MODAL COMPACT HAUSDORFF SPACES

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Abstract. We introduce modal compact Hausdorff spaces as generalizations of modal spaces, and show these are coalgebras for the Vietoris functor on compact Hausdorff spaces. Modal compact regular frames and modal de Vries algebras are introduced as algebraic counterparts of modal compact Hausdorff spaces, and dualities are given for the categories involved. These extend the familiar Isbell and de Vries dualities for compact Hausdorff spaces, as well as the duality between modal spaces and modal algebras. As the first step in the logical treatment of modal compact Hausdorff spaces, a version of Sahlqvist correspondence is given for the positive modal language.

Contents

1. Introduction

A duality between modal algebras and modal spaces (descriptive Kripke frames) is of crucial importance in modal logic. Modal algebras are obtained by extending Boolean algebras with a normal additive unary operation. Modal spaces are Stone spaces equipped with a binary relation satisfying certain conditions. This duality is an extension of the celebrated Stone duality between Boolean algebras and Stone spaces. Every axiomatically defined system of modal logic, via this duality, is sound and complete with respect to modal spaces. In contrast to this, there exist relationally (Kripke) incomplete systems of modal logic. This bridge between modal algebras and modal spaces has been instrumental in investigations of wide range of phenomena in modal logic and led to a resolution of many open problems in the area (see, e.g., [10, 32, 8]).

Modal spaces also admit a coalgebraic representation as coalgebras for the Vietoris functor on the category Stone of Stone spaces and continuous maps [16, 33, 1]. The category of Vietoris coalgebras on Stone is isomorphic to the category of modal spaces and corresponding morphisms (continuous p-morphisms). This isomorphism brings an extra dimension, as well as a host of methods and techniques developed in coalgebra and coalgebraic logic, into modal logic investigations [43]. From the coalgebraic point of view, however, the Vietoris functor, as well as the notion of a Vietoris coalgebra, can be defined in a more general setting of

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2000 *Mathematics Subject Classification.* 54E05; 06D22; 54B20; 06E15; 03B45.

Key words and phrases. Compact Hausdorff space, Vietoris functor, compact regular frame, Isbell duality, de Vries duality, modal algebra, Sahlqvist correspondence.

The work of the second author was partially supported by the EPSRC grants EP/F032102/1 and EP/F031173/1 and by the GNSF grant GNSF/ST08/3-397. The second author would like to thank Drew Moshier, Ian Hodkinson, and Dirk Pattinson for many interesting and encouraging discussions on the topic of this paper.

compact Hausdorff spaces. Stone spaces are compact Hausdorff spaces that, in addition, are zero-dimensional (have a basis of clopen sets). Ubiquitous mathematical structures such as $[0, 1]$ provide examples of compact Hausdorff spaces that are not zero-dimensional. Moreover, there are many logical formalisms (e.g., probabilistic systems) that have as their models structures based on (not necessarily zero-dimensional) compact Hausdorff spaces.

In this paper we study compact Hausdorff analogues of modal spaces that are obtained by extending the Vietoris functor from Stone to the category KHaus of compact Hausdorff spaces and continuous maps. This results in the notion of a modal compact Hausdorff space. This is a structure (X, R), where X is a compact Hausdorff space and R is a continuous relation on X, meaning the corresponding map from X to its Vietoris space is continuous. The category MKHaus of modal compact Hausdorff spaces and continuous p-morphisms is then isomorphic to the category of coalgebras for the Vietoris functor on KHaus.

Apart from their connection to coalgebras, modal compact Hausdorff spaces have an interesting role as part of the wider study of topological spaces with additional binary relations. Examples of such include Nachbin's ordered topological spaces. These are pairs $(X, \leq)$, where X is a topological space and $\leq$ is a quasi-order on X whose graph is closed in the product [38]. The continuity of a relation in the above sense implies its graph is closed, so modal compact Hausdorff spaces with a reflexive and transitive relation fall in scope of Nachbin's theory. Continuous relations on topological spaces also play an important role in logical considerations, in particular, in interpreting quantifiers/modalities in topological spaces [24, 42, 36, 11, 13, 12].

We extend the duality between modal algebras and modal spaces to the setting of modal compact Hausdorff spaces. As noted above, the key part of the duality between modal algebras and modal spaces is Stone duality between Boolean algebras and Stone spaces. The main ingredient of our new duality will be a duality between compact Hausdorff spaces and certain algebraic structures. There exist a number of dualities for compact Hausdorff spaces, including Gelfand-Stone duality via real C^*-algebras [20, 21, 39, 40], Kakutani-Yosida duality via real vector lattices [30, 31, 44], de Vries duality via complete Boolean algebras with proximity [14], Isbell duality via compact regular frames [27, 2, 28], and Jung-Sünderhauf-Moshier duality via proximity lattices [29, 37]. We base our duality on de Vries and Isbell dualities as we view these as the closest to Stone duality. Moreover, many tools and techniques from the duality for modal algebras can be adapted to these settings.

A frame [28] is a complete lattice where finite meets distribute over infinite joins, and a frame homomorphism is a map that preserves finite meets and infinite joins. Primary examples of frames are the lattices of open sets of a topological space. A certain class of frames, the compact regular ones, were shown by Isbell to be exactly the ones that are isomorphic to the open set lattices of compact Hausdorff spaces, leading to a dual equivalence between KHaus and the category KRFrm of compact regular frames and frame homomorphisms.

We define a modal compact regular frame to be a compact regular frame with two modal operators $\Box, \Diamond$ that satisfy the conditions that are used in [28, Ch. III.4] to describe the point-free analogue of the Vietoris functor for KHaus. We note that each of $\Box$ and $\Diamond$ is determined by the other, as is the case for modal algebras. A morphism between modal compact regular frames is a frame homomorphism that preserves the modal operators. We show the resulting category MKRFrm of modal compact regular frames is dually equivalent to MKHaus, thereby extending Isbell duality to the modal setting.

A de Vries algebra [14, 3] is a complete Boolean algebra with additional relation $\prec$, called a proximity, satisfying certain conditions. The motivating example is the complete Boolean algebra of all regular open sets of a compact Hausdorff space, with $U \prec V$ if the closure of U is contained in V. From a de Vries algebra, one contructs a compact Hausdorff space by topologizing its maximal round filters, much as one does in Stone duality. Morphisms between de Vries algebras are functions satisfying certain conditions with respect to the Boolean algebra structure and proximities involved. These form a category DeV of de Vries algebras when equipped with a composition of morphisms $\star$ that, importantly, is different from usual function composition. Then de Vries duality shows DeV is dually equivalent to KHaus.

We define a modal de Vries algebra to be a de Vries algebra with an additional unary operation $\Diamond$ that, in a certain sense, is finitely additive with respect to the proximity relation $\prec$. Modal de Vries morphisms are de Vries morphisms satisfying a condition involving $\prec$ and $\Diamond$ that is similar to, but weaker than, the homomorphism property. With the same composition $\star$ as de Vries algebras, this yields a category MDV of modal de Vries algebras that we show is dually equivalent to MKHaus, thereby extending de Vries duality to the modal setting.

Behavior in MDV is not exactly as one might expect. While modal operators $\Diamond$ do preserve proximity, they need not preserve order. Also, isomorphisms need not be homomorphisms with respect to the modal operators involved, and it is possible to have two different modal de Vries operators on the same de Vries algebra giving isomorphic modal de Vries algebras.

We identify two full subcategories of MDV where the behavior is better. The categories LMDV and UMDV of lower and upper continuous modal de Vries algebras are those where the modal operators can be approximated, respectively, from below and from above. Here, the modal operators are order-preserving and isomorphisms do have the usual homomorphism property. We show each member of MDV is isomorphic to a member of LMDV and to a member of UMDV. So LMDV and UMDV are equivalent to MDV, hence are dually equivalent to MKHaus. Topology lends understanding to the situation. For a modal compact Hausdorff space (X,R), there are two natural ways to define a modal de Vries operator $\Diamond$ on its de Vries algebra of regular open sets, a lower continuous one given by $\mathbf{IC}R^{-1}$ and an upper continuous one given by $\mathbf{I}R^{-1}\mathbf{C}$. Here $\mathbf{I}$ and $\mathbf{C}$ denote topological interior and closure.

It is instructive to consider these extensions of Isbell and de Vries dualities as they apply to the classical setting of modal spaces where we have the familiar modal algebras as algebraic duals. For a modal space $\mathfrak{X}=(X,R)$ with corresponding modal algebra $\mathfrak{B}=(B,\Box,\Diamond)$, the associated modal compact regular frame is the ideal completion of $\mathfrak{B}$ where the modal operators $\Box$ and $\Diamond$ are extended in the usual way; and the lower and upper continuous modal de Vries algebras associated to $\mathfrak{X}$ are given by the MacNeille completion of B, considered as a de Vries algebra, with the modal operator $\Diamond$ given by either the lower or upper extension of the operator on $\mathfrak{B}$.

Finally, we begin development of a logical theory for modal compact Hausdorff spaces $\mathfrak{X}$, modal compact regular frames $\mathcal{L}$, and modal de Vries algebras $\mathfrak{A}$. We restrict to the positive fragment of modal logic using the operators $\Box,\Diamond$. For formulas φ and ψ in this language, we define what it means for each type of structure to satisfy a sequent $\varphi\vdash\psi$, and show that if $\mathfrak{X}$, $\mathcal{L}$, and $\mathfrak{A}$ are related by our dualities, then they satisfy the same sequents $\varphi\vdash\psi$. We also develop a version of Sahlqvist correspondence. Stronger languages, including negation and/or infinite disjunctions, may be more suitable for these structures, but this will have to be undertaken at a future time.

The paper is organized as follows. In Section 2 we recall the standard duality between modal algebras and modal spaces, the Vietoris functor, and the coalgebraic representation of modal spaces. We also introduce modal compact Hausdorff spaces and show MKHaus is isomorphic to the category of coalgebras for the Vietoris functor on KHaus. In Section 3 we recall Isbell duality, introduce modal compact regular frames, develop their basic properties, and show MKRFrm is dually equivalent to MKHaus. In Section 4.1 we recall de Vries duality. In Section 4.2 we introduce modal de Vries algebras and develop their basic properties. In Section 4.3 we consider lower and upper continuous modal de Vries algebras, and show MDV is equivalent to each of its subcategories LMDV and UMDV of lower and upper continuous modal de Vries algebras. In Section 5 we lift de Vries duality to a duality between MKHaus and LMDV, and between MKHaus and UMDV. It follows that MKHaus is also dually equivalent to MDV. In Section 6 we summarize our duality results. We also show how Isbell, de Vries, and modal algebra dualities follow as particular cases of our dualities, and make links in the modal space setting to ideal and MacNeille completions of modal algebras. In Section 7 we provide an interpretation of the positive modal language in our structures and establish a version of Sahlqvist correspondence. Section 8 concludes with a few brief comments on directions for further study.

2. Modal compact Hausdorff spaces

In this section we introduce our primary object of study, modal compact Hausdorff spaces, and show they play the role for the Vietoris functor on compact Hausdorff spaces that modal spaces play for the Vietoris functor on the category of Stone spaces. We begin recalling some basics.

Definition 2.1. (see, e.g., [43]) *Let* C *be a category and let* $\mathcal{T}:\mathsf{C}\to\mathsf{C}$ *be an endofunctor. A* $\mathcal{T}$-coalgebra *is a pair* (X,σ)*, where* $\sigma:X\to\mathcal{T}X$ *is a morphism in* C*. A* morphism *between two coalgebras* (X,σ) *and* (X',σ') *is a morphism* f *in* C *such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \sigma'} \\ \mathcal{T}X & \xrightarrow[\mathcal{T}f]{} & \mathcal{T}X' \end{array}$$

Consider the power set functor $\mathcal{P}$ on the category of sets. Recall that $\mathcal{P}$ maps a set W to its power set $\mathcal{P}(W)$, and a function $f : W \to W'$ to the direct image function $\mathcal{P}f$ given by $\mathcal{P}f(U) = f[U]$ for all $U \subseteq W$.

Definition 2.2. (see, e.g., [8, 10]) *A* Kripke frame *is a pair (W, R), where W is a nonempty set and R is a binary relation on W. For Kripke frames (W, R) and (V, R), a function $f : W \to V$ is a* p-morphism *if (i) wRw' implies $f(w)Rf(w')$ and (ii) $f(w)Rv$ implies there is $w' \in W$ with wRw' and $f(w') = v$.*

The following well-known result (see, e.g., [17, Prop. I.4.14]) is useful when dealing with p-morphisms. For $w \in W$, we recall that $R[w] = \{v \in W : wRv\}$ and $R^{-1}[w] = \{v \in W : vRw\}$; also, for $S \subseteq W$, $R[S] = \{w \in W : R^{-1}[w] \cap S \neq \varnothing\}$ and $R^{-1}[S] = \{w \in W : R[w] \cap S \neq \varnothing\}$.

Lemma 2.3. *For (W, R) and (V, R) Kripke frames and $f : W \to V$, the following are equivalent.*

(1) *f is a p-morphism.*
(2) *For each $A \subseteq W$, we have $f(R[A]) = R[f(A)]$.*
(3) *For each $B \subseteq V$, we have $f^{-1}(R^{-1}[B]) = R^{-1}[f^{-1}(B)]$.*

Kripke frames can naturally be viewed as coalgebras for the power set functor on sets as any relation R on W can be viewed as a function $\rho_R : W \to \mathcal{P}(W)$ that maps a point w to the set $R[w]$. The following is a basic result of the coalgebraic treatment of modal logic (see, e.g., [43, Ex. 9.4]).

Theorem 2.4. *The category of Kripke frames and p-morphisms is isomorphic to the category of $\mathcal{P}$-coalgebras.*

As many modal logics are incomplete with respect to Kripke semantics, there is an obvious need to generalize Kripke semantics in such a way as to yield completeness. This results in the concept of *general frames*, which are triples $(W, R, \mathfrak{B})$, where (W, R) is a Kripke frame and $\mathfrak{B}$ is a Boolean subalgebra of $\mathcal{P}(W)$ closed under the modal operator associated with R. *Descriptive frames* are those general frames, where $\mathfrak{B}$ is the Boolean algebra of clopen sets of a Stone topology on W, and it is well known (see, e.g., [8, 10]) that each modal logic is complete with respect to the semantics of general or descriptive frames. Descriptive frames are equivalent to the modal spaces described below.

Definition 2.5. *A* modal space *is a pair $\mathfrak{X} = (X, R)$ where X is a Stone space and R is a binary relation on X satisfying (i) $R[x]$ is closed for each $x \in X$ and (ii) $R^{-1}[U]$ is clopen for each clopen $U \subseteq X$. Let* MS *be the category of modal spaces and continuous p-morphisms.*

Modal spaces can also be represented as coalgebras, but on the category Stone of Stone spaces and continuous maps. The analogue of the power set functor on the category of Stone spaces is given by the *Vietoris construction*, which may be defined as follows.

Definition 2.6. *For a topological space X and $U \subseteq X$ an open set, consider the sets*

$$\begin{aligned} \Box U &= \{F \subseteq X : F \text{ is closed and } F \subseteq U\} \\ \Diamond U &= \{F \subseteq X : F \text{ is closed and } F \cap U \neq \varnothing\}. \end{aligned}$$

Then the Vietoris space *$\mathcal{V}(X)$ of X is defined to have the closed sets of X as its points, and the collection of all sets $\Box U, \Diamond U$, where $U \subseteq X$ is open, as a subbasis for its topology.*

It is a standard result in topology (see, e.g., [15, p. 380]) that if X is a Stone space, then so is $\mathcal{V}(X)$, and note that if X is a Stone space, then in Definition 2.6 we could take U to be clopen. The Vietoris construction $\mathcal{V}$ extends to a functor $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$, which sends a Stone space X to $\mathcal{V}(X)$ and a continuous map $f : X \to Y$ to $\mathcal{V}(f)$ where $\mathcal{V}(f)(F) = f[F]$ for all closed sets $F \subseteq X$. In considering $\mathcal{V}$-colagebras, note that if R is a relation on X, then $\rho_R : X \to \mathcal{P}(X)$ given by $\rho_R(x) = R[x]$ is a well-defined continuous map from X to $\mathcal{V}(X)$ iff (X, R) is a modal space. This leads to the following theorem.

Theorem 2.7. ([16, 33, 1]) MS *is isomorphic to the category of $\mathcal{V}$-coalgebras on* Stone.

Modal spaces have an algebraic realization that lies at the heart of the algebraic treatment of modal logic.

Definition 2.8. (see, e.g., [8, 10]) *A* modal algebra *is a pair* $\mathfrak{B} = (B, \Diamond)$*, where* B *is a Boolean algebra and* $\Diamond$ *is a unary operation on* B *satisfying (i)* $\Diamond 0 = 0$ *and (ii)* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ *for each* $a, b \in B$*. For modal algebras* $\mathfrak{A} = (A, \Diamond)$ *and* $\mathfrak{B} = (B, \Diamond)$*, a map* $h : A \to B$ *is a* modal homomorphism *if* h *is a Boolean homomorphism and* $h(\Diamond a) = \Diamond h(a)$ *for each* $a \in A$*. Let* MA *be the category of modal algebras and modal homomorphisms.*

Before proceeding, we recall Stone duality, and introduce our notation for the functors involved. For a Boolean algebra B, let $\mathsf{Sp}\, B$ be the Stone space X of B, that is, the space of maximal filters of B topologized by the basis $\{\varphi(a) : a \in B\}$, where $\varphi(a) = \{x \in X : a \in x\}$, and for a Boolean homomorphism h, let $\mathsf{Sp}\, h = h^{-1}$. For a Stone space X, let $\mathsf{Clop}\, X$ be the Boolean algebra of clopen subsets of X, and for a continuous map f, let $\mathsf{Clop}\, f = f^{-1}$. Then Sp and Clop are contravariant functors giving a duality with adjunctions $\varphi : B \to \mathsf{Clop}(\mathsf{Sp}\, B)$ and $\varepsilon : X \to \mathsf{Sp}(\mathsf{Clop}\, X)$ where $\varepsilon(x) = \{U \in \mathsf{Clop}(X) : x \in U\}$.

Theorem 2.9. *The duality between Boolean algebras and Stone spaces lifts to a duality between the categories* MA *of modal algebras and* MS *of modal spaces.*

We do not reproduce the proof of this standard result [8, 10], but recall how the functors Sp and Clop are lifted. For a modal algebra $\mathfrak{B} = (B, \Diamond)$, let $\mathsf{Sp}\, \mathfrak{B} = (X, R)$, where X is the Stone space of B and R is given by xRy iff $\Diamond[y] \subseteq x$, where $\Diamond[y] = \{\Diamond a : a \in y\}$; and for a modal space $\mathfrak{X} = (X, R)$, let $\mathsf{Clop}\, \mathfrak{X} = (\mathsf{Clop}\, X, R^{-1})$. The action of Sp and Clop on morphisms remains as before, and φ, ε remain adjunctions.

We now consider matters in the more general setting of compact Hausdorff spaces.

Theorem 2.10. (see, e.g., [15, p. 244]) *The Vietoris construction yields a functor* $\mathcal{V} : \mathsf{KHaus} \to \mathsf{KHaus}$ *where a continuous map* $f : X \to Y$ *is taken to* $\mathcal{V}(f)$ *with* $\mathcal{V}(f)(F) = f[F]$ *for all closed sets* $F \subseteq X$*.*

It is natural to consider coalgebras for this functor.

Definition 2.11. *For* R *a relation on a compact Hausdorff space* X*, we say* R *is* point closed *if the relational image* $R[x]$ *is a closed set for each* $x \in X$*. We say* R *is* continuous *if it is point closed and the associated map* $\rho_R : X \to \mathcal{V}(X)$ *taking a point* x *to* $R[x]$ *is a continuous map from* X *into its Vietoris space* $\mathcal{V}(X)$*. In other words,* R *is continuous if* (X, ρ_R) *is a Vietoris coalgebra.*

For a subset S of X, we use $-S$ for the complement of S in X.

