(\mathfrak{F}) > 2$), a slight modification of the construction of 4 can produce a Tychonoff HED-space $X_{\mathfrak{F}}$ in which the isolated points are dense. Let $Y = X_{\mathfrak{F}\setminus\max(\mathfrak{F})}$ be as in the recursive step defining $X_{\mathfrak{F}}$. Up to homeomorphism, Y is a subspace of $\beta\omega \setminus \omega$ (see Figure 7). Identify D with ω and $\mathbf{c}_{E(2^{\mathfrak{c}})}(D)$ with $\beta\omega$. Take $X_{\mathfrak{F}}$ to be the subspace $Y \cup \omega$ of $\beta\omega$. Then the isolated points of $X_{\mathfrak{F}}$ are dense.

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Guram Bezhanishvili: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA, guram@math.nmsu.edu

Nick Bezhanishvili: Institute for Logic, Language and Computation, University of Amsterdam, 1090 GE Amsterdam, The Netherlands, N.Bezhanishvili@uva.nl

Joel Lucero-Bryan: Department of Applied Mathematics and Sciences, Khalifa University, Abu Dhabi, UAE, joel.lucero-bryan@kustar.ac.ae

Jan van Mill: Korteweg-de Vries Institute for Mathematics, University of Amsterdam, 1098 XG Amsterdam, The Netherlands, j.vanMill@uva.nl