Proposition 2.12. *A relation* R *on a compact Hausdorff space* X *is continuous iff* R *satisfies the following conditions:*

(1) $R[x]$ *is closed for each* $x \in X$*.*
(2) $R^{-1}[F]$ *is closed for each closed* $F \subseteq X$*.*
(3) $R^{-1}[U]$ *is open for each open* $U \subseteq X$*.*

Proof. The function $\rho_R : X \to \mathcal{V}(X)$ is well defined iff $R[x]$ is closed for each $x \in X$. Recall that the Vietoris space has as a subbasis all sets $\{H : H \subseteq U\}$ and $\{H : H \cap U \neq \varnothing\}$, where U is open. Note $x \in R^{-1}[U]$ iff $R[x] \cap U \neq \varnothing$, so $R^{-1}[U] = \rho_R^{-1}(\{H : H \cap U \neq \varnothing\})$. Also, $x \notin R^{-1}[F]$ iff $R[x] \subseteq -F$, so $-R^{-1}[F] = \rho_R^{-1}(\{H : H \subseteq -F\})$. Therefore, if ρ_R is continuous, items 2 and 3 hold; and if items 2 and 3 hold, then the inverse image of each set in the subbasis is open, so ρ_R is continuous. □

Remark 2.13. It is obvious that if R is continuous, then $R^{-1}[x]$ is closed for each $x \in X$. It is also not difficult to verify that $R[F]$ is closed for each closed $F \subseteq X$ (see the proof of Lemma 7.10.2). However, $R[U]$ may not always be open for an open $U \subseteq X$.

We come now to our key notion, which amounts to a concrete realization of colagebras for the Vietoris functor on compact Hausdorff spaces.

Definition 2.14. *We call a pair* (X, R) *a* modal compact Hausdorff space *(abbreviated:* MKH*-space) if* X *is compact Hausdorff and* R *is a continuous relation on* X*.*

Proposition 2.15. *The collection* MKHaus *of* MKH*-spaces and continuous p-morphisms forms a category under usual function composition, and the isomorphisms in* MKHaus *are the continuous p-morphisms* f *that are homeomorphisms between the underlying spaces and satisfy* xRz *iff* $f(x)Rf(z)$*.*

Proof. Suppose X, Y, and Z are MKH-spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous p-morphisms. Surely $g \circ f$ is continuous. If xRu, then $f(x)Rf(u)$, hence $gf(x)Rgf(u)$. Suppose $gf(x)Rz$. Then there is $y \in Y$ with $f(x)Ry$ and $g(y) = z$, and this gives $u \in X$ with xRu and $f(u) = y$, hence with $gf(u) = z$. So $g \circ f$ is a continuous p-morphism. Clearly the identity map is a continuous p-morphism as well, and so MKHaus forms a category. If f is a continuous p-morphism from X to Y that has an inverse, then surely f is a homeomorphism as its inverse is continuous. But xRz implies $f(x)Rf(z)$, and this implies $f^{-1}f(x)Rf^{-1}f(z)$, hence xRz. □

The proof of the next theorem uses Proposition 2.14 and is similar to the proof of Theorem 2.7, so we omit its proof.

Theorem 2.16. MKHaus *is isomorphic to the category of Vietoris coalgebras on* KHaus.

While this is a primary motivation for our study of MKH-spaces, these are interesting mathematical objects in their own right, related to areas such as ordered topological spaces. In the next several sections we create algebraic equivalents to MKH-spaces, along the lines of the algebraic realization of modal spaces provided by modal algebras. A primary tool will be the following result known as Esakia's Lemma.

Lemma 2.17 (Esakia)**.** *If R is a point-closed relation on a compact Hausdorff space X, then for each down-directed family of closed sets F_i $(i \in I)$ of X we have $R^{-1}[\bigcap_I F_i] = \bigcap_I R^{-1}[F_i]$.*

Proof. That $R^{-1}[\bigcap_I F_i] \subseteq \bigcap_I R^{-1}[F_i]$ is trivial. If $x \notin R^{-1}[\bigcap_I F_i]$, then $R[x]$ is disjoint from $\bigcap_I F_i$. As $R[x]$ and the F_i are closed, compactness gives some finite intersection is empty, so the down-directed assumption gives $R[x]$ is disjoint from some F_i. Therefore, $x \notin R^{-1}[F_i]$ for some $i \in I$, hence $x \notin \bigcap_I R^{-1}[F_i]$. □

3. Modal compact regular frames

In this section we generalize the concept of compact regular frame to that of modal compact regular frame, and extend Isbell duality between compact Hausdorff spaces and compact regular frames to a duality between modal compact Hausdorff spaces and modal compact regular frames.

Definition 3.1. (see, e.g., [28]) *A* frame *L is a complete lattice that satisfies $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$. It is* compact *if whenever $\bigvee S = 1$, there is a finite subset $T \subseteq S$ with $\bigvee T = 1$. A function $f : L \to M$ between frames is a* frame homomorphism *if it preserves finite meets and arbitrary joins.*

The prime example of a frame is the collection ΩX of all open sets of a topological space X, and this frame is compact iff the space X is compact. For $f : X \to Y$ a continuous map between spaces, the map $f^{-1} : \Omega Y \to \Omega X$ is a frame homomorphism. In fact, setting $\Omega f = f^{-1}$ gives a contravariant functor Ω from the category of topological spaces to the category of frames.

Definition 3.2. (see, e.g., [28]) *A* point *of a frame L is a frame homomorphism $p : L \to 2$ to the 2-element frame. The set of points $\mathfrak{p}L$ forms a topological space when topologized by the sets $\varphi(a) = \{p : p(a) = 1\}$ where $a \in L$.*

For a frame homomorphism $h : L \to M$, the map $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ sending a point p of M to the point $p \circ h$ of L is continuous. This gives a contravariant functor $\mathfrak{p}$ from the category of frames to that of topological spaces. Further, there is an adjunction between Ω and $\mathfrak{p}$ given by $\varphi : L \to \Omega\mathfrak{p}L$ and $\varepsilon : X \to \mathfrak{p}\Omega X$ where $\varepsilon(x)$ is the point of ΩX with $\varepsilon(x)(U) = 1$ iff $x \in U$. For further details see [28, Ch. II.1].

Definition 3.3. (see, e.g., [28]) *Suppose L is a frame. For each $a \in L$ there is a largest element of L whose meet with a is zero, called the* pseudocomplement *of a and written $\neg a$. For $a, b \in L$ we say a is* well inside *b and write $a \prec b$ if $\neg a \vee b = 1$. We say L is* regular *if $a = \bigvee\{b : b \prec a\}$ for each $a \in L$.*

For a topological space X we use $\mathbf{I}$ and $\mathbf{C}$ for interior and closure in X. Also recall that , $-A$ denotes the complement of a subset $A \subseteq X$. In the frame ΩX we have $\neg A = \mathbf{I} - A$ for each open $A \subseteq X$. It follows that $A \prec B$ iff $\mathbf{C}A \subseteq B$. If X is compact Hausdorff, then ΩX is a compact regular frame. For the next theorem see, e.g., [27, 2, 28].

Theorem 3.4 (Isbell)**.** *The functors Ω and $\mathfrak{p}$ restrict to provide a dual equivalence between the category* KHaus *of compact Hausdorff spaces and continuous maps and the category* KRFrm *of compact regular frames and frame homomorphisms.*

We will lift this duality to one involving the category of modal compact Hausdorff spaces. We first describe how to enrich the structure of compact regular frames to incorporate modality.

Definition 3.5. *A* modal compact regular frame *(abbreviated:* MKR*-frame) is a triple* $\mathcal{L} = (L, \Box, \Diamond)$ *where* L *is a compact regular frame, and* $\Box, \Diamond$ *are unary operations on* L *satisfying the following conditions.*

(1) $\Box$ *preserves finite meets, so* $\Box 1 = 1$ *and* $\Box(a \wedge b) = \Box a \wedge \Box b$.
(2) $\Diamond$ *preserves finite joins, so* $\Diamond 0 = 0$ *and* $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.
(3) $\Box(a \vee b) \le \Box a \vee \Diamond b$ *and* $\Box a \wedge \Diamond b \le \Diamond(a \wedge b)$.
(4) $\Box, \Diamond$ *preserve directed joins, so* $\Diamond \bigvee S = \bigvee\{\Diamond s : s \in S\}$, $\Box \bigvee S = \bigvee\{\Box s : s \in S\}$ *for any up-directed* S.

Note, as $\Diamond$ preserves finite and directed joins, $\Diamond$ preserves arbitrary joins, however $\Box$ need not preserve either finite joins or arbitrary (directed) meets.

Lemma 3.6. *Let* $\mathcal{L} = (L, \Box, \Diamond)$ *be an* MKR*-frame and* $a, b \in L$. *Then*

(1) $\Diamond a \le \neg\Box\neg a$ *and* $\Box a \le \neg\Diamond\neg a$.
(2) *If* $a \prec b$, *then* $\Diamond a \prec \Diamond b$ *and* $\Box a \prec \Box b$.
(3) *If* $a \prec b$, *then* $\neg\Box\neg a \prec \Diamond b$ *and* $\neg\Diamond\neg a \prec \Box b$.
(4) *If* $a \prec b$, *then* $\Box a \prec \neg\Diamond\neg b$ *and* $\Diamond a \prec \neg\Box\neg b$.

Proof. (1) As $\neg a \wedge a = 0$ we have $\Diamond(\neg a \wedge a) = 0$, and the definition of an MKR-frame gives $\Box\neg a \wedge \Diamond a \le \Diamond(\neg a \wedge a)$, hence $\Diamond a \wedge \Box\neg a = 0$, so $\Diamond a \le \neg\Box\neg a$. Similarly, $\Box a \wedge \Diamond\neg a = 0$, so $\Box a \le \neg\Diamond\neg a$.

(2) Suppose $a \prec b$. Then $\neg a \vee b = 1$, so $\Box(\neg a \vee b) = 1$, and as the definition of an MKR-frame gives $\Box(\neg a \vee b) \le \Box\neg a \vee \Diamond b$, we have $\Box\neg a \vee \Diamond b = 1$. In any frame, $x \le \neg y$ iff $y \le \neg x$, so (1) gives $\Box\neg a \le \neg\Diamond a$. Thus $\neg\Diamond a \vee \Diamond b = 1$, which gives $\Diamond a \prec \Diamond b$. Also, $\Box(\neg a \vee b) = 1$ gives $\Box b \vee \Diamond\neg a = 1$. By (1), $\Diamond\neg a \le \neg\Box a$, so $\Box b \vee \neg\Box a = 1$. Thus $\Box a \prec \Box b$.

(3) In proving the previous item we showed $a \prec b$ implies $\Box\neg a \vee \Diamond b = 1$. Thus $\neg\neg\Box\neg a \vee \Diamond b = 1$, showing $\neg\Box\neg a \prec \Diamond b$. We also showed that $\Diamond\neg a \vee \Box b = 1$, so $\neg\neg\Diamond\neg a \vee \Box b = 1$, showing $\neg\Diamond\neg a \prec \Box b$.

(4) If $a \prec b$, then by (1) and (2), $\Box a \prec \Box b \le \neg\Diamond\neg b$ and $\Diamond a \prec \Diamond b \le \neg\Box\neg b$. The result follows. $\Box$

Remark 3.7. Just as with modal algebras, the operations $\Box$ and $\Diamond$ on an MKR-frame are definable from each other. Using the above lemma, that each element in a compact regular frame is the directed join of the elements way below it, and the fact that $\Box$ and $\Diamond$ preserve directed joins, one can show that $\Diamond b = \bigvee\{\neg\Box\neg a : a \prec b\}$ and $\Box b = \bigvee\{\neg\Diamond\neg a : a \prec b\}$. We have taken both $\Diamond$ and $\Box$ as primitive for a tidier definition.

Definition 3.8. *For* MKR*-frames* $\mathcal{L} = (L, \Box, \Diamond)$ *and* $\mathcal{M} = (M, \Box, \Diamond)$, *an* MKR-morphism *from* $\mathcal{L}$ *to* $\mathcal{M}$ *is a frame homomorphism* $h : L \to M$ *that satisfies* $h(\Box a) = \Box h(a)$ *and* $h(\Diamond a) = \Diamond h(a)$ *for each* $a \in L$. *Let* MKRFrm *be the category whose objects are* MKR*-frames and whose morphisms are* MKR*-morphisms.*

We now describe the lifting of the functors Ω and $\mathfrak{p}$ to provide a duality between MKHaus and MKRFrm.

Definition 3.9. *For* $\mathfrak{X} = (X, R)$ *an* MKH*-space, let* $\Omega\mathfrak{X} = (\Omega X, \Box, \Diamond)$ *where* ΩX *is the frame of open sets of* X *and* $\Box, \Diamond$ *are defined by setting for each open* $U \subseteq X$,

(1) $\Box U = -R^{-1}[-U]$.
(2) $\Diamond U = R^{-1}[U]$.

For $f : \mathfrak{X} \to \mathfrak{Y}$ *a continuous p-morphism, define* $\Omega f : \Omega\mathfrak{Y} \to \Omega\mathfrak{X}$ *by* $\Omega f = f^{-1}$.

Proposition 3.10. $\Omega :$ MKHaus $\to$ MKRFrm *is a functor.*

Proof. For $\mathfrak{X} = (X, R)$ an MKH-space, $\Omega(X)$ is a compact regular frame. For $U \subseteq X$ open, we have $\Box U = -R^{-1}[-U]$ and $\Diamond U = R^{-1}[U]$ are open since the continuity of R provides that the inverse image of an open set is open, and of a closed set is closed. So $\Box$ and $\Diamond$ are unary operations on ΩX. The fact that R^{-1} preserves arbitrary unions shows $\Diamond$ preserves arbitrary joins and $\Box$ preserves finite meets. That $\Box$ preserves directed joins follows from Esakia's Lemma. To show $\Omega\mathfrak{X}$ is an MKR-frame, it remains to verify the third condition of Definition 3.5. Let $x \in \Box(U \cup V)$. Then $R[x] \subseteq U \cup V$. If $x \notin \Box U$, then there is $y \notin U$ with xRy. Then $y \in V$, so $x \in R^{-1}[V] = \Diamond V$. Thus $\Box(U \cup V) \subseteq \Box U \cup \Diamond V$. Suppose $x \in \Box U \cap \Diamond V$. Then $R[x] \subseteq U$ and there is $y \in V$ with xRy. Then $y \in U \cap V$, so $x \in \Diamond(U \cap V)$. This shows $\Omega\mathfrak{X}$ is an MKR-frame.

For $f : \mathfrak{X} \to \mathfrak{Y}$ a continuous p-morphism, $\Omega f = f^{-1} : \Omega X \to \Omega Y$ is a frame homomorphism and $\Omega(g \circ f) = (\Omega f) \circ (\Omega g)$. To show Ωf is an MKR-morphism, we require $f^{-1}(\Box U) = \Box(f^{-1}U)$ and $f^{-1}(\Diamond U) = \Diamond(f^{-1}U)$. As f is a p-morphism, Lemma 2.3 shows R^{-1} and f^{-1} commute, and the result follows. $\Box$

Definition 3.11. *For $\mathcal{L} = (L, \Box, \Diamond)$ an* MKR*-frame, let $\mathfrak{p}\mathcal{L} = (X, R)$ where*

(1) *X is the space of points of L.*
(2) *R is the relation on X defined by pRq iff $q(a) = 1$ implies $p(\Diamond a) = 1$ for all $a \in L$.*

For $h : \mathcal{L} \to \mathcal{M}$ an MKR*-morphism, define $\mathfrak{p}h : \mathfrak{p}\mathcal{M} \to \mathfrak{p}\mathcal{L}$ by $(\mathfrak{p}h)(q) = q \circ h$.*

Lemma 3.12. *Let $\mathcal{L} = (L, \Box, \Diamond)$ be an* MKR*-frame with $\mathfrak{p}\mathcal{L} = (X, R)$. Then*

(1) *pRq iff $q(c_p) = 0$ where $c_p = \bigvee\{b : p(\Diamond b) = 0\}$.*
(2) *$R^{-1}[\varphi a] = \varphi(\Diamond a)$.*
(3) *$-R^{-1}[-\varphi a] = \varphi(\Box a)$.*

Proof. (1) As $\Diamond$ and p preserve arbitrary joins, $p(\Diamond c_p) = 0$, so c_p is largest in $\{b : p(\Diamond b) = 0\}$. Then as pRq iff $p(\Diamond a) = 0$ implies $q(a) = 0$, we have pRq iff $q(c_p) = 0$.

(2) If $p \in R^{-1}[\varphi a]$, then pRq for some q with $q(a) = 1$. The definition of R then implies $p(\Diamond a) = 1$, so $p \in \varphi(\Diamond a)$. If $p \in \varphi(\Diamond a)$, then $p(\Diamond a) = 1$, so $a \nleq c_p$. Then there is a point q with $q(a) = 1$ and $q(c_p) = 0$, so $p \in R^{-1}[\varphi a]$. (3) If $p \in -R^{-1}[-\varphi a]$, then $R[p]$ is disjoint from $-\varphi a$. (1) shows $R[p] = -\varphi c_p$, so $(-\varphi c_p) \cap (-\varphi a) = \varnothing$, giving $c_p \vee a = 1$. By Definition 3.5 we have $1 = \Box(a \vee c_p) \leq \Box a \vee \Diamond c_p$. Then $1 = p(\Box a \vee \Diamond c_p) = p(\Box a) \vee p(\Diamond c_p) = p(\Box a)$ since the construction of c_p provides $p(\Diamond c_p) = 0$. Thus $p \in \varphi(\Box a)$.

Conversely, suppose $p \in \varphi(\Box a)$. As a is the join of the directed set $\{b : b \prec a\}$ and $\Box$ preserves directed joins, we have $\varphi(\Box a) = \bigcup\{\varphi(\Box b) : b \prec a\}$. So there is $b \prec a$ with $p(\Box b) = 1$. Definition 3.5 gives $\Box b \wedge \Diamond\neg b \leq \Diamond(b \wedge \neg b) = 0$, and as $p(\Box b) = 1$, this yields $p(\Diamond\neg b) = 0$. Then if pRq, the definition of R gives $q(\neg b) = 0$, and as $b \prec a$ means $\neg b \vee a = 1$, we have $q(a) = 1$. Thus $R[p]$ is contained in φa, showing $p \in -R^{-1}[-\varphi a]$. □

Proposition 3.13. $\mathfrak{p} : \mathsf{MKRFrm} \to \mathsf{MKHaus}$ *is a functor.*

Proof. Suppose $\mathcal{L} = (L, \Box, \Diamond)$ is an MKR-frame, and $\mathfrak{p}\mathcal{L} = (X, R)$. Then X is a compact Hausdorff space. The conditions for the continuity of R (Proposition 2.12) are given by Lemma 3.12 since the open subsets of X are exactly the φa where $a \in L$.

Suppose $\mathcal{M} = (M, \Box, \Diamond)$ is an MKR-frame with $\mathfrak{p}\mathcal{M} = (Y, R)$ and $h : \mathcal{L} \to \mathcal{M}$ is an MKR-morphism. From Isbell duality $\mathfrak{p}h : Y \to X$ is continuous and $\mathfrak{p}$ preserves composition. It remains to show $\mathfrak{p}h$ is a p-morphism (Definition 2.2). For readability we use f in place of $\mathfrak{p}h$.

We first show if $U \subseteq X$ is open, then $f^{-1}R^{-1}[U] = R^{-1}f^{-1}[U]$. Each open U is φa for some $a \in L$, and it is well known that $f^{-1}[\varphi a] = \varphi(ha)$ for each $a \in L$. Lemma 3.12 shows $f^{-1}R^{-1}[\varphi a] = \varphi(h(\Diamond a))$ and $R^{-1}f^{-1}[\varphi a] = \varphi(\Diamond h(a))$. As h is an MKR-morphism, $h(\Diamond a) = \Diamond h(a)$, and the result follows.

Suppose $f(y')$ does not belong to the closed set $R[f(y)]$. Then there is an open neighborhood U of $f(y')$ disjoint from $R[f(y)]$, so $f(y) \notin R^{-1}[U]$. Then $y \notin f^{-1}R^{-1}[U] = R^{-1}f^{-1}[U]$. But $f(y') \in U$ implies $y' \in f^{-1}[U]$, so $y \notin R^{-1}[y']$. Thus yRy' implies $f(y)Rf(y')$, condition (i) of Definition 2.2.

Suppose x does not belong to $f[R[y]]$. Note that $R[y]$ closed and f a continuous map between compact Hausdorff spaces gives $f[R[y]]$ closed. So there are disjoint open sets U and V with $f[R[y]] \subseteq U$ and $x \in V$. Then $R[y]$ is contained in $f^{-1}[U]$, so is disjoint from $f^{-1}[V]$, and this implies $y \notin R^{-1}f^{-1}[V] = f^{-1}R^{-1}[V]$. Thus $f(y) \notin R^{-1}[x]$. So $f(y)Rx$ implies $x = f(y')$ for some yRy', condition (ii) of Definition 2.2. □

Theorem 3.14. *The functors Ω and $\mathfrak{p}$ provide a dual equivalence between* MKHaus *and* MKRFrm.

Proof. For $\mathfrak{X} = (X, R)$ an MKH-space and $\mathcal{L} = (L, \Box, \Diamond)$ an MKR-frame, define $\varepsilon : \mathfrak{X} \to \mathfrak{p}\,\Omega\,\mathfrak{X}$ by setting $\varepsilon(x)$ to be the point with $\varepsilon(x)(U) = 1$ iff $x \in U$ for each open $U \subseteq X$, and let $\varphi : \mathcal{L} \to \Omega\,\mathfrak{p}\,\mathcal{L}$ be given by $\varphi(a) = \{p : p(a) = 1\}$. From Isbell duality, ε and φ are natural isomorphisms on the level of compact Hausdorff spaces and compact regular frames. It remains to show ε is a continuous p-morphism and φ is an MKR-morphism.

As φ is a frame isomorphism, to show it is an MKR-isomorphism we must show $\varphi(\Box a) = \Box\varphi(a)$ and $\varphi(\Diamond a) = \Diamond\varphi(a)$ for each $a \in L$. This is immediate from the definition of $\Box, \Diamond$ on $\Omega\,\mathfrak{p}\mathcal{L}$ and Lemma 3.12. As ε is known to be a homeomorphism, we must show it is a p-morphism. Assume xRy. Then for each open U with $y \in U$ we have $x \in R^{-1}[U] = \Diamond U$, hence $\varepsilon(y)(U) = 1$ implies $\varepsilon(x)(\Diamond U) = 1$, so $\varepsilon(x)R\varepsilon(y)$. Assume q is a point of ΩX and $\varepsilon(x)Rq$. As ε is a homeomorphism, there is a unique $y \in X$ with $\varepsilon(y) = q$. If $x \not R\, y$, then as $R[x]$ is closed, there is an open neighborhood U of y disjoint from $R[x]$. Then $x \notin R^{-1}[U] = \Diamond U$. This gives $\varepsilon(y)(U) = 1$ and $\varepsilon(x)(\Diamond U) \neq 1$, contradicting $\varepsilon(x)Rq$. □

Remark 3.15. There are connections between MKR-frames and the construction of Vietoris frames of compact regular frames [28, 5]. In fact, MKR-frames are algebras for the Vietoris functor on KRFrm. This is an alternate route to the duality of Theorem 3.14.

4. Modal de Vries algebras

In this section we begin our effort to lift de Vries duality from the setting of compact Hausdorff spaces to modal compact Hausdorff spaces. We introduce the category of modal de Vries algebras, and two of its subcategories that play interesting roles. The extension of de Vries duality is in the following section.

4.1. de Vries duality.

Definition 4.1. *A* de Vries algebra *is a pair* $\mathfrak{A} = (A, \prec)$ *consisting of a complete Boolean algebra* A *and a binary relation* $\prec$ *on* A *satisfying the following.*

(1) $1 \prec 1$.
(2) $a \prec b$ *implies* $a \leq b$.
(3) $a \leq b \prec c \leq d$ *implies* $a \prec d$.
(4) $a \prec b, c$ *implies* $a \prec b \wedge c$.
(5) $a \prec b$ *implies* $\neg b \prec \neg a$.
(6) $a \prec b$ *implies there exists* c *with* $a \prec c \prec b$.
(7) $a \neq 0$ *implies there exists* $b \neq 0$ *with* $b \prec a$.

It follows from the definition that if a is an element of a de Vries algebra, then $a = \bigvee\{b : b \prec a\}$. This is reminiscent of the fact that each element of a compact regular frame is the join of the elements that are well inside it.

Definition 4.2. *Let* $\mathfrak{A} = (A, \prec)$ *and* $\mathfrak{B} = (B, \prec)$ *be de Vries algebras. A map* $\alpha : A \to B$ *is a* de Vries morphism *if*

(1) $\alpha(0) = 0$.
(2) $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$.
(3) $a \prec b$ *implies* $\neg\alpha(\neg a) \prec \alpha(b)$.
(4) $\alpha(a) = \bigvee\{\alpha(b) : b \prec a\}$.

For de Vries morphisms $\alpha : \mathfrak{A} \to \mathfrak{B}$ *and* $\beta : \mathfrak{B} \to \mathfrak{C}$*, define their composite* $(\beta \star \alpha)(a) = \bigvee\{\beta\alpha(b) : b \prec a\}$.

Then $\beta \star \alpha$ is a de Vries morphism, and with this definition of composition the collection DeV of de Vries algebras and de Vries morphisms forms a category with the identity functions serving as identity morphisms. We recall two facts that will be used in later sections. The first is trivial to verify, the second is in [14].

Lemma 4.3. *If* $\beta : \mathfrak{B} \to \mathfrak{C}$ *is a de Vries morphism that preserves arbitrary joins, then for any de Vries morphism* $\alpha : \mathfrak{A} \to \mathfrak{B}$ *we have* $\beta \star \alpha$ *is equal to the function composite* $\beta \circ \alpha$.

Proposition 4.4. *If* $\alpha : \mathfrak{A} \to \mathfrak{B}$ *is a de Vries morphism, then* α *is an isomorphism in* DeV *iff* α *is a Boolean algebra isomorphism and* $x \prec y \Leftrightarrow \alpha(x) \prec \alpha(y)$.

A subset U of a topological space X is *regular open* if $U = \mathbf{IC}U$. It is well known that the set $\mathcal{RO}(X)$ of regular open subsets of X is a complete Boolean algebra where $\bigvee U_i = \mathbf{IC}\bigcup U_i$, $\bigwedge U_i = \mathbf{I}\bigcap U_i$, and $\neg U = \mathbf{I} - U$. Define $\prec$ on $\mathcal{RO}(X)$ by $U \prec V$ iff $\mathbf{C}U \subseteq V$. Then if X is compact Hausdorff, its regular open sets are a basis of its topology and $X^* = (\mathcal{RO}(X), \prec)$ is a de Vries algebra. Moreover, if $f : X \to Y$ is continuous, then $f^* : Y^* \to X^*$ given by $f^*(U) = \mathbf{IC}f^{-1}(U)$ is a de Vries morphism, and so $(-)^* : \mathsf{KHaus} \to \mathsf{DeV}$ is a functor.

Definition 4.5. *For a de Vries algebra* $\mathfrak{A} = (A, \prec)$ *and* $S \subseteq A$ *let* ${\uparrow\!\!\!\!\uparrow} S = \{b : a \prec b \text{ for some } a \in S\}$ *and* ${\downarrow\!\!\!\!\downarrow} S = \{b : b \prec a \text{ for some } a \in S\}$. *Note any* ${\uparrow\!\!\!\!\uparrow} S$ *is a filter of* $\mathfrak{A}$ *and* ${\downarrow\!\!\!\!\downarrow} S$ *is an ideal of* $\mathfrak{A}$. *We call a filter* F *of* $\mathfrak{A}$ *a* round filter *if* $F = {\uparrow\!\!\!\!\uparrow} F$ *and an ideal* I *of* $\mathfrak{A}$ *a* round ideal *if* $I = {\downarrow\!\!\!\!\downarrow} I$. *Maximal round filters are called* ends.

For a de Vries algebra $\mathfrak{A} = (A, \prec)$, let $X = \mathfrak{A}_*$ be the set of its ends, and for $a \in A$ set $\varphi(a) = \{x \in X : a \in x\}$. Then $\{\varphi(a) : a \in A\}$ is a basis of a compact Hausdorff topology on X; the sets $\varphi(a)$ are exactly the regular open sets of this topology; and $\mathbf{C}\varphi(a) \subseteq \varphi(b)$ iff $a \prec b$. If $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a de Vries morphism, then $\alpha_* : \mathfrak{B}_* \to \mathfrak{A}_*$ given by $\alpha_*(x) = {\uparrow\!\!\!\!\uparrow}\alpha^{-1}(x)$ is continuous. Therefore, $(-)_* : \mathsf{DeV} \to \mathsf{KHaus}$ is a functor. Moreover, $\varphi : \mathfrak{A} \to {\mathfrak{A}_*}^*$ is a de Vries isomorphism, and $\varepsilon : X \to {X^*}_*$, given by $\varepsilon(x) = \{U \in \mathcal{RO}(X) : x \in U\}$, is a homeomorphism. Thus, we arrive at the following theorem [14].

Theorem 4.6 (De Vries)**.** DeV *is dually equivalent to* KHaus*.*

4.2. **Modal de Vries algebras.**

Definition 4.7. *Let* $\mathfrak{A} = (A, \prec)$ *be a de Vries algebra. We call* $\Diamond : A \to A$ de Vries additive *if (i)* $\Diamond 0 = 0$ *and (ii)* $a_1 \prec a_2$*,* $b_1 \prec b_2$ *imply* $\Diamond(a_1 \vee b_1) \prec \Diamond a_2 \vee \Diamond b_2$*. A de Vries additive operation on* $\mathfrak{A}$ *is called a* modal de Vries operator*, and a de Vries algebra with a de Vries additive operator is a* modal de Vries algebra *(abbreviated:* MDV*-algebra).*

Proposition 4.8. *If* $\mathfrak{A} = (A, \prec, \Diamond)$ *is an* MDV*-algebra, then* $\Diamond$ *is proximity preserving, meaning* $a \prec b$ *implies* $\Diamond a \prec \Diamond b$ *for each* $a, b \in A$*.*

Proof. If $a \prec b$, then $\Diamond a = \Diamond(a \vee a) \prec \Diamond b \vee \Diamond b = \Diamond b$. □

Example 4.9.
(1) A finitely additive operation on a de Vries algebra need not be de Vries additive. To see this, let $\mathfrak{A} = (A, \prec)$ be a de Vries algebra, $a \in A$, and define $\Diamond$ on A by $\Diamond 0 = 0$ and $\Diamond x = a$ for all $x \neq 0$. Then $\Diamond$ is finitely additive, but it is not de Vries additive unless $a \prec a$.
(2) A de Vries additive operator need not be finitely additive. To see this, consider the de Vries algebra $\mathcal{P}(\omega)$, where $a \prec b$ iff $a \subseteq b$ and either a is finite or b is cofinite. This is isomorphic to the de Vries algebra of regular open sets of the one-point compactification of the natural numbers. Define $\Diamond$ on $\mathcal{P}(\omega)$ by $\Diamond a = 1$ if a is cofinite and $\Diamond a = 0$ otherwise. To see $\Diamond$ is de Vries additive, suppose $a_1 \prec a_2$ and $b_1 \prec b_2$. If either a_2, b_2 is cofinite, then $\Diamond a_2 \vee \Diamond b_2 = 1$. Otherwise, as $a_1 \prec a_2$ and $b_1 \prec b_2$, we must have both a_1, b_1 are finite, so $a_1 \vee b_1$ is finite, hence $\Diamond(a_1 \vee b_1) = 0$. So $\Diamond$ is de Vries additive, but surely it is not finitely additive.
(3) A de Vries additive operator need not be order-preserving. Proceed as in (2), but define $\Diamond a = 1$ if a is cofinite, $\Diamond a = 0$ if a is finite, and define $\Diamond$ in some random but not order-preserving way on the remainder.

The above examples show neither finite additivity nor de Vries additivity imply the other. However, we do have the following.

Proposition 4.10. *If* $\Diamond$ *is a finitely additive, proximity preserving operation on a de Vries algebra, then* $\Diamond$ *is de Vries additive.*

Proof. Suppose $a_1 \prec a_2$ and $b_1 \prec b_2$. Then by finite additivity, $\Diamond(a_1 \vee b_1) = \Diamond a_1 \vee \Diamond b_1$. As $\Diamond$ is proximity preserving, $\Diamond a_1 \prec \Diamond a_2$ and $\Diamond b_1 \prec \Diamond b_2$, hence $\Diamond(a_1 \vee b_1) \prec \Diamond a_2 \vee \Diamond b_2$. □

Definition 4.11. *Let* $\mathfrak{A} = (A, \prec, \Diamond)$ *and* $\mathfrak{B} = (B, \prec, \Diamond)$ *be* MDV*-algebras. We call* $\alpha : A \to B$ *a* modal de Vries morphism *(abbreviated:* MDV*-morphism) if*
(1) α *is a de Vries morphism.*
(2) $a \prec b$ *implies* $\alpha(\Diamond a) \prec \Diamond \alpha(b)$*.*
(3) $a \prec b$ *implies* $\Diamond \alpha(a) \prec \alpha(\Diamond b)$*.*

Proposition 4.12. *For* MDV*-morphisms* $\alpha : \mathfrak{A} \to \mathfrak{B}$*,* $\beta : \mathfrak{B} \to \mathfrak{C}$*, define the composite* $\beta \star \alpha$ *by*

$$(\beta \star \alpha)(a) = \bigvee \{\beta\alpha(b) : b \prec a\}.$$

Then with this definition of composition, the collection MDV *of modal de Vries algebras and modal de Vries morphisms forms a category with the identity functions serving as the identity morphisms.*

Proof. As the $\star$ composite of de Vries morphisms is a de Vries morphism, to show $\beta \star \alpha$ is an MDV-morphism, we must show

$$a \prec b \quad \Rightarrow \quad \Diamond(\beta \star \alpha)(a) \prec (\beta \star \alpha)(\Diamond b).$$
$$a \prec b \quad \Rightarrow \quad (\beta \star \alpha)(\Diamond a) \prec \Diamond(\beta \star \alpha)(b).$$

To establish these, find p, q, r with $a \prec p \prec q \prec r \prec b$. Then

$$\Diamond(\beta \star \alpha)(a) \ \prec \ \Diamond\beta\alpha(p) \ \prec \ \beta(\Diamond\alpha(q)) \ \prec \ \beta\alpha(\Diamond r) \ \prec \ (\beta \star \alpha)(\Diamond b).$$

The first step follows from $x \prec y \Rightarrow (\beta \star \alpha)(x) \prec \beta\alpha(y)$ and the fact that $\Diamond$ preserves proximity; the second follows from the fact that α preserves proximity and β being an MDV-morphism; the third follows from

α being an MDV-morphism and the fact that β preserves proximity; the fourth follows from $\Diamond$ preserving proximity and $x \prec y \Rightarrow \beta\alpha(x) \prec (\beta * \alpha)(y)$. This gives the first formula.

For the second formula,

$$(\beta * \alpha)(\Diamond a) \prec \beta\alpha(\Diamond p) \prec \beta(\Diamond\alpha(q)) \prec \Diamond\beta\alpha(r) \prec \Diamond(\beta * \alpha)(b).$$

Here, the first step follows from the fact that $\Diamond$ preserves proximity and $x \prec y \Rightarrow (\beta * \alpha)(x) \prec \beta\alpha(y)$; the second follows from α being an MDV-morphism and the fact that β preserves proximity; the third follows from the fact that α preserves proximity and β being an MDV-morphism; and the fourth follows from $x \prec y \Rightarrow \beta\alpha(x) \prec (\beta * \alpha)(y)$ and the fact that $\Diamond$ preserves proximity.

This establishes that $\beta * \alpha$ is an MDV-morphism. That $*$ is associative follows as it is associative when applied to de Vries algebra morphisms. The identity map on an MDV-algebra is known to be a de Vries algebra morphism, and it is easily seen that it satisfies the additional conditions to be an MDV-morphism since $\Diamond$ satisfies $a \prec b \Rightarrow \Diamond a \prec \Diamond b$. It follows that the collection of modal de Vries algebras with modal de Vries morphisms forms a category. □

Isomorphisms in the category of de Vries algebras are Boolean isomorphisms α that satisfy $x \prec y$ iff $\alpha(x) \prec \alpha(y)$ [14, Ch. I.5]. So isomorphisms in MDV also have these properties. However, as the following example shows, isomorphisms in MDV need not satisfy $\alpha(\Diamond a) = \Diamond\alpha(a)$. The reason for this is that composition $*$ is not function composition.

Example 4.13. Let $\mathfrak{A} = (\mathcal{P}(\omega), \prec, \Diamond)$ be the MDV-algebra of Example 4.9.2, where $a \prec b$ iff $a \subseteq b$ and either a is finite or b is cofinite, and $\Diamond a = 1$ if a is cofinite and $\Diamond a = 0$ otherwise. Let $\mathfrak{A}' = (\mathcal{P}(\omega), \prec, \Diamond')$, where $\Diamond' a = 1$ if a is cofinite, $\Diamond' a = 0$ if a is finite, and $\Diamond' a = a$ otherwise. It is easy to see that $\mathfrak{A}'$ is an MDV-algebra. Clearly $\mathfrak{A}$ and $\mathfrak{A}'$ have the same de Vries algebra structure, but different modal operators. Let $\alpha : \mathfrak{A} \to \mathfrak{A}'$ be the identity map. Obviously α is a de Vries isomorphism. It is also an MDV-morphism as $a \prec b$ implies either a is finite or b is cofinite, so either $\Diamond a = \Diamond' a = 0$ or $\Diamond b = \Diamond' b = 1$. So $a \prec b$ implies $\alpha(\Diamond a) \prec \Diamond'\alpha(b)$ and $\Diamond\alpha(a) \prec \alpha\Diamond'(b)$. The same argument shows that the identity map $\beta : \mathfrak{A}' \to \mathfrak{A}$ is also an MDV-morphism. Therefore, α and β are inverses of each other, and so are MDV-isomorphisms. On the other hand, $\alpha(\Diamond x) \neq \Diamond'\alpha(x)$ and $\beta(\Diamond' x) \neq \Diamond\beta(x)$ when x is infinite but not cofinite.

4.3. Lower and upper continuity.

Definition 4.14. *Let $\mathfrak{A} = (A, \prec, \Diamond)$ be an* MDV*-algebra. We say*

(1) $\Diamond$ *is* lower continuous *if $\Diamond a = \bigvee\{\Diamond b : b \prec a\}$ for each $a \in A$.*
(2) $\Diamond$ *is* upper continuous *if $\Diamond a = \bigwedge\{\Diamond b : a \prec b\}$ for each $a \in A$.*

We say $\mathfrak{A}$ is lower (upper) continuous *if $\Diamond$ is lower (upper) continuous.*

Proposition 4.15. *Let $\mathfrak{A} = (A, \prec, \Diamond)$ be an* MDV*-algebra.*

(1) *If $\Diamond$ is lower continuous, then $\Diamond$ is order-preserving.*
(2) *If $\Diamond$ is upper continuous, then $\Diamond$ is both order-preserving and finitely additive.*

Proof. (1) Let $a \le b$. As $\Diamond$ is lower continuous, $\Diamond a = \bigvee\{\Diamond c : c \prec a\}$. But $c \prec a$ implies $c \prec b$, and as $\Diamond b = \bigvee\{\Diamond c : c \prec b\}$, we have $\Diamond c \le \Diamond b$. It follows that $\Diamond a \le \Diamond b$.

(2) Let $\Diamond$ be upper continuous. An argument dual to (1) shows $\Diamond$ is order-preserving. So for any $a, b \in A$ we have $\Diamond a \vee \Diamond b \le \Diamond(a \vee b)$. For the other inequality, note by upper continuity we have $\Diamond a \vee \Diamond b = \bigwedge\{\Diamond c : a \prec c\} \vee \bigwedge\{\Diamond d : b \prec d\}$. By infinite distributivity in any complete Boolean algebra, we have $\Diamond a \vee \Diamond b = \bigwedge\{\Diamond c \vee \Diamond d : a \prec c,\ b \prec d\}$. But if $a \prec c$ and $b \prec d$, de Vries additivity of $\Diamond$ gives $\Diamond(a \vee b) \prec \Diamond c \vee \Diamond d$, providing the other inequality. □

We provide several examples to show that lower continuity of $\Diamond$ does not imply finite additivity, and that neither lower nor upper continuity of $\Diamond$ implies finite multiplicity.

Example 4.16.

(1) The MDV-algebra of Example 4.9.2 has a lower continuous operation $\Diamond$ that is not finitely additive.
(2) For any modal algebra $(B, \Diamond)$ whose underlying Boolean algebra is complete, $(B, \le, \Diamond)$ is an MDV-algebra that is both lower and upper continuous. However, $\Diamond$ need not preserve finite meets.

Definition 4.17. *Let* LMDV *be the category of lower continuous modal de Vries algebras and modal de Vries morphisms between them, and let* UMDV *be the category of upper continuous modal de Vries algebras and modal de Vries morphisms between them.*

Clearly LMDV and UMDV are full subcategories of MDV. We next consider the nature of isomorphisms in these categories.

Definition 4.18. *For* MDV*-algebras* $\mathfrak{A} = (A, \prec, \Diamond)$ *and* $\mathfrak{B} = (B, \prec, \Diamond)$*, a set map* $\alpha : A \to B$ *is a* structure-preserving bijection *if it satisfies the following conditions.*

(1) *α is a Boolean isomorphism.*
(2) *$a \prec b$ iff $\alpha(a) \prec \alpha(b)$ for all $a, b \in A$.*
(3) *$\alpha(\Diamond a) = \Diamond\alpha(a)$ for all $a \in A$.*

Note that isomorphisms between de Vries algebras are exactly the set mappings that satisfy items (1) and (2) of Definition 4.18, and Example 4.13 gives an isomorphism between MDV-algebras that does not satisfy item (3) of Definition 4.18.

Proposition 4.19.

(1) *A structure preserving bijection between* MDV*-algebras is an isomorphism in* MDV*.*
(2) *Isomorphisms in* LMDV *are exactly the structure-preserving bijections.*
(3) *Isomorphisms in* UMDV *are exactly the structure-preserving bijections.*

Proof. (1) It is trivial to verify that a structure-preserving bijection α between MDV-algebras is an MDV-morphism, and that the set inverse α^{-1} of a structure-preserving bijection is a structure-preserving bijection, hence also an MDV-morphism. As structure-preserving bijections preserve arbitrary joins, we have $\alpha \star \alpha^{-1} = \alpha \circ \alpha^{-1}$ and $\alpha^{-1} \star \alpha = \alpha^{-1} \circ \alpha$ where $\circ$ is ordinary function composition. Thus, both evaluate to the identity morphisms in MDV, showing they are mutually inverse isomorphisms.

(2) As isomorphisms between de Vries algebras are set mappings satisfying items (1) and (2) of Definition 4.18, it is enough to show that if $\Diamond$ and $\Diamond'$ are two lower continuous de Vries additive operators on the same de Vries algebra $(A, \prec)$ with $\mathrm{id} : (A, \prec, \Diamond) \to (A, \prec, \Diamond')$ an MDV-morphism, then $\Diamond = \Diamond'$. Having id be a de Vries morphism means $a \prec b$ implies $\Diamond a \prec \Diamond' b$ and $\Diamond' a \prec \Diamond b$. Then using lower continuity, $\Diamond b = \bigvee\{\Diamond a : a \prec b\} \le \Diamond' b$ and similarly $\Diamond' b \le \Diamond b$. Thus, $\Diamond = \Diamond'$.

(3) Dual argument to (2). $\square$

This shows that isomorphisms between lower, or upper, continuous MDV-algebras behave like homomorphisms with respect to the modal operators. One might wonder if this property extends to all de Vries morphisms between lower, or upper, continuous MDV-algebras. The following example shows it does not.

Example 4.20. Let $\mathfrak{A} = (\mathcal{P}(\omega), \prec, \Diamond)$ be the MDV-algebra of Example 4.9.2, where $a \prec b$ iff $a \subseteq b$ and either a is finite or b is cofinite, and $\Diamond a = 1$ if a is cofinite and $\Diamond a = 0$ otherwise. As we have already noted, $\mathfrak{A} \in \mathsf{LMDV}$. Let I be a non-principal maximal ideal of $\mathcal{P}(\omega)$, and let $\mathfrak{A}' = (\mathcal{P}(\omega), \le, \Diamond')$, where $\le$ is the inclusion order, $\Diamond' a = 0$ if $a \in I$, and $\Diamond' a = 1$ otherwise. Then $\Diamond'$ is finitely additive and proximity preserving (that is, order-preserving), so $\mathfrak{A}'$ is an MDV-algebra. Also, $\mathfrak{A}' \in \mathsf{LMDV}$ trivially as the proximity on $\mathfrak{A}'$ is $\le$ so each element is proximal to itself. Let $\alpha : \mathfrak{A} \to \mathfrak{A}'$ be the identity map. Then α is an MDV-morphism. To see this, suppose $a \prec b$. Then either a is finite or b is cofinite. We must show (i) $\alpha(\Diamond a) \le \Diamond'\alpha(b)$, which means $\Diamond a \le \Diamond' b$ and (ii) $\Diamond'\alpha(a) \le \alpha(\Diamond b)$, which means $\Diamond' a \le \Diamond b$. If a is finite, then $\Diamond a = \Diamond' a = 0$, so both follow. If b is cofinite, then $\Diamond b = \Diamond' b = 1$, and again both follow. On the other hand, $\alpha(\Diamond a) \ne \Diamond'\alpha(a)$. To see this, note that $\alpha(\Diamond a) = \Diamond'\alpha(a)$ means $\Diamond a = \Diamond' a$, which is clearly not the case.

Modifying $\Diamond$ on $\mathcal{P}(\omega)$ so that $\Diamond a = 0$ if a is finite and $\Diamond a = 1$ otherwise, gives an example of a de Vries morphism α between upper continuous MDV-algebras with $\alpha(\Diamond a) \ne \Diamond'\alpha(a)$.

Theorem 4.21. *Let* $\mathfrak{A} = (A, \prec, \Diamond)$ *be an* MDV*-algebra. Define* $\Diamond_L$ *on* A *by setting*

$$\Diamond_L a = \bigvee\{\Diamond b : b \prec a\} \qquad \textit{for each } a \in A.$$

Then $\mathfrak{A}_L = (A, \prec, \Diamond_L)$ *is a lower continuous* MDV*-algebra, and the identity maps* $i_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}_L$ *and* $j_{\mathfrak{A}} : \mathfrak{A}_L \to \mathfrak{A}$ *are mutually inverse* MDV*-isomorphisms.*

Proof. First we show that $\Diamond_L$ is a modal de Vries operator on $(A, \prec)$. Obviously $\Diamond_L 0 = 0$. Suppose $a_1 \prec a_2$ and $b_1 \prec b_2$. Then there exist $a, b \in A$ such that $a_1 \prec a \prec a_2$ and $b_1 \prec b \prec b_2$. Since $\Diamond$ is proximity preserving, we have $\Diamond_L x \le \Diamond x$ for each $x \in A$. Therefore, as $\Diamond$ is de Vries additive,

$$\Diamond_L(a_1 \vee b_1) \le \Diamond(a_1 \vee b_1) \prec \Diamond a \vee \Diamond b \le \Diamond_L a_2 \vee \Diamond_L b_2. \tag{1}$$

Thus, $\Diamond_L$ is de Vries additive, and so $\mathfrak{A}_L$ is an MDV-algebra. Next we show that $\mathfrak{A}_L$ is lower continuous. For this we must show that $\Diamond_L b = \bigvee\{\Diamond_L a : a \prec b\}$ for each $b \in A$. As $\Diamond_L$ is de Vries additive, $a \prec b$ gives $\Diamond_L a \prec \Diamond_L b$, hence $\Diamond_L a \le \Diamond_L b$. It follows that $\bigvee\{\Diamond_L a : a \prec b\} \le \Diamond_L b$. For the other inequality we recall that $\Diamond_L b = \bigvee\{\Diamond c : c \prec b\}$. Suppose $c \prec b$. Then there exists $a \in A$ such that $c \prec a \prec b$. By definition of $\Diamond_L$, we have $\Diamond c \le \Diamond_L a$. It follows that $\Diamond_L b \le \bigvee\{\Diamond_L a : a \prec b\}$. Thus, $\mathfrak{A}_L$ is a lower continuous MDV-algebra.

Clearly $i_{\mathfrak{A}}$ and $j_{\mathfrak{A}}$ are mutually inverse de Vries morphisms. It remains to show they are MDV-morphisms. For this, we must show $a \prec b$ implies $\Diamond a \prec \Diamond_L b$ and $\Diamond_L a \prec \Diamond b$. If $a \prec b$, then as $\Diamond$ is proximity preserving, $\Diamond a \prec \Diamond b$, and as $\Diamond_L a \le \Diamond a$, we have $\Diamond_L a \prec \Diamond b$. To show $\Diamond a \prec \Diamond_L b$, choose $c \in A$ with $a \prec c \prec b$. By definition of $\Diamond_L$, we have $\Diamond a \le \Diamond_L c$, and as $\Diamond_L$ is proximity preserving, $\Diamond_L c \prec \Diamond_L b$. Thus, $\Diamond a \prec \Diamond_L b$. □

A dual condition holds for upper continuous MDV-algebras.

Theorem 4.22. *Let $\mathfrak{A} = (A, \prec, \Diamond)$ be an* MDV*-algebra. Define $\Diamond_U$ on A by setting*

$$\Diamond_U a = \bigwedge\{\Diamond b : a \prec b\} \quad \text{for each } a \in A.$$

Then $\mathfrak{A}_U = (A, \prec, \Diamond_U)$ is an upper continuous MDV*-algebra, and the identity maps $\mu_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}_U$ and $\nu_{\mathfrak{A}} : \mathfrak{A}_U \to \mathfrak{A}$ are mutually inverse* MDV*-isomorphisms.*

Proof. The proof is dual to that of Theorem 4.21 with the exception of the step corresponding to equation (1). Here we first note $\Diamond x \le \Diamond_U x$ for each $x \in A$. Then for $a_1 \prec a \prec a_2$ and $b_1 \prec b \prec b_2$, we use the definition of $\Diamond_U$ to obtain $\Diamond_U(a_1 \vee b_1) \le \Diamond(a \vee b)$, then use the de Vries additivity of $\Diamond$ to obtain $\Diamond(a \vee b) \prec \Diamond a_2 \vee \Diamond b_2$, and then note $\Diamond a_2 \vee \Diamond b_2 \le \Diamond_U a_2 \vee \Diamond_U b_2$. Thus, $\Diamond_U(a_1 \vee a_2) \prec \Diamond_U a_2 \vee \Diamond_U b_2$, providing de Vries additivity. □

Note that if $\mathfrak{A}$ is a lower continuous MDV-algebra, the definition of lower continuity gives $\mathfrak{A} = \mathfrak{A}_L$, and if $\mathfrak{A}$ is an upper continuous MDV-algebra, then $\mathfrak{A} = \mathfrak{A}_U$. The following is then immediate from [34, p. 92].

Theorem 4.23. *There is an equivalence $L : \mathsf{MDV} \to \mathsf{LMDV}$ where $L\mathfrak{A} = \mathfrak{A}_L$ for each object $\mathfrak{A}$, and there is an equivalence $U : \mathsf{MDV} \to \mathsf{UMDV}$ where $U\mathfrak{A} = \mathfrak{A}_U$ for each object $\mathfrak{A}$.*

While this result might seem counterintuitive, we recall that the categories involved are not concrete categories, as composition of morphisms is not given by function composition. This allows isomorphisms in MDV to be more general than the existence of a structure preserving bijection, and this is precisely what allows each MDV-algebra to be isomorphic to a lower and upper continuous one.

Corollary 4.24. LMDV *and* UMDV *are equivalent to each other.*

Remark 4.25. More can be said about these equivalences. Let I and J be the inclusion functors of LMDV and UMDV into MDV. Then [34, p. 92] shows $(L, I, i, 1)$ is an adjoint equivalence from MDV to LMDV and $(U, J, \mu, 1)$ is an adjoint equivalence from MDV to UMDV.

The restrictions of L and U to LMDV and UMDV provide more than an equivalence, they are inverse isomorphisms. To see this, one shows that for any $\alpha : \mathfrak{A} \to \mathfrak{B}$, that $L\alpha = i_{\mathfrak{B}} \star \alpha \star j_{\mathfrak{A}}$ is the same set mapping as α, as is $U\alpha$. Then Proposition 4.19 shows $\mathfrak{A} = LU\mathfrak{A}$ for each $\mathfrak{A} \in \mathsf{LMDV}$ since $\mathfrak{A}$ and $LU\mathfrak{A}$ are isomorphic via a de Vries morphism that is the identity map as a set mapping. Similarly $\mathfrak{B} = UL\mathfrak{B}$ for each $\mathfrak{B} \in \mathsf{UMDV}$.

5. Lifting de Vries duality

In this section we extend de Vries duality to a duality between MKHaus and MDV.

Lemma 5.1. *If $\mathfrak{A} = (A, \prec, \Diamond)$ is an* MDV*-algebra and x is an end of $\mathfrak{A}$, then $I_x = \{a \in A : \Diamond[\Uparrow a] \notin x\}$ is an ideal of $\mathfrak{A}$.*

Proof. Surely I_x is a downset of $\mathfrak{A}$ and $0 \in I_x$. Suppose $a, b \in I_x$. Then there are a_1, b_1 with $a \prec a_1$, $b \prec b_1$ and $\Diamond a_1 \notin x, \Diamond b_1 \notin x$. Using interpolation, there are a_2, b_2 with $a \prec a_2 \prec a_1$ and $b \prec b_2 \prec b_1$. As $\Diamond$ is proximity preserving, we have $\Diamond a_2 \prec \Diamond a_1$ and $\Diamond b_2 \prec \Diamond b_1$, and this shows $\Uparrow\Diamond[\Uparrow a] \notin x$ and $\Uparrow\Diamond[\Uparrow b] \notin x$. Since $\Uparrow\Diamond[\Uparrow a]$, $\Uparrow\Diamond[\Uparrow b]$ are round filters and x is an end, $\Uparrow\Diamond[\Uparrow a] \cap \Uparrow\Diamond[\Uparrow b] \notin x$. So there is some $c \notin x$ and $a \prec a', b \prec b'$ with $\Diamond a' \prec c$ and $\Diamond b' \prec c$. Use interpolation to find a'', b'' with $a \prec a'' \prec a'$ and $b \prec b'' \prec b'$. Then $a \vee b \prec a'' \vee b''$, and using the de Vries additivity of $\Diamond$ we have $\Diamond(a'' \vee b'') \prec \Diamond a' \vee \Diamond b' \prec c$. It follows that $\Diamond[{\Uparrow}(a \vee b)] \notin x$, hence $a \vee b \in I_x$. □

Theorem 5.2. *Let $\mathfrak{A} = (A, \prec, \Diamond)$ be an* MDV*-algebra and let X be its space of ends. Define $R_\Diamond$ on X by setting $x\, R_\Diamond\, y$ iff $\Diamond[y] \subseteq x$. Then $R_\Diamond$ is point-closed and for any $a \in A$, we have:*

(1) $R_\Diamond^{-1}\varphi(a) = \bigcup\{\varphi(\Diamond b) : b \prec a\}$.
(2) $R_\Diamond^{-1}\mathbf{C}\phi(a) = \bigcap\{\varphi(\Diamond b) : a \prec b\} = \bigcap\{\mathbf{C}\varphi(\Diamond b) : a \prec b\}$.

Consequently, $\mathfrak{A}_* = (X, R_\Diamond)$ is an MKH*-space.*

Proof. To see $R_\Diamond$ is point-closed, suppose $x \in X$ and $y \notin R_\Diamond[x]$. Then $\Diamond[y] \notin x$, so there is $a \in y$ with $\Diamond a \notin x$. Then y is in the basic open set $\varphi(a)$, and if $z \in \varphi(a)$, we have $a \in z$, so $\Diamond[z] \notin x$, hence $z \notin R_\Diamond[x]$. So $\varphi(a)$ is an open neighborhood of y disjoint from $R_\Diamond[x]$, showing $R_\Diamond[x]$ is closed.

Claim 5.3. *For any $a, b \in A$:*

(i) $R_\Diamond^{-1}\varphi(a) \subseteq \varphi(\Diamond a)$.
(ii) *If $b \prec a$, then $\varphi(\Diamond b) \subseteq R_\Diamond^{-1}\varphi(a)$.*

Proof of claim. (i) If $x \in R_\Diamond^{-1}\varphi(a)$, then there is some $y \in \varphi(a)$ with $x\, R_\Diamond\, y$. So $a \in y$ and $\Diamond[y] \subseteq x$, so $\Diamond a \in x$. Thus, $x \in \varphi(\Diamond a)$.

(ii) Let $x \in \varphi(\Diamond b)$ so $\Diamond b \in x$. Consider the ideal I_x of Lemma 5.1. As $\Diamond b \in x$, the definition of I_x gives $b \notin I_x$. Therefore, there is an ultrafilter u of $\mathfrak{A}$ with $b \in u$ and $u \cap I_x = \varnothing$. Set $y = {\Uparrow}u$. Then y is an end of $\mathfrak{A}$, and as $b \prec a$ and $b \in u$, we have $a \in {\Uparrow}u = y$. So $y \in \varphi(a)$. Suppose $d \in y$. Then there is $e \in u$ with $e \prec d$. As u is disjoint from I_x, we have $e \notin I_x$ so $\Diamond[{\Uparrow}e] \subseteq x$. This shows $\Diamond d \in x$. So we have $\Diamond[y] \subseteq x$, hence $xR_\Diamond\, y$. Therefore, $x \in R_\Diamond^{-1}\varphi(a)$. □

Continuing the proof of Theorem 5.2. (1) Since $\varphi(a) = \bigcup\{\varphi(b) : b \prec a\}$ and $R_\Diamond^{-1}$ commutes with arbitrary unions, $R_\Diamond^{-1}\varphi(a) = \bigcup\{R_\Diamond^{-1}\varphi(b) : b \prec a\}$. But for $b \prec a$, Claim 5.3 gives $R_\Diamond^{-1}\varphi(b) \subseteq \varphi(\Diamond b) \subseteq R_\Diamond^{-1}\varphi(a)$, giving $R_\Diamond^{-1}\varphi(a) = \bigcup\{\varphi(\Diamond b) : b \prec a\}$.

(2) If $a \prec b$, then $\mathbf{C}\varphi(a) \subseteq \varphi(b)$. So $R_\Diamond^{-1}\mathbf{C}\varphi(a) \subseteq R_\Diamond^{-1}\varphi(b)$, and the first part of Claim 5.3 shows $R_\Diamond^{-1}\varphi(b) \subseteq \varphi(\Diamond b)$. So $R_\Diamond^{-1}\mathbf{C}\varphi(a) \subseteq \bigcap\{\varphi(\Diamond b) : a \prec b\}$. For the other containment, since $\mathbf{C}\varphi(a) = \bigcap\{\varphi(d) : a \prec d\}$, where the intersection is down-directed, and as $R_\Diamond$ is point-closed, we may apply Esakia's Lemma to obtain $R_\Diamond^{-1}\mathbf{C}\varphi(a) = \bigcap\{R_\Diamond^{-1}\varphi(d) : a \prec d\}$. If $a \prec d$, then by interpolation there is b with $a \prec b \prec d$. By the second part of Claim 5.3, $\varphi(\Diamond b) \subseteq R_\Diamond^{-1}\varphi(d)$. This establishes the other containment, giving $R_\Diamond^{-1}\mathbf{C}\varphi(a) = \bigcap\{\varphi(\Diamond b) : a \prec b\}$.

For the second equality in (2), clearly $\bigcap\{\varphi(\Diamond b) : a \prec b\} \subseteq \bigcap\{\mathbf{C}\varphi(\Diamond b) : a \prec b\}$. For the other containment, suppose $a \prec b$. Use interpolation to find d with $a \prec d \prec b$. Then as $\Diamond$ is proximity preserving, $\Diamond d \prec \Diamond b$, giving $\mathbf{C}\varphi(\Diamond d) \subseteq \varphi(\Diamond b)$.

We already saw that $R_\Diamond$ is point-closed. Let $U \subseteq X$ be open. Then $U = \bigcup\{\varphi(a) : \varphi(a) \subseteq U\}$, so $R_\Diamond^{-1}U = \bigcup\{R_\Diamond^{-1}\varphi(a) : \varphi(a) \subseteq U\}$. By (1), each $R_\Diamond^{-1}\varphi(a)$ is open, implying that $R_\Diamond^{-1}U$ is open. Let $F \subseteq X$ be closed. Then $F = \bigcap\{\mathbf{C}\varphi(a) : F \subseteq \mathbf{C}\varphi(a)\}$. As this is a down-directed intersection, by Esakia's Lemma, $R_\Diamond^{-1}F = \bigcap\{R_\Diamond^{-1}\mathbf{C}\varphi(a) : F \subseteq \mathbf{C}\varphi(a)\}$. By (2), each $R_\Diamond^{-1}\mathbf{C}\varphi(a)$ is closed. So $R_\Diamond^{-1}F$ is closed. Thus, $\mathfrak{A}_*$ is an MKH-space. □

Theorem 5.4. *Let $\mathfrak{A}, \mathfrak{B}$ be* MDV*-algebras and $\alpha : \mathfrak{A} \to \mathfrak{B}$ be an* MDV*-morphism. Define $\alpha_* : \mathfrak{B}_* \to \mathfrak{A}_*$ by $\alpha_*(x) = {\Uparrow}\alpha^{-1}(x)$. Then α_* is a continuous p-morphism.*

Proof. By de Vries duality, α_* is a well-defined continuous map. To show α_* is a p-morphism, we must show $xR_\Diamond z$ implies $\alpha_*(x)R_\Diamond\alpha_*(z)$, and $\alpha_*(x)R_\Diamond y$ implies there exits $z \in \mathfrak{B}_*$ with $xR_\Diamond z$ and $\alpha_*(z) = y$. Applying the definitions of the terms involved, we must show:

$$\Diamond[z] \subseteq x \quad\Rightarrow\quad \Diamond[{\Uparrow}\alpha^{-1}(z)] \subseteq {\Uparrow}\alpha^{-1}(x).$$
$$\Diamond[y] \subseteq {\Uparrow}\alpha^{-1}(x) \quad\Rightarrow\quad \text{there exists } z \in \mathfrak{B}_* \text{ with } \Diamond[z] \subseteq x \text{ and } {\Uparrow}\alpha^{-1}(z) = y.$$

For the first formula, suppose $\Diamond[z] \subseteq x$ and $a \in {\Uparrow}\alpha^{-1}(z)$. Then there is $b \in \alpha^{-1}(z)$ with $b \prec a$, so there is $c \in A$ with $b \prec c \prec a$. As $b \in \alpha^{-1}(z)$, we have $\alpha(b) \in z$, and as $\Diamond[z] \subseteq x$, we have $\Diamond\alpha(b) \in x$. Since $b \prec c$ and α is an MDV-morphism, $\Diamond\alpha(b) \prec \alpha(\Diamond c)$, so $\alpha(\Diamond c) \in x$. This gives $\Diamond c \in \alpha^{-1}(x)$. But $c \prec a$ gives $\Diamond c \prec \Diamond a$, hence $\Diamond a \in {\Uparrow}\alpha^{-1}(x)$. We have shown $\Diamond[{\Uparrow}\alpha^{-1}(z)] \subseteq {\Uparrow}\alpha^{-1}(x)$.

For the second formula, suppose x is an end of $\mathfrak{B}$ and y is an end of $\mathfrak{A}$ with $\Diamond[y] \subseteq \Uparrow\alpha^{-1}(x)$. We must show there exists $z \in \mathfrak{B}_*$ with $\Diamond[z] \subseteq x$ and $\Uparrow\alpha^{-1}(z) = y$. We begin with two claims. Let $\alpha(y) = \{\alpha(a) : a \in y\}$ and $\uparrow\alpha(y)$ be the upset of $\alpha(y)$.

Claim 5.5. *$\uparrow\alpha(y)$ is a round filter of $\mathfrak{B}$ and $\Diamond[\uparrow\alpha(y)] \subseteq x$.*

Proof of claim. If $a, b \in y$, then as y is a round filter of $\mathfrak{A}$, there is $c \in y$ with $c \prec a, b$. Then as α is proximity preserving, $\alpha(c) \prec \alpha(a), \alpha(b)$. From this it follows that $\uparrow\alpha(y)$ is a round filter of $\mathfrak{B}$. Next, let $a \in \uparrow\alpha(y)$. Then there exist $b, c, d \in y$ with $b \prec c \prec d$ and $\alpha(d) \le a$. As α is an MDV-morphism, we have $\alpha(\Diamond b) \prec \Diamond\alpha(c)$ and $\alpha(c) \prec \alpha(d) \le a$. As $\Diamond$ is proximity preserving, $\Diamond\alpha(c) \prec \Diamond a$. Thus, $\alpha(\Diamond b) \prec \Diamond a$. Since $\Diamond b \in \Diamond[y] \subseteq \Uparrow\alpha^{-1}(x)$, there is $e \in \alpha^{-1}(x)$ with $e \prec \Diamond b$. Therefore, $\alpha(e) \in x$ and $\alpha(e) \prec \alpha(\Diamond b)$, implying $\alpha(\Diamond b) \in x$. Thus, $\Diamond a \in x$, and so $\Diamond[\uparrow\alpha(y)] \subseteq x$. □

Claim 5.6. *For $T = \{a : \Diamond a \notin x\}$ we have $\Downarrow T$ is a round ideal of $\mathfrak{B}$.*

Proof of claim. Note that $\Downarrow T$ is a downset and as $\Diamond 0 = 0$, we have $0 \in \Downarrow T$. Suppose $a, b \in \Downarrow T$. Then there are $a \prec a_1 \prec a_2 \prec a_3$ and $b \prec b_1 \prec b_2 \prec b_3$ with $a_3, b_3 \in T$. So $\Diamond a_3, \Diamond b_3 \notin x$. This gives $\Uparrow\Diamond a_2 \notin x$ and $\Uparrow\Diamond b_2 \notin x$. As x is an end, $\Uparrow\Diamond a_2 \cap \Uparrow\Diamond b_2 \notin x$. It is easy to see that this intersection equals $\Uparrow(\Diamond a_2 \vee \Diamond b_2)$, so $\Uparrow(\Diamond a_2 \vee \Diamond b_2) \notin x$, and this implies $\Diamond a_2 \vee \Diamond b_2 \notin x$. As $\Diamond$ is de Vries additive, we have $\Diamond(a_1 \vee b_1) \prec \Diamond a_2 \vee \Diamond b_2$, and this gives $\Diamond(a_1 \vee b_1) \notin x$. So $a_1 \vee b_1 \in T$, and as $a \vee b \prec a_1 \vee b_1$ we have $a \vee b \in \Downarrow T$. Thus, $\Downarrow T$ is an ideal, and is clearly round. □

Continuing the proof of Theorem 5.4. As $\Diamond[\uparrow\alpha(y)] \subseteq x$, we have $\uparrow\alpha(y)$ is disjoint from T, hence disjoint from the round ideal $\Downarrow T$. As $\uparrow\alpha(y)$ is a (round) filter, we can find an ultrafilter u of $\mathfrak{B}$ that contains $\uparrow\alpha(y)$ and is disjoint from $\Downarrow T$. Let $z = \Uparrow u$. Then z is an end of $\mathfrak{B}$. As $\uparrow\alpha(y)$ is a round filter contained in u, we have $\uparrow\alpha(y)$ is contained in $z = \Uparrow u$. Then $y \subseteq \alpha^{-1}(z)$, and as y is a round filter, $y \subseteq \Uparrow\alpha^{-1}(z)$. But de Vries duality gives $\Uparrow\alpha^{-1}(z)$ is an end of $\mathfrak{A}$, and we assumed y was an end of $\mathfrak{A}$, so the containment $y \subseteq \Uparrow\alpha^{-1}(z)$ gives equality $y = \Uparrow\alpha^{-1}(z)$. It remains only to show $\Diamond[z] \subseteq x$. Suppose $a \in z$ and $\Diamond a \notin x$. Then $a \in T$. As z is round, we can find $b \in z$ with $b \prec a$. Then $b \in \Downarrow T$. But $z \subseteq u$ and u was chosen to be disjoint from $\Downarrow T$, a contradiction. Consequently, $\alpha_* : \mathfrak{B}_* \to \mathfrak{A}_*$ is a continuous p-morphism. □

Theorem 5.7. *There is a functor $(-)_* : \mathsf{MDV} \to \mathsf{MKHaus}$ taking an* MDV*-algebra $\mathfrak{A}$ to the* MKH*-space $\mathfrak{A}_*$ of its ends, and an* MDV*-morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$ to the continuous p-morphism $\alpha_* : \mathfrak{B}_* \to \mathfrak{A}_*$.*

Proof. Theorem 5.2 shows that $(-)_*$ maps objects of MDV to objects of MKHaus and Theorem 5.4 shows $(-)_*$ maps morphisms of MDV to morphisms of MKHaus. As the map α_* produced is exactly the map produced in de Vries duality from the underlying de Vries algebras of $\mathfrak{A}$ and $\mathfrak{B}$, and the rules of composition in MDV and MKHaus are exactly as they are in DeV and KHaus, it follows that $(-)_*$ is compatible with composition and identities, so is indeed a functor. □

Having a functor to go from MDV to MKHaus we consider the other direction. In fact, we construct two functors from MKHaus to MDV; one will land in LMDV and the other in UMDV.

Theorem 5.8. *Let $\mathfrak{X} = (X, R)$ be an* MKH*-space, and define unary operations $\Diamond^L$ and $\Diamond^U$ on the de Vries algebra $\mathcal{RO}(X)$ of regular open sets of X by*

$$\begin{aligned} \Diamond^L S &= \mathbf{IC}R^{-1}(S), \\ \Diamond^U S &= \mathbf{I}R^{-1}\mathbf{C}(S). \end{aligned}$$

Then $\mathfrak{X}^L = (\mathcal{RO}(X), \prec, \Diamond^L)$ is a lower continuous MDV*-algebra and $\mathfrak{X}^U = (\mathcal{RO}(X), \prec, \Diamond^U)$ is an upper continuous* MDV*-algebra.*

Proof. Since the interior of the closure of any set is regular open, $\Diamond^L$ is well-defined. As $\mathfrak{X}$ is an MKH-space, $R^{-1}\mathbf{C}A$ is closed, implying that $\Diamond^U$ is also well-defined.

Clearly $\Diamond^L\varnothing = \varnothing$. Suppose $A_1 \prec A_2$ and $B_1 \prec B_2$, so $\mathbf{C}A_1 \subseteq A_2$ and $\mathbf{C}B_1 \subseteq B_2$. To show $\Diamond^L$ is de Vries additive, we must show $\mathbf{C}\Diamond^L(A_1 \vee B_1) \subseteq \Diamond^L A_2 \vee \Diamond^L B_2$.

$$\begin{aligned}
\mathbf{C}\Diamond^L(A_1 \vee B_1) &= \mathbf{CIC}R^{-1}\mathbf{IC}(A_1 \cup B_1)\\
&\subseteq \mathbf{C}R^{-1}\mathbf{C}(A_1 \cup B_1)\\
&= R^{-1}\mathbf{C}A_1 \cup \mathbf{R}^{-1}\mathbf{C}B_1\\
&\subseteq R^{-1}A_2 \cup R^{-1}B_2\\
&\subseteq \mathbf{IC}(\mathbf{IC}R^{-1}A_2 \cup \mathbf{IC}R^{-1}B_2)\\
&= \Diamond^L A_2 \vee \Diamond^L B_2.
\end{aligned}$$

The first step is from the definitions of $\Diamond^L$ and the join in $\mathcal{RO}(X)$; the second is obvious; the third as R^{-1} of a closed set is closed and both R^{-1} and $\mathbf{C}$ distribute over finite unions; the fourth as $A_1 \prec A_2$ and $B_1 \prec B_2$; the fifth as R^{-1} of an open set is open and $\mathbf{IC}$ is increasing on open sets; and the final step is from the definitions of $\Diamond^L$ and the join in $\mathcal{RO}(X)$.

We show $\Diamond^L$ is lower continuous. As $\Diamond^L$ is de Vries additive, it is proximity preserving, so $\Diamond^L A \geq \bigvee\{\Diamond^L B : B \prec A\}$. For the other inequality, note that $A = \bigcup\{B : B \prec A\}$, so $R^{-1}A = \bigcup\{R^{-1}B : B \prec A\}$. For B open, the continuity of R gives $R^{-1}B = \mathbf{I}R^{-1}B \subseteq \mathbf{IC}R^{-1}B = \Diamond^L B$. It then follows that $R^{-1}A \subseteq \bigcup\{\Diamond^L B : B \prec A\}$. Thus $\mathbf{IC}R^{-1}A \subseteq \mathbf{IC}\bigcup\{\Diamond^L B : B \prec A\}$, showing $\Diamond^L A \leq \bigvee\{\Diamond^L B : B \prec A\}$. This shows $\mathfrak{X}^L = (\mathcal{RO}(X), \prec, \Diamond^L)$ is a lower continuous MDV-algebra.

We next show $\Diamond^U$ is de Vries additive. Surely $\Diamond^U\varnothing = \varnothing$. Suppose we have regular open sets $A_1 \prec A_2$ and $B_1 \prec B_2$, so $\mathbf{C}A_1 \subseteq A_2$ and $\mathbf{C}B_1 \subseteq B_2$. We show $\mathbf{C}\Diamond^U(A_1 \vee B_1) \subseteq \Diamond^U A_2 \vee \Diamond^U B_2$.

$$\begin{aligned}
\mathbf{C}\Diamond^U(A_1 \vee B_1) &= \mathbf{CI}R^{-1}\mathbf{CIC}(A_1 \cup B_1)\\
&\subseteq \mathbf{C}R^{-1}\mathbf{C}(A_1 \cup B_1)\\
&= R^{-1}\mathbf{C}A_1 \cup R^{-1}\mathbf{C}B_1\\
&\subseteq R^{-1}A_2 \cup R^{-1}B_2\\
&\subseteq \mathbf{IC}(\mathbf{I}R^{-1}\mathbf{C}A_2 \cup \mathbf{I}R^{-1}\mathbf{C}B_2)\\
&= \Diamond^U A_2 \vee \Diamond^U B_2.
\end{aligned}$$

The first step is from the definitions of $\Diamond^U$ and the join in $\mathcal{RO}(X)$; the second is obvious; the third as R^{-1} of a closed set is closed and both R^{-1} and $\mathbf{C}$ distribute over finite unions; the fourth as $A_1 \prec A_2$ and $B_1 \prec B_2$; for the fifth, as $R^{-1}A_2$ and $R^{-1}B_2$ are open, they are contained in $\mathbf{I}R^{-1}\mathbf{C}A_2$ and $\mathbf{I}R^{-1}\mathbf{C}B_2$, and then we use that $\mathbf{IC}$ is increasing on open sets; the final step is from the definitions of $\Diamond^U$ and the join in $\mathcal{RO}(X)$.

To see $\Diamond^U$ is upper continuous, we must show $\Diamond^U A = \bigwedge\{\Diamond^U B : A \prec B\}$. As $\Diamond^U$ is de Vries additive, it is proximity preserving, so $\Diamond^U A \subseteq \bigwedge\{\Diamond^U B : A \prec B\}$. For the other containment, it is enough to show

$$R^{-1}\mathbf{C}A \supseteq \bigcap\{\mathbf{I}R^{-1}\mathbf{C}D : A \prec D\}.$$

For this, note $\mathbf{C}A = \bigcap\{B : A \prec B\} = \bigcap\{\mathbf{C}B : A \prec B\}$. Therefore, $R^{-1}\mathbf{C}A = R^{-1}\bigcap\{\mathbf{C}B : A \prec B\}$. As this intersection is down-directed, by Esakia's Lemma, $R^{-1}\mathbf{C}A = \bigcap\{R^{-1}\mathbf{C}B : A \prec B\}$. Consequently, $R^{-1}\mathbf{C}A = \bigcap\{R^{-1}B : A \prec B\}$. Suppose $x \in \bigcap\{\mathbf{I}R^{-1}\mathbf{C}D : A \prec D\}$ and that $A \prec B$. Then there is D with $A \prec D \prec B$. So $x \in \mathbf{I}R^{-1}\mathbf{C}D$. As $D \prec B$ we have $\mathbf{C}D \subseteq B$, so $x \in \mathbf{I}R^{-1}B$, hence $x \in R^{-1}B$. As this holds for each B with $A \prec B$, we have $x \in R^{-1}\mathbf{C}A$. This shows $\mathfrak{X}^U = (\mathcal{RO}(X), \prec, \Diamond^U)$ is an upper continuous MDV-algebra. □

Theorem 5.9. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be* MKH*-spaces and let $f : X \to Y$ be a continuous p-morphism. Define $f^* : \mathcal{RO}(Y) \to \mathcal{RO}(X)$ by setting $f^*(S) = \mathbf{IC}f^{-1}(S)$ for each $S \in \mathcal{RO}(Y)$.*

1. *$f^* : \mathfrak{Y}^L \to \mathfrak{X}^L$ is an* MDV*-morphism.*
2. *$f^* : \mathfrak{Y}^U \to \mathfrak{X}^U$ is an* MDV*-morphism.*

Proof. It follows from de Vries duality that f^* is a de Vries morphism. We have to show f^* satisfies the modal de Vries morphism conditions for compatibility with the operations involved. To verify the conditions for f^* to be an MDV-morphism from $\mathfrak{Y}^L$ to $\mathfrak{X}^L$, we must show that for regular open subsets A, B of Y, that $A \prec B$ implies $f^*(\Diamond^L A) \prec \Diamond^L f^*(B)$ and $\Diamond^L f^*(A) \prec f^*(\Diamond^L B)$. Upon substituting the definitions involved, we must show:

$$\mathbf{C}A \subseteq B \quad \Rightarrow \quad \mathbf{CIC}f^{-1}\mathbf{IC}R^{-1}A \subseteq \mathbf{IC}R^{-1}\mathbf{IC}f^{-1}B.$$
$$\mathbf{C}A \subseteq B \quad \Rightarrow \quad \mathbf{CIC}R^{-1}\mathbf{IC}f^{-1}A \subseteq \mathbf{IC}f^{-1}\mathbf{IC}R^{-1}B.$$

For the first formula we have:

$$\begin{aligned} \mathbf{CIC}f^{-1}\mathbf{IC}R^{-1}A &= \mathbf{C}f^{-1}\mathbf{IC}R^{-1}A \\ &\subseteq \mathbf{C}f^{-1}R^{-1}\mathbf{C}A \\ &= f^{-1}R^{-1}\mathbf{C}A \\ &= R^{-1}f^{-1}\mathbf{C}A \\ &\subseteq R^{-1}f^{-1}B \\ &= \mathbf{I}R^{-1}\mathbf{I}f^{-1}B \\ &\subseteq \mathbf{IC}R^{-1}\mathbf{IC}f^{-1}B. \end{aligned}$$

The first step is as $\mathbf{CIC}U = \mathbf{C}U$ for any open set U; the second as $\mathbf{IC}R^{-1}A \subseteq \mathbf{C}R^{-1}A \subseteq R^{-1}\mathbf{C}A$, which holds as R^{-1} of a closed set is closed; the third as R^{-1} of a closed set is closed and f^{-1} of a closed set is closed; the fourth by Lemma 2.3; the fifth as $A \prec B$; the sixth as f^{-1} of an open set is open and R^{-1} of an open set is open; and the final step is trivial.

Similarly, for the second formula the reasoning is nearly identical.

$\mathbf{CIC}R^{-1}\mathbf{IC}f^{-1}A = \mathbf{C}R^{-1}\mathbf{IC}f^{-1}A \subseteq \mathbf{C}R^{-1}f^{-1}\mathbf{C}A = R^{-1}f^{-1}\mathbf{C}A = f^{-1}R^{-1}\mathbf{C}A \subseteq f^{-1}R^{-1}B = \mathbf{I}f^{-1}\mathbf{I}R^{-1}B \subseteq \mathbf{IC}f^{-1}\mathbf{IC}R^{-1}B$. This establishes (1), that $f^* : \mathfrak{Y}^L \to \mathfrak{X}^L$ is an MDV-morphism.

For (2), to show f^* is an MDV-morphism from $\mathfrak{Y}^U$ to $\mathfrak{X}^U$, we must show that for regular open subsets A, B of Y, that $A \prec B$ implies $f^*(\Diamond^U A) \prec \Diamond^U f^*(B)$ and $\Diamond^U f^*(A) \prec f^*(\Diamond^U B)$. Upon substituting the definitions involved, we must show:

$$\mathbf{C}A \subseteq B \quad \Rightarrow \quad \mathbf{CIC}f^{-1}\mathbf{I}R^{-1}\mathbf{C}A \subseteq \mathbf{I}R^{-1}\mathbf{CIC}f^{-1}B.$$
$$\mathbf{C}A \subseteq B \quad \Rightarrow \quad \mathbf{CI}R^{-1}\mathbf{CIC}f^{-1}A \subseteq \mathbf{IC}f^{-1}\mathbf{I}R^{-1}\mathbf{C}B.$$

For the first formula we have $\mathbf{CIC}f^{-1}\mathbf{I}R^{-1}\mathbf{C}A = \mathbf{C}f^{-1}\mathbf{I}R^{-1}\mathbf{C}A \subseteq \mathbf{C}f^{-1}R^{-1}\mathbf{C}A = f^{-1}R^{-1}\mathbf{C}A = R^{-1}f^{-1}\mathbf{C}A \subseteq R^{-1}f^{-1}B = \mathbf{I}R^{-1}\mathbf{I}f^{-1}B \subseteq \mathbf{I}R^{-1}\mathbf{CIC}f^{-1}B$. The reasoning for the steps involved is similar to that above. For the second formula, by similar reasoning, we have $\mathbf{CI}R^{-1}\mathbf{CIC}f^{-1}A = \mathbf{CI}R^{-1}\mathbf{C}f^{-1}A \subseteq \mathbf{C}R^{-1}\mathbf{C}f^{-1}A \subseteq \mathbf{C}R^{-1}f^{-1}\mathbf{C}A = R^{-1}f^{-1}\mathbf{C}A = f^{-1}R^{-1}\mathbf{C}A \subseteq f^{-1}R^{-1}B = \mathbf{I}f^{-1}\mathbf{I}R^{-1}B \subseteq \mathbf{IC}f^{-1}\mathbf{I}R^{-1}\mathbf{C}B$. This shows f^* is an MDV-morphism from $\mathfrak{Y}^U$ to $\mathfrak{X}^U$. □

Theorem 5.10. *There are functors* $(-)^L : \mathsf{MKHaus} \to \mathsf{LMDV}$ *and* $(-)^U : \mathsf{MKHaus} \to \mathsf{UMDV}$*, where* $(-)^L$ *and* $(-)^U$ *take an* MKH*-space* $\mathfrak{X} = (X, R)$ *to* $\mathfrak{X}^L = (\mathcal{RO}(X), \prec, \Diamond^L)$ *and* $\mathfrak{X}^U = (\mathcal{RO}(X), \prec, \Diamond^U)$*, respectively, and for* $f : X \to Y$ *a continuous p-morphism,* $f^L = f^U = f^* = \mathbf{IC}f^{-1}$.

Proof. Theorem 5.8 shows $(-)^L$ and $(-)^U$ take objects of MKHaus to objects of LMDV and UMDV, respectively. As LMDV and UMDV are defined to be full subcategories of MDV, Theorem 5.9 shows both $(-)^L$ and $(-)^U$ take continuous p-morphisms to MDV-morphisms, hence to morphisms in the categories LMDV and UMDV. Again, the maps produced are exactly the ones produced in de Vries duality for the underlying de Vries algebras, and as the rules of composition involved are the same as those involved in de Vries duality, it follows that these are functors. □

We next show these functors provide dualities between LMDV, UMDV, and MKHaus, hence between MDV and MKHaus. To begin, recall that for a de Vries algebra $\mathfrak{A} = (A, \prec)$, the regular open sets of the space of ends of $\mathfrak{A}$ are exactly the sets $\varphi(a) = \{x : a \in x\}$ for $a \in A$. Further, φ is a de Vries isomorphism between $\mathfrak{A}$ and the de Vries algebra of regular open sets of its space of ends.

Theorem 5.11. *Suppose* $\mathfrak{A} = (A, \prec, \Diamond)$ *is an* MDV*-algebra.*

(1) *If* $\mathfrak{A}$ *is lower continuous, then* φ *is an* MDV*-isomorphism from* $\mathfrak{A}$ *to* ${\mathfrak{A}_*}^L$.
(2) *If* $\mathfrak{A}$ *is upper continuous, then* φ *is an* MDV*-isomorphism from* $\mathfrak{A}$ *to* ${\mathfrak{A}_*}^U$.

Proof. We know that in both cases φ is a de Vries algebra isomorphism. So by Proposition 4.19, to show the first statement we must show that if $\Diamond$ is lower continuous, then $\varphi(\Diamond a) = \Diamond^L \varphi(a)$, which means $\varphi(\Diamond a) =$

$\mathbf{IC}R_\diamond^{-1}\varphi(a)$. Also by Proposition 4.19, to show the second statement, we must show that if $\diamond$ is upper continuous, then $\varphi(\diamond a) = \diamond^U \varphi(a)$, which means $\varphi(\diamond a) = \mathbf{I}R_\diamond^{-1}\mathbf{C}\varphi(a)$.

(1) Suppose $\diamond$ is lower continuous. By Claim 5.3, $R_\diamond^{-1}\varphi(a) \subseteq \varphi(\diamond a)$, and as $\varphi(\diamond a)$ is regular open, $\mathbf{IC}R_\diamond^{-1}\varphi(a) \subseteq \varphi(\diamond a)$. For the other containment it is enough to show $\varphi(\diamond a) \subseteq \mathbf{C}R_\diamond^{-1}\varphi(a)$. As each closed set is the intersection of the regular open sets that contain it, it is sufficient to show that if $\mathbf{C}R_\diamond^{-1}\varphi(a)$ is contained in $\varphi(c)$, then $\varphi(\diamond a)$ is contained in $\varphi(c)$. For such c, making use of the fact that $R_\diamond^{-1}\varphi(a) = \bigcup\{\varphi(\diamond b) : b \prec a\}$ (see Theorem 5.2), we have $\diamond b \le c$ for each $b \prec a$. Then as $\diamond$ is lower continuous, we have $\diamond a = \bigvee\{\diamond b : b \prec a\}$, giving $\diamond a \le c$, hence $\varphi(\diamond a) \subseteq \varphi(c)$ as required.

(2) Suppose $\diamond$ is upper continuous. By Theorem 5.2, $\mathbf{I}R_\diamond^{-1}\mathbf{C}\varphi(a) = \mathbf{I}\bigcap\{\varphi(\diamond b) : a \prec b\}$. If $a \prec b$, then as $\diamond$ is proximity preserving, $\diamond a \prec \diamond b$, so $\varphi(\diamond a) \subseteq \varphi(\diamond b)$, so $\varphi(\diamond a)$ is an open set contained in $\bigcap\{\varphi(\diamond b) : a \prec b\}$, hence $\varphi(\diamond a)$ is contained in $\mathbf{I}\bigcap\{\varphi(\diamond b) : a \prec b\}$. For the other containment, suppose $\varphi(c) \subseteq \bigcap\{\varphi(\diamond b) : a \prec b\}$. Then $c \le \diamond b$ for each b with $a \prec b$. Thus, $c \le \bigwedge\{\diamond b : a \prec b\}$. As $\diamond$ is upper continuous, $\diamond a = \bigwedge\{\diamond b : a \prec b\}$, so $c \le \diamond a$. So $\varphi(\diamond a) = \mathbf{I}\bigcap\{\varphi(\diamond b) : a \prec b\}$ as required. □

Recall that for X a compact Hausdorff space, de Vries duality gives $\varepsilon(x) = \{U \in \mathcal{RO}(X) : x \in U\}$ is an end of $\mathcal{RO}(X)$, and ε is a homeomorphism from X onto the space of ends of $\mathcal{RO}(X)$.

Proposition 5.12. *Let $\mathfrak{X} = (X, R)$ be an* MKH*-space and let $x, y \in X$.*

(1) *xRy iff $\varepsilon(x) R_{\diamond^L} \varepsilon(y)$.*
(2) *xRy iff $\varepsilon(x) R_{\diamond^U} \varepsilon(y)$.*

Proof. Recall $\varepsilon(x) R_{\diamond^L} \varepsilon(y)$ means $\diamond^L[\varepsilon(y)] \subseteq \varepsilon(x)$, and $\varepsilon(x) R_{\diamond^U} \varepsilon(y)$ means $\diamond^U[\varepsilon(y)] \subseteq \varepsilon(x)$. So

$$\varepsilon(x) R_{\diamond^L} \varepsilon(y) \quad \text{iff} \quad y \in A \Rightarrow x \in \mathbf{IC}R^{-1}A,$$
$$\varepsilon(x) R_{\diamond^U} \varepsilon(y) \quad \text{iff} \quad y \in A \Rightarrow x \in \mathbf{I}R^{-1}\mathbf{C}A.$$

In these formulas, A ranges over all regular open sets. As $\mathfrak{X} = (X, R)$ is an MKH-space, the inverse image of a closed set is closed, so for any set A we have $\mathbf{IC}R^{-1}A$ is contained in $\mathbf{I}R^{-1}\mathbf{C}A$. It follows that $\varepsilon(x) R_{\diamond^L} \varepsilon(y)$ implies $\varepsilon(x) R_{\diamond^U} \varepsilon(y)$. So it is enough to show xRy implies $\varepsilon(x) R_{\diamond^L} \varepsilon(y)$ and that $\varepsilon(x) R_{\diamond^U} \varepsilon(y)$ implies xRy.

Suppose xRy and y belongs to the regular open set A. Then xRy gives $x \in R^{-1}A$. As R is continuous and A is open, $R^{-1}A$ is open, so $x \in \mathbf{I}R^{-1}A$, hence $x \in \mathbf{IC}R^{-1}A$. This shows xRy implies $\varepsilon(x) R_{\diamond^L} \varepsilon(y)$.

Suppose $x \not R\, y$. Then $y \notin R[x]$. As R is continuous, $R[x]$ is closed. As X is a compact Hausdorff space, there are disjoint open sets U, V with $R[x] \subseteq U$ and $y \in V$. As the regular open sets form a basis, there is a regular open set $A \subseteq V$ with $y \in A$. Then $\mathbf{C}A \subseteq -U$, so $\mathbf{I}R^{-1}\mathbf{C}A \subseteq R^{-1} - U$, and as $R[x] \subseteq U$, we have $x \notin \mathbf{I}R^{-1}\mathbf{C}A$. So $x \not R\, y$ implies $\varepsilon(x) \not R_{\diamond^U}\, \varepsilon(y)$. □

Corollary 5.13. *For an* MKH*-space $\mathfrak{X}$, we have ${\mathfrak{X}^L}_*$ is equal to ${\mathfrak{X}^U}_*$, and that the map ε is an isomorphism in* MKHaus *from $\mathfrak{X}$ onto these equal structures.*

Using the above results, we obtain the following.

Theorem 5.14. *The functors $(-)_* :$ LMDV $\to$ MKHaus and $(-)^L :$ MKHaus $\to$ LMDV give a dual equivalence between* LMDV *and* MKHaus*, and the functors $(-)_* :$ UMDV $\to$ MKHaus and $(-)^U :$ MKHaus $\to$ UMDV give a dual equivalence between* UMDV *and* MKHaus*.*

Proof. To show that $(-)_*$ and $(-)^L$ give a dual equivalence, it is enough to show $\varphi : 1 \to (-)^L \circ (-)_*$ and $\varepsilon : 1 \to (-)_* \circ (-)^L$ are natural isomorphisms. By Theorem 5.11 and Corollary 5.13, φ and ε are isomorphisms. It remains to show that for each $\alpha : \mathfrak{A} \to \mathfrak{B}$ in LMDV, that ${\alpha_*}^L \star \varphi_{\mathfrak{A}} = \varphi_{\mathfrak{B}} \star \alpha$; and for each $f : \mathfrak{X} \to \mathfrak{Y}$ in MKHaus, that ${f^L}_* \circ \varepsilon_{\mathfrak{X}} = \varepsilon_{\mathfrak{Y}} \circ f$. This follows as the set mappings involved, and the rules of composition, are exactly those in de Vries duality between de Vries algebras and compact Hausdorff spaces. That $(-)_*, (-)^U$ give a duality between UMDV and MKHaus is similar. □

Corollary 5.15. *The categories* MDV *and* MKHaus *are dually equivalent.*

Proof. Apply Theorems 4.23 and 5.14. □

Corollary 5.16. *The categories* MKRFrm*,* MDV*,* LMDV*, and* UMDV *are dually equivalent to the category of Vietoris coalgebras on* KHaus*.*

Proof. Apply Theorems 2.16, 3.14, 5.14 and Corollary 5.15. □

6. Summary of the dualities

In this section we collect our duality results and describe how they can be viewed as extensions of Isbell and de Vries dualities. We also consider their restrictions to the zero-dimensional case, which implies the standard duality between modal algebras and modal spaces, and show these have links to ideal and MacNeille completions of modal algebras. We begin with Figure 1 that summarizes the results we have so far obtained. For readability, the identical embeddings of LMDV and UMDV into MDV are not shown on this figure, nor are the composites of these with $(-)_*$. We remark that the dualities involving MKHaus use some version of the axiom of choice. In [5] choice-free equivalences between MKRFrm and both LMDV and UMDV, and hence MDV, are given.

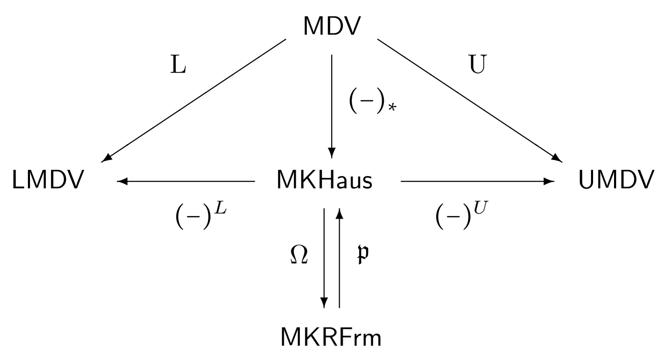

Figure 1

Our results are easily seen to extend the Isbell and de Vries dualities on which they are based. Call an MKR-frame *trivial* if $\Box, \Diamond$ are the identity maps, call an MDV-algebra $\mathfrak{A} = (A, \prec, \Diamond)$ *trivial* if $\Diamond$ is the identity map, and call an MKH-space $\mathfrak{X} = (X, R)$ *trivial* if R is the identity relation. Then the obvious forgetful functors provide isomorphisms from the full subcategories of trivial MKR-frames, trivial MDV-algebras, and trivial MKH-spaces to the categories KRFrm, DeV, and KHaus, respectively. Moreover, the restrictions of the functors above provide the usual functors giving Isbell and de Vries dualities.

Theorem 6.1. *The dualities from* MKRFrm *and* MDV *to* MKHaus *naturally extend the Isbell and de Vries dualities from* KRFrm *and* DeV *to* KHaus.

We next consider our dualities in the zero-dimensional setting. We require some definitions [28, 3].

Definition 6.2. *A frame L is* zero-dimensional *if its complemented elements are join-dense in L. In a de Vries algebra $(A, \prec)$, we say c is* reflexive *if $c \prec c$, and we say $(A, \prec)$ is* zero-dimensional *if $a \prec b$ implies there exists a reflexive $c \in A$ with $a \prec c \prec b$.*

Isbell and de Vries dualities restrict to give dualities between the category zKFrm of compact zero-dimensional frames, the category zDeV of zero-dimensional de Vries algebras, and the category Stone of Stone spaces [28, 3, 4]. Note that any zero-dimensional compact frame is regular, so zKFrm is a full subcategory of KRFrm (see, e.g., [4, Sec. 4]). Defining zMKFrm, zMDV, zLMDV, and zUMDV to be the obvious full subcategories of MKRFrm, MDV, LMDV, and UMDV, and noting that the MKH-spaces whose underlying topologies are zero-dimensional are exactly the modal spaces, our dualities restrict to provide the following.

Theorem 6.3. *The category* MS *is dually equivalent to* zMKFrm*,* zMDV*,* zLMDV*, and* zUMDV*.*

Using Stone duality between Stone and the category BA of Boolean algebras and Boolean homomorphisms, the restrictions of Isbell and de Vries dualities to the zero-dimensional case can be given an algebraic form [4]. The main tools will be the ideal completion $\mathfrak{I}B$ and MacNeille completion $\overline{B}$ of a Boolean algebra B. We view both as complete lattices that contain B. In $\mathfrak{I}B$ each element is the join of the elements of B beneath it, and in $\overline{B}$ each element is both the join of the elements of B beneath it, and the meet of the elements of B above it. Finally, we recall the clopen set functor $\mathsf{Clop} : \mathsf{Stone} \to \mathsf{BA}$ and the Stone space functor $\mathsf{Sp} : \mathsf{BA} \to \mathsf{Stone}$ provide a dual equivalence that lifts to a dual equivalence between MA and MS.

Definition 6.4. *The ideal frame functor* $\mathfrak{I} : \mathsf{BA} \to \mathsf{zKFrm}$ *takes a Boolean algebra to its ideal frame, and a Boolean homomorphism* h *to the frame homomorphism* $\mathfrak{I}h$ *where* $\mathfrak{I}h(x) = \bigvee\{h(a) : a \in B \text{ and } a \leq x\}$*. The complemented element functor* $\mathfrak{C} : \mathsf{zKFrm} \to \mathsf{BA}$ *takes a frame to its Boolean algebra of complemented elements, and a frame homomorphism to its restriction to these complemented elements.*

For a Boolean algebra B, the frame of open sets of its Stone space is isomorphic to the ideal frame of B, and the complemented elements of this ideal frame form a Boolean algebra isomorphic to B. This leads to the results [28] that $\mathfrak{I} \simeq \Omega \circ \mathsf{Sp}$ and $\mathfrak{C} \simeq \mathsf{Clop} \circ \mathfrak{p}$ where $\simeq$ indicates naturally isomorphic functors.

Definition 6.5. *The MacNeille completion functor* $\mathfrak{M} : \mathsf{BA} \to \mathsf{zDeV}$ *sends a Boolean algebra* B *to the zero-dimensional de Vries algebra* $(\overline{B}, \prec)$*, where* $\overline{B}$ *is the MacNeille completion of* B *and* $x \prec y$ *if* $x \leq a \leq y$ *for some* $a \in B$*; and sends a Boolean homomorphism* h *to the de Vries morphism* $\overline{h}$ *where* $\overline{h}(x) = \bigvee\{h(a) : a \in B \text{ and } a \leq x\}$*. The reflexive element functor* $\mathfrak{R} : \mathsf{zDeV} \to \mathsf{BA}$ *sends a zero-dimensional de Vries algebra to its Boolean algebra of reflexive elements, and a de Vries morphism to its restriction to the reflexive elements.*

For a Boolean algebra B, the MacNeille completion of B is isomorphic to the regular open subsets of its Stone space, and for such U and V regular open, we have $\mathbf{C}U \subseteq V$ iff there is clopen C with $U \subseteq C \subseteq V$. Conversely, the regular open sets U with $\mathbf{C}U \subseteq U$ are the clopen ones. This leads to the results [3] that $\mathfrak{M} \simeq (-)^* \circ \mathsf{Sp}$ and $\mathfrak{R} \simeq \mathsf{Clop} \circ (-)_*$ where $(-)^*$ and $(-)_*$ are the restrictions of the functors providing de Vries duality.

Definition 6.6. *For a modal algebra* $\mathfrak{B} = (B, \Diamond)$ *with* $\Box = \neg\Diamond\neg$ *define its* ideal completion $\mathfrak{I}\mathfrak{B} = (\mathfrak{I}B, \boxdot, \diamonddot)$, lower MacNeille completion $\mathfrak{M}^L\mathfrak{B} = (\overline{B}, \prec, \Diamond_L)$*, and* upper MacNeille completion $\mathfrak{M}^U = (\overline{B}, \prec, \Diamond_U)$ *where*

(1) $\boxdot x = \bigvee\{\Box a : a \in B \text{ and } a \leq x\}$ *and* $\diamonddot x = \bigvee\{\Diamond a : a \in B \text{ and } a \leq x\}$.
(2) $\Diamond_L x = \bigvee\{\Diamond a : a \in B \text{ and } a \leq x\}$.
(3) $\Diamond_U x = \bigwedge\{\Diamond a : a \in B \text{ and } x \leq a\}$.

Theorem 6.7. *The ideal frame and complemented element functors lift to functors* $\mathfrak{I} : \mathsf{MA} \to \mathsf{zMKFrm}$ *and* $\mathfrak{C} : \mathsf{zMKFrm} \to \mathsf{MA}$ *with* $\mathfrak{I}$ *taking a modal algebra to its ideal completion and* $\mathfrak{C}$ *taking a zero-dimensional* MKR*-frame to the modal algebra obtained by restricting* $\Diamond$ *to the complemented elements. Further,* $\mathfrak{I} \simeq \Omega \circ \mathsf{Sp}$ *and* $\mathfrak{C} \simeq \mathsf{Clop} \circ \mathfrak{p}$*, so* $\mathfrak{I}$ *and* $\mathfrak{C}$ *give an equivalence between* MA *and* zMKFrm*.*

Proof. The statements for the ideal functor will follow if we show that for a modal algebra $\mathfrak{B} = (B, \Diamond)$ with modal space $\mathfrak{X}$, the modal operators $\boxdot$ and $\diamonddot$ on $\mathfrak{I}B$ are those transferred from $\Omega\mathfrak{X}$ by the frame isomorphism e from ΩX to $\mathfrak{I}B$ sending U to $\bigvee\{a : \varphi(a) \subseteq U\}$. For a clopen set φa of X, it is standard from modal logic that $R^{-1}\varphi a = \varphi\Diamond a$ and $-R^{-1} - \varphi a = \varphi\Box a$, so $e\Box\varphi a = \Box a = \boxdot e\varphi a$ and $e\Diamond\varphi a = \Diamond a = \diamonddot e\varphi a$. The result then follows as in $\Omega\mathfrak{X}$ the actions of $\Box$ and $\Diamond$ on U are the joins of their actions on the φa with $\varphi a \subseteq U$, and in $\mathfrak{I}\mathfrak{B}$ the actions of $\boxdot$ and $\diamonddot$ on eU are the joins of their actions on the a with $\varphi a \subseteq U$.

For the complemented element functor, let $\mathcal{L} = (L, \Box, \Diamond)$ be a zero-dimensional MKR-frame with modal space $\mathfrak{p}\mathcal{L} = (X, R)$. The restriction of φ is a Boolean algebra isomorphism from the complemented elements of L to $\mathsf{Clop}X$. For $a \in L$, Lemma 3.12 gives $\varphi\Box a = -R^{-1} - \varphi a$ and $\varphi\Diamond a = R^{-1}\varphi a$, so if a is complemented, $\Box a$ and $\Diamond a$ are complemented. Therefore, the modal operators of $\mathcal{L}$ restrict to its complemented elements, and the isomorphism φ from these complemented elements to the modal algebra $\mathsf{Clop}\,\mathfrak{X}$ of clopen subsets of $\mathfrak{X}$ is a modal isomorphism. Our statements about $\mathfrak{C}$ follow. $\square$

Corollary 6.8. *The equivalence between* MA *and* zMKFrm*, together with the dual equivalence between* zMKFrm *and* MS*, yields, up to natural isomorphism, the usual duality between* MA *and* MS*.*

Proof. Informally, the duality between MA and MS is obtained from the duality between zMKFrm and MS by restricting to the complemented elements of a zero-dimensional MKR-frame. More formally, for the functors Clop and Sp between MA and MS, the above results show $\mathsf{Clop} \simeq \mathfrak{C} \circ \Omega$ and $\mathsf{Sp} \simeq \mathfrak{p} \circ \mathfrak{I}$. $\square$

Theorem 6.9. *The MacNeille completion and reflexive element functors lift to functors* $\mathfrak{M}^L : \mathsf{MA} \to \mathsf{zLMDV}$ *and* $\mathfrak{R} : \mathsf{zMDV} \to \mathsf{MA}$ *with* $\mathfrak{M}^L$ *taking a modal algebra to its lower MacNeille completion and* $\mathfrak{R}$ *taking a zero-dimensional* MDV*-algebra to the modal algebra obtained by restricting* $\Diamond$ *to its reflexive elements. Further,* $\mathfrak{M}^L \simeq (-)^L \circ \mathsf{Sp}$ *and* $\mathfrak{R} \simeq \mathsf{Clop} \circ (-)_*$*, so* $\mathfrak{M}^L$ *and* $\mathfrak{R}$ *give an equivalence between* MA *and* zLMDV*.*

Proof. For a modal algebra $\mathfrak{B} = (B, \Diamond)$ with modal space $\mathfrak{X}$, we show the operation $\Diamond_L$ of $\mathfrak{M}^L\mathfrak{B}$ is transferred from $\mathfrak{X}^L$ via the de Vries isomorphism α from the regular open sets of X to $\overline{B}$. As we saw in the proof of Theorem 6.7, this is the case for clopen $\varphi(a)$. For arbitrary $U \in \mathfrak{X}^L$, the lower continuity of $\mathfrak{X}^L$ gives $\Diamond^L U = \bigvee\{\Diamond^L V : V \prec U\}$. As X is Stone, $V \prec U$ means there is a clopen set φa with $V \subseteq \varphi a \subseteq U$, so $\Diamond^L U = \bigvee\{\Diamond^L \varphi a : \varphi a \prec U\}$. Then the action of $\Diamond^L$ on U is the join of its actions on the φa where $\varphi a \subseteq U$, and by definition the action of $\Diamond_L$ on αU is the join of its actions on the $\alpha\varphi a$ where $\varphi a \subseteq U$. The result follows.

For the reflexive element functor, let $\mathfrak{A} = (A, \prec \Diamond)$ be a zero-dimensional MDV-algebra with $\mathfrak{A}_* = (X, R)$ its modal space. Then the φa with a reflexive are exactly the clopen sets of X. Claim 5.3 shows that if a is reflexive, then $\varphi\Diamond a = R^{-1}\varphi a$. This shows that $\Diamond$ restricts to an operator on the reflexive elements of A, and the isomorphism φ from these reflexive elements to $\mathsf{Clop}\,\mathfrak{X}$ is a modal isomorphism. ☐

Theorem 6.10. *The MacNeille completion functor lifts to a functor $\mathfrak{M}^U : \mathsf{MA} \to \mathsf{zUMDV}$ with $\mathfrak{M}^U$ taking a modal algebra to its upper MacNeille completion. Further, for $\mathfrak{R}$ the reflexive element functor of Theorem 6.9, we have $\mathfrak{M}^U \simeq (-)^U \circ \mathsf{Sp}$ and $\mathfrak{R} \simeq \mathsf{Clop} \circ (-)_*$, so $\mathfrak{M}^U$ and $\mathfrak{R}$ give an equivalence between MA and zUMDV.*

Proof. This follows as in the proof of Theorem 6.9, using the upper continuity of $\mathfrak{X}^U$ and the definition of the modal operator in the upper MacNeille completion as an approximation from above. ☐

Corollary 6.11. *The category MA is equivalent to each of zLMDV, zUMDV, zMDV, and zMKFrm.*

Of course this result follows from Theorem 6.3, but our purpose was to point out the direct description of the functors realizing these equivalences and especially their connections to ideal and MacNeille completions.

Corollary 6.12. *The equivalence between MA and zMDV, together with the dual equivalence between zMDV and MS, yields, up to natural isomorphism, the usual duality between MA and MS.*

Proof. Informally, the usual duality between MA and MS is obtained from the duality between zMDV and MS by restricting to the reflexive elements of a zero-dimensional MDV-algebra. More formally, for the functors Clop and Sp between MA and MS, the above results show $\mathsf{Clop} \simeq \mathfrak{R} \circ (-)^L$ and $\mathsf{Sp} \simeq (-)_* \circ \mathfrak{M}^L$ (the upper extensions $(-)^U$ and $\mathfrak{M}^U$ give similar results). ☐

Remark 6.13. The ideal and MacNeille completions occur in studies of modal logic [35, 22, 18, 26, 41, 6], but neither to the extent that the canonical completion (ultrafilter extension) occurs. For the reader surprised to see the ideal and MacNeille completions take a more prominent role in these studies than the canonical completion, we comment that the underlying reason is their closer connection to the topology that underscores our effort.

In the next section we begin a study of logical properties of MKH-spaces, MKR-frames, and MDV-algebras. This is related to equational properties of various related algebras. We conclude this section by pointing out the standard result [25] that the ideal completion of any lattice with additional order-preserving operations satisfies the same equations as the original. This, in particular, applies to the ideal completion of modal algebras, provided we consider only the operations $\Box$ and $\Diamond$ and not negation. Equational properties of lower and upper MacNeille completions of modal algebras have also been studied, we direct the reader to [35, 22, 18, 26, 41].

7. Logical aspects

In this section we consider the various structures discussed above as models of a positive fragment of propositional modal logic. Here we consider the set $\mathcal{F}$ of formulas φ built from a set $\mathcal{V} = \{v_1, v_2, v_3, \ldots\}$ of propositional variables, using the constants $\top, \bot$, connectives $\wedge, \vee$, and modal operators $\Box, \Diamond$. For such formulas φ, ψ, we define below what it means for a model of a certain type to satisfy the sequent $\varphi \vdash \psi$.

For each basic type of structure, an MKR-frame $\mathcal{L} = (L, \Box, \Diamond)$, an MDV-algebra $\mathfrak{A} = (A, \prec, \Diamond)$, and an MKH-space $\mathfrak{X} = (X, \tau, R)$, we have an associated algebraic structure with underlying lattice, top and bottom, and two unary operations $\Box, \Diamond$. For $\mathcal{L}$ this is simply the structure itself, for $\mathfrak{A}$ we use the structure $\mathfrak{A}$ but define $\Box = \neg\Diamond\neg$, and for $\mathfrak{X}$ we use the structure $\Omega\mathfrak{X} = (\Omega X, \Box, \Diamond)$.

Definition 7.1. *Let $\mathcal{L}$ be an* MKR*-frame, $\mathfrak{A}$ be an* MDV*-algebra, $\mathfrak{X}$ be an* MKH*-space, and let $\varphi(\vec{v})$ be a formula whose variables are among $\vec{v} = v_1, \ldots, v_n$.*

1. *For $\vec{a}$ in $\mathcal{L}^n$ let $\varphi^{\mathcal{L}}(\vec{a})$ be the result of substituting $\vec{a}$ for $\vec{v}$ in the term φ over $\mathcal{L}$.*
2. *For $\vec{a}$ in $\mathfrak{A}^n$ let $\varphi^{\mathfrak{A}}(\vec{a})$ be the result of substituting $\vec{a}$ for $\vec{v}$ in the term φ over $\mathfrak{A}$.*

(3) *For $\vec{U}$ in $\Omega(X)^n$ let $\varphi^{\Omega\mathfrak{X}}(\vec{U})$ be the result of substituting $\vec{U}$ for $\vec{v}$ in the term φ over $\Omega\mathfrak{X}$.*

We often simply write $\varphi(\vec{a})$ or $\varphi(\vec{U})$ with the algebra clear from the context.

For sequences of elements $\vec{a} = a_1,\dots,a_n$ and $\vec{b} = b_1,\dots,b_n$ in an MKR-frame $\mathcal{L}$ or an MDV-algebra $\mathfrak{A}$, we write $\vec{a} \prec \vec{b}$ if $a_i \prec b_i$ for each $i = 1,\dots,n$.

Definition 7.2. *Let $\mathcal{L}$ be an* MKR*-frame, $\mathfrak{A}$ be an* MDV*-algebra, $\mathfrak{X}$ be an* MKH*-space, and φ,ψ be formulas whose variables are among $\vec{v} = v_1,\dots,v_n$. Define*

1. *$\mathcal{L} \vDash \varphi \vdash \psi$ iff $\varphi(\vec{a}) \le \psi(\vec{a})$ for each $\vec{a} \in \mathcal{L}^n$.*
2. *$\mathfrak{A} \vDash \varphi \vdash \psi$ iff $\varphi(\vec{a}) \prec \psi(\vec{b})$ for each $\vec{a} \prec \vec{b} \in \mathfrak{A}^n$.*
3. *$\mathfrak{X} \vDash \varphi \vdash \psi$ iff $\varphi(\vec{U}) \subseteq \psi(\vec{U})$ for each $\vec{U} \in \Omega(X)^n$.*

Before proceeding, we recall a few basics about pseudocomplements and the well-inside relation in compact regular frames (see Definition 3.3). In any frame, we have $a \le \neg\neg a$ and $\neg a = \neg\neg\neg a$. So $a \prec b$ implies $\neg b \prec \neg a$ and $\neg\neg a \prec b$. An element $a \in L$ is called *regular* if $a = \neg\neg a$. If L is the frame of open sets of a space X, then pseudocomplement in L is the interior of set-theoretic complement, so the regular elements of L are exactly the regular open sets of X, hence the name. In a compact regular frame L, the well-inside relation has the interpolation property: if $a \prec b$, then there is c with $a \prec c \prec b$, and from the remarks above this c can be chosen regular. Thus, if L is compact regular and $b \in L$ we have $b = \bigvee\{a : a \text{ is regular and } a \prec b\}$.

Lemma 7.3. *Let $\mathfrak{A}$ be an* MDV*-algebra, $\mathcal{L}$ be an* MKR*-frame, and $\varphi(\vec{v})$ be a formula. Then*

1. *If $\vec{a} \prec \vec{b}$ in $\mathfrak{A}^n$, then $\varphi^{\mathfrak{A}}(\vec{a}) \prec \varphi^{\mathfrak{A}}(\vec{b})$.*
2. *If $\vec{a} \prec \vec{b}$ in $\mathcal{L}^n$, then $\varphi^{\mathcal{L}}(\vec{a}) \prec \varphi^{\mathcal{L}}(\vec{b})$.*
3. *For any $\vec{b} \in \mathcal{L}^n$, we have $\varphi^{\mathcal{L}}(\vec{b}) = \bigvee\{\varphi^{\mathcal{L}}(\vec{a}) : \vec{a}$ is regular and $\vec{a} \prec \vec{b}\}$.*

Proof. (1) In any de Vries algebra, $x_1 \prec y_1$ and $x_2 \prec y_2$ imply $x_1 \wedge x_2 \prec y_1 \wedge y_2$ and $x_1 \vee x_2 \prec y_1 \vee y_2$ and $x \prec y$ implies $\neg y \prec \neg x$. Proposition 4.8 gives $x \prec y$ implies $\Diamond x \prec \Diamond y$, hence $\neg\Diamond\neg x \prec \neg\Diamond\neg y$, so $\Box x \prec \Box y$. Then as $0 \prec x$ and $x \prec 1$ for each x, the result follows by an induction on the complexity of φ. (2) In any compact regular frame, $x_1 \prec y_1$ and $x_2 \prec y_2$ imply $x_1 \wedge x_2 \prec y_1 \wedge y_2$ and $x_1 \vee x_2 \prec y_1 \vee y_2$, and in an MKR-frame, Lemma 3.6 shows $a \prec b$ implies $\Diamond a \prec \Diamond b$ and $\Box a \prec \Box b$. Again, the result follows by induction. (3) By (2) we have $\bigvee\{\varphi(\vec{a}) : \vec{a} \text{ is regular and } \vec{a} \prec \vec{b}\} \le \varphi(\vec{b})$. To show equality induct on the complexity of φ. For constants this is trivial, and in any compact regular frame $b = \bigvee\{a : a \text{ is regular and } a \prec b\}$. By the infinite distributive law and inductive hypothesis, $\varphi_1(\vec{b}) \wedge \varphi_2(\vec{b})$ equals $\bigvee\{\varphi_1(\vec{c}) \wedge \varphi_2(\vec{d}) : \vec{c},\vec{d} \text{ are regular and } \vec{c},\vec{d} \prec \vec{b}\}$. For any $\vec{c},\vec{d} \prec \vec{b}$ there is regular $\vec{a}$ with $\vec{c},\vec{d} \prec \vec{a} \prec \vec{b}$, giving $\varphi_1(\vec{b}) \wedge \varphi_2(\vec{b}) \le \bigvee\{\varphi_1(\vec{a}) \wedge \varphi_2(\vec{a}) : \vec{a} \text{ is regular and } \vec{a} \prec \vec{b}\}$, hence equality. The argument for $\varphi_1(\vec{b}) \vee \varphi_2(\vec{b})$ is similar. The cases for the modal operators $\Diamond\varphi(\vec{b})$ and $\Box\varphi(\vec{b})$ follow as $\Diamond$ and $\Box$ by definition preserve directed joins in any MKR-frame. □

For convenience, we recall some earlier definitions. Suppose $\mathfrak{X} = (X,R)$ is an MKH-space with $\mathcal{L} = \Omega\mathfrak{X}$, $\mathfrak{L} = \mathfrak{X}^L$, and $\mathfrak{U} = \mathfrak{X}^U$ its associated MKR-frame and lower and upper continuous MDV-algebras. Then $\mathcal{L} = (\Omega X, \Box, \Diamond)$ where $\Box S = -R^{-1} - S$ and $\Diamond S = R^{-1}S$, $\mathfrak{L} = (\mathcal{RO}(X), \prec, \Diamond^L)$ where $\Diamond^L S = \mathbf{IC}R^{-1}S$, and $\mathfrak{U} = (\mathcal{RO}(X), \prec, \Diamond^U)$ where $\Diamond^U S = \mathbf{I}R^{-1}\mathbf{C}S$. The derived operations on $\mathfrak{L}$ and $\mathfrak{U}$ respectively, are given by $\Box^L = \neg\Diamond^L\neg$ and $\Box^U = \neg\Diamond^U\neg$. The relation $\prec$ on $\mathcal{L}$ is given by $S \prec T$ iff $\mathbf{C}S \subseteq T$, and the restriction of this relation to the regular open sets is the proximity on each of $\mathfrak{L}$ and $\mathfrak{U}$.

Lemma 7.4. *Let $\mathfrak{X} = (X,R)$ be an* MKH*-space with $\mathcal{L} = \Omega\mathfrak{X}$, $\mathfrak{L} = \mathfrak{X}^L$, and $\mathfrak{U} = \mathfrak{X}^U$. For a formula $\varphi(\vec{v})$, if $\vec{S}$, $\vec{T}$ are regular open and $\vec{S} \prec \vec{T}$, then*

$$\varphi^{\mathfrak{L}}(\vec{S}),\, \varphi^{\mathfrak{U}}(\vec{S}),\, \varphi^{\mathcal{L}}(\vec{S}) \;\prec\; \varphi^{\mathfrak{L}}(\vec{T}),\, \varphi^{\mathfrak{U}}(\vec{T}),\, \varphi^{\mathcal{L}}(\vec{T}).$$

Proof. We first show that if S,T are regular open with $S \prec T$, then $\Diamond^L S, \Diamond^U S, \Diamond S \prec \Diamond^L T, \Diamond^U T, \Diamond T$. Indeed, the definitions show $\Diamond S \le \Diamond^L S \le \Diamond^U S$, so it is enough to show $\Diamond^U S \prec \Diamond T$. But $S \prec T$ implies $\mathbf{C}S \subseteq T$, so $\Diamond^U S = \mathbf{I}R^{-1}\mathbf{C}S \subseteq R^{-1}T = \Diamond T$.

We next show $\Box^L S, \Box^U S, \Box S \prec \Box^L T, \Box^U T, \Box T$. The above definitions, together with $\neg = \mathbf{I}-$, give $\Diamond^U S = \neg\Box\neg S$. So $\Box S \le \neg\neg\Box\neg\neg S = \Box^U S$, and as $\Diamond^L S \le \Diamond^U S$ we have $\Box^U S \le \Box^L S$. So it is enough to show

$\Box^L S \prec \Box T$. As $\mathbf{IC} = \neg\neg$ we have $\Box^L S = \neg\neg\neg R^{-1}\neg S = \mathbf{I} - R^{-1}\mathbf{I} - S$. Then $S \prec T$ gives $\mathbf{C}S \subseteq T$, hence $-T \subseteq \mathbf{I} - S$. So $\Box^L S \subseteq -R^{-1} - T = \Box T$. The result then follows by induction on the complexity of φ as in the proof of Lemma 7.3. □

Theorem 7.5. *Let $\mathfrak{X}$ be an* MKH*-space, $\mathcal{L} = \Omega\mathfrak{X}$ its associated* MKR*-frame, $\mathfrak{L} = \mathfrak{X}^L$ its associated lower continuous* MDV*-algebra, and $\mathfrak{U} = \mathfrak{X}^U$ its associated upper continuous* MDV*-algebra. For formulas φ, ψ these are equivalent.*

(1) $\mathfrak{X} \vDash \varphi \vdash \psi$.
(2) $\mathcal{L} \vDash \varphi \vdash \psi$.
(3) $\mathfrak{L} \vDash \varphi \vdash \psi$.
(4) $\mathfrak{U} \vDash \varphi \vdash \psi$.

Proof. The equivalence of (1) and (2) is obvious. For (2) implies (3) suppose $\vec{S}$ and $\vec{T}$ are regular with $\vec{S} \prec \vec{T}$. By interpolation there is a regular $\vec{V}$ with $\vec{S} \prec \vec{V} \prec \vec{T}$. Then by Lemma 7.4 and the assumption $\mathcal{L} \vDash \varphi \vdash \psi$ we have $\varphi^{\mathfrak{L}}(\vec{S}) \prec \varphi^{\mathcal{L}}(\vec{V}) \le \psi^{\mathcal{L}}(\vec{V}) \prec \psi^{\mathfrak{L}}(\vec{T})$, showing $\mathfrak{L} \vDash \varphi \vdash \psi$. The argument that (2) implies (4) is nearly identical. To see (3) implies (2) suppose $\vec{T} \in \mathcal{L}^n$. If $\vec{S}$ is regular and $\vec{S} \prec \vec{T}$, then by interpolation there are regular $\vec{U}, \vec{V}$ with $\vec{S} \prec \vec{U} \prec \vec{V} \prec \vec{T}$. Then by Lemma 7.4 and the assumption $\mathfrak{L} \vDash \varphi \vdash \psi$ we have $\varphi^{\mathcal{L}}(\vec{S}) \prec \varphi^{\mathfrak{L}}(\vec{U}) \prec \psi^{\mathfrak{L}}(\vec{V}) \prec \psi^{\mathcal{L}}(\vec{T})$. In particular $\varphi^{\mathcal{L}}(\vec{S}) \le \psi^{\mathcal{L}}(\vec{T})$ for each regular $\vec{S} \prec \vec{T}$, and it follows from Lemma 7.3.3 that $\varphi^{\mathcal{L}}(\vec{T}) \le \psi^{\mathcal{L}}(\vec{T})$. Showing (4) implies (2) is nearly identical. □

Remark 7.6. Isomorphisms for MKR-frames and MKH-spaces are structure-preserving bijections, thus isomorphic MKR-frames and isomorphic MKH-spaces satisfy the same sequents $\varphi \vdash \psi$. An isomorphism α between MDV-algebras $\mathfrak{A}$ and $\mathfrak{A}'$ is a bijection preserving the de Vries structure, but not necessarily the modal structure. It need only satisfy $a \prec b$ implies $\Diamond\alpha(a) \prec \alpha(\Diamond b)$ and $\alpha(\Diamond a) \prec \Diamond\alpha(b)$. But it is a simple matter to use this condition to see $\mathfrak{A} \vDash \varphi \vdash \psi$ iff $\mathfrak{A}' \vDash \varphi \vdash \psi$. This lends explanation to the result of Theorem 7.5 that the lower and upper MDV-algebras $\mathfrak{L}$ and $\mathfrak{U}$ associated to $\mathfrak{X}$ satisfy the same sequents $\varphi \vdash \psi$.

This result is striking when considered in the context of a modal space $\mathfrak{X}$. Here, as seen in Section 6, the lower and upper MDV-algebras associated with $\mathfrak{X}$ are the lower and upper MacNeille completions of the modal algebra corresponding to $\mathfrak{X}$. The upper MacNeile completion of a modal algebra is always a modal algebra [26, Thm. 3.5], but the lower MacNeille completion need not be [26, Thm. 3.3]. However, when we consider these lower and upper MacNeille completions as MDV-algebras, they are isomorphic and satisfy exactly the same sequents $\varphi \vdash \psi$. But of course, they do not satisfy the same modal equations.

Definition 7.7. *For an* MKH*-space $\mathfrak{X} = (X, R)$ let $\mathcal{M}\mathfrak{X} = (\mathcal{P}(X), \Box, \Diamond)$ be the modal algebra, where $\mathcal{P}(X)$ is the power set of X, $\Box A = -R^{-1} - A$, and $\Diamond A = R^{-1}A$ for any $A \subseteq X$. Then for formulas φ, ψ whose variables are among $\vec{v}$ define $\mathcal{M}\mathfrak{X} \vDash \varphi \vdash \psi$ iff $\varphi(\vec{U}) \subseteq \psi(\vec{U})$ for each $\vec{U} \in \mathcal{P}(X)^n$.*

Lemma 7.8. *For $\mathfrak{X} = (X, R)$ an* MKH*-space, $\varphi(\vec{v})$ a formula, and $\vec{F}$ closed in X^n,*

(1) $\varphi(\vec{F}) = \bigcap\{\varphi(\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$.
(2) $\varphi(\vec{F}) = \bigcap\{\varphi(\mathbf{C}\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$.

Here $\mathbf{C}\vec{U}$ is the termwise closure of $\vec{U}$, and all formulas are interpreted in the modal algebra $\mathcal{M}\mathfrak{X}$.

Proof. As intersection, union, and the modal operations $\Box = -R^{-1}-$ and $\Diamond = R^{-1}$ are order-preserving, $\vec{U} \subseteq \vec{V}$ implies $\varphi(\vec{U}) \subseteq \varphi(\vec{V})$. So $\varphi(\vec{F}) \subseteq \bigcap\{\varphi(\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\} \subseteq \bigcap\{\varphi(\mathbf{C}\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$. We show equality by induction on the complexity of φ.

For constants this is trivial, and for a variable v this is the well-known fact that for F a closed set in a compact Hausdorff space, that $F = \bigcap\{U : F \subseteq U \text{ open}\} = \bigcap\{\mathbf{C}U : F \subseteq U \text{ open}\}$. For $\varphi_1 \wedge \varphi_2$, we must show $\bigcap\{\varphi_1(\mathbf{C}\vec{U}) \cap \varphi_2(\mathbf{C}\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$ is contained in $\varphi_1(\vec{F}) \cap \varphi_2(\vec{F})$, which follows from the inductive hypothesis. For $\varphi_1 \vee \varphi_2$, using the inductive hypothesis and the infinite distributive law, we must show $\bigcap\{\varphi_1(\mathbf{C}\vec{U}) \cup \varphi_2(\mathbf{C}\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$ is contained in $\bigcap\{\varphi_1(\mathbf{C}\vec{V}) \cup \varphi_2(\mathbf{C}\vec{W}) : \vec{F} \subseteq \vec{V}, \vec{W} \text{ open}\}$. For $\vec{F} \subseteq \vec{V}, \vec{W}$ open, set $\vec{U} = \vec{V} \cap \vec{W}$. Then $\vec{F} \subseteq \vec{U}$ open, and $\varphi_1(\mathbf{C}\vec{U}) \cup \varphi_2(\mathbf{C}\vec{U})$ is contained in $\varphi_1(\mathbf{C}\vec{V}) \cup \varphi_2(\mathbf{C}\vec{W})$.

Finally, we consider the cases for the modal operators. Note first that $\Box$ and $\Diamond$ applied to a closed set yield a closed set as in an MKH-space R^{-1} of an open set is open and R^{-1} of a closed set is closed. So for any formula ψ and sequence of closed subsets $\vec{F}$, a simple induction shows $\psi(\vec{F})$ is closed. By definition, $\Diamond\varphi(\vec{F})$ is given by $R^{-1}\varphi(\vec{F})$, and the inductive hypothesis gives $\varphi(\vec{F})$ is equal to the down-directed intersection of closed sets $\bigcap\{\varphi(\mathbf{C}\vec{U}) : \vec{F} \subseteq \vec{U} \text{ open}\}$. So Esakia's Lemma gives $R^{-1}[\varphi(\vec{F})]$ is equal to $\bigcap\{R^{-}[\varphi(\mathbf{C}\vec{U})] : \vec{F} \subseteq \vec{U} \text{ open}\}$. Matters for $\Box$ are simpler, and follow from the basic fact that R^{-1} commutes with infinite unions for any relation R. ☐

Following [9], we next define analogues of Sahlqvist formulas (Definition 7.9) and Sahlqvist sequents (Definition 7.12) for the positive modal language.

Definition 7.9. *Define $\Box^0 v = v$, and $\Box^{n+1}v = \Box(\Box^n v)$ for each $n \geq 0$. A formula φ is called a* basic Sahlqvist formula *if it is of the form $\Box^n\top$, $\Box^n\bot$ or $\Box^n v$, for some variable v and $n \geq 0$. A* Sahlqvist formula *is one obtained from basic Sahlqvist formulas by applying $\wedge$ and $\Diamond$.*

Lemma 7.10. *Let $\mathfrak{X} = (X, R)$ be an* MKH*-space.*

1. *For $x \in X$ and $S \subseteq X$, we have $x \in \Box^n S$ iff $R^n[x] \subseteq S$.*
2. *For $F \subseteq X$ closed, $R^n[F]$ is closed.*

Proof. (1) Note $x \in \Box S$ iff $x \in -R^{-1} - S$ iff $R[x] \subseteq S$. So $x \in \Box^{n+1}S$ iff $R[x] \subseteq \Box^n S$. This is equivalent to saying xRy implies $R^n[y] \subseteq S$, hence that $R^{n+1}[x] \subseteq S$. (2) It is enough to show F closed implies $R[F]$ is closed. Note that as X is compact Hausdorff, closed sets are the same as compact sets. Suppose $\mathcal{A}$ is a collection of open sets that is closed under finite unions, and $R[F] \subseteq \bigcup\mathcal{A}$. Then for each $x \in F$, since $R[x]$ is closed, hence compact, there is $U_x \in \mathcal{A}$ with $R[x] \subseteq U_x$. Thus by (1), $x \in \Box U_x$ which we have previously noted is open. Then as F is compact, there is a finite $\mathcal{A}' \subseteq \mathcal{A}$ so that each $x \in F$ belongs to $\Box U$ for some $U \in \mathcal{A}'$. Then, again by (1), we have $R[F] \subseteq \bigcup\mathcal{A}'$. So $R[F]$ is compact, hence closed. ☐

The following is an adaptation of the well-known technique of minimal closed assignments (see, e.g., [7, 8, 9, 10]).

Lemma 7.11. *Suppose $\mathfrak{X} = (X, R)$ is an* MKH*-space, $\varphi(\vec{v})$ is a Sahlqvist formula, and $\vec{S}$ is any sequence of subsets of X. Then for any $x \in \varphi(\vec{S})$ there is a closed $\vec{F}$ with $\vec{F} \subseteq \vec{S}$ and $x \in \varphi(\vec{F})$.*

Proof. The proof is by induction on the number of applications of $\wedge$ and $\Diamond$ to obtain φ from basic Sahlqvist formulas. If φ is a basic Sahlqvist formula of the form $\Box^n\top$ or $\Box^n\bot$ this is trivial. If φ is of the form $\Box^n v$, then Lemma 7.10 states $x \in \varphi(S)$ iff $R^n[x] \subseteq S$ and that $R^n[x]$ is closed. So we may use $R^n[x]$ for F. Having established the result for basic Sahlqvist formulas, suppose $\varphi(\vec{v}) = \varphi_1(\vec{v}) \wedge \varphi_2(\vec{v})$ where φ_1 and φ_2 are Sahlqvist. Then $x \in \varphi(\vec{S})$ implies $x \in \varphi_1(\vec{S})$ and $x \in \varphi_2(\vec{S})$, so by the inductive hypothesis, there are closed $\vec{G}, \vec{H} \subseteq \vec{S}$ with $x \in \varphi_1(\vec{G})$ and $x \in \varphi_2(\vec{H})$. Set $\vec{F} = \vec{G} \cup \vec{H}$. Finally, suppose $\varphi = \Diamond\varphi_1$. Then $x \in \Diamond\varphi(\vec{S})$ means $x \in R^{-1}\varphi_1(\vec{S})$, so there is some $y \in \varphi_1(\vec{S})$ with xRy. By the inductive hypothesis, there is closed $\vec{F} \subseteq \vec{S}$ with $y \in \varphi_1(\vec{F})$, hence $x \in \Diamond\varphi_1(\vec{F})$. ☐

Definition 7.12. *We say $\varphi \vdash \psi$ is a* Sahlqvist sequent *if φ is Sahlqvist.*

Theorem 7.13. *For $\mathfrak{X} = (X, R)$ an* MKH*-space and $\varphi \vdash \psi$ a Sahlqvist sequent, these are equivalent.*

1. *$\varphi(\vec{S}) \subseteq \psi(\vec{S})$ for any sequence $\vec{S}$ of subsets of X.*
2. *$\varphi(\vec{U}) \subseteq \psi(\vec{U})$ for any sequence $\vec{U}$ of open subsets of X.*

Proof. (1) implies (2) is trivial. Suppose (1) does not hold. Then there is a sequence $\vec{S}$ of sets and $x \in X$ with $x \in \varphi(\vec{S})$ and $x \notin \psi(\vec{S})$. Then by Lemma 7.11, there is a closed $\vec{F}$ with $\vec{F} \subseteq \vec{S}$ and $x \in \varphi(\vec{F})$. Note, $\vec{F} \subseteq \vec{S}$ implies $\psi(\vec{F}) \subseteq \psi(\vec{S})$, so $x \notin \psi(\vec{F})$. Then, by Lemma 7.8.1, there is an open $\vec{U}$ with $\vec{F} \subseteq \vec{U}$ and $x \notin \psi(\vec{U})$. Since $\vec{F} \subseteq \vec{U}$ implies $\varphi(\vec{F}) \subseteq \varphi(\vec{U})$, we have $x \in \varphi(\vec{U})$. So $x \in \varphi(\vec{U})$ and $x \notin \psi(\vec{U})$, so (2) does not hold. ☐

Corollary 7.14. *If $\mathfrak{X}$ is an* MKH*-space, then $\mathfrak{X}$ and $\mathcal{M}\mathfrak{X}$ satisfy exactly the same Sahlqvist sequents $\varphi \vdash \psi$.*

It is well known from modal logic [8, 10] that for any Sahlqvist sequent $\varphi \vdash \psi$, there is a corresponding first-order formula Φ in the language having a single binary relation symbol R, so that for a relational structure (X, R), the modal algebra $(\mathcal{P}(X), \Box, \Diamond)$ satisfies $\varphi \vdash \psi$ iff (X, R) satisfies Φ. Then from the above result and Theorem 7.5, the following is immediate.

Corollary 7.15. *For $\varphi \vdash \psi$ a Sahlqvist sequent, there is a first-order sentence Φ in the language with a single binary relation symbol, so that for an* MKH*-space $\mathfrak{X} = (X, R)$, its associated* MKR*-frame $\mathcal{L} = \Omega\mathfrak{X}$, its associated lower continuous* MDV*-algebra $\mathfrak{L} = \mathfrak{X}^L$, and its associated upper continuous* MDV*-algebra $\mathfrak{U} = \mathfrak{X}^U$, these are equivalent.*

(1) (X, R) *satisfies* Φ.
(2) $\mathfrak{X} \vDash \varphi \vdash \psi$.
(3) $\mathcal{L} \vDash \varphi \vdash \psi$.
(4) $\mathfrak{L} \vDash \varphi \vdash \psi$.
(5) $\mathfrak{U} \vDash \varphi \vdash \psi$.

For generalizations of Sahlqvist correspondence in different contexts see [9, 19, 23, 7].

Remark 7.16. The above results may be combined with standard results from modal logic. For example, let $\mathfrak{B}$ be a modal algebra with dual space $\mathfrak{X} = (X, R)$. Then [8, 10] $a \le \Diamond a$ holds in $\mathfrak{B}$ iff R is reflexive, $\Diamond\Diamond a \le \Diamond a$ holds in $\mathfrak{B}$ iff R is transitive, and $a \le \Box\Diamond a$ holds in $\mathfrak{B}$ iff R is symmetric. As all these are Sahlqvist, it follows that if $\mathcal{L}$ is an MKR-frame and $\mathfrak{X}$ is its dual MKH-space, then $a \le \Diamond a$ holds in $\mathcal{L}$ iff R is reflexive, $\Diamond\Diamond a \le \Diamond a$ holds in $\mathcal{L}$ iff R is transitive, and $a \le \Box\Diamond a$ holds in $\mathcal{L}$ iff R is symmetric. So reflexive and transitive MKH-spaces correspond to MKR-frames satisfying $a \le \Diamond a$ and $\Diamond\Diamond a \le \Diamond a$, and MKH-spaces whose relations are equivalence relations correspond to MKR-frames satisfying $a \le \Diamond a$, $\Diamond\Diamond a \le \Diamond a$, and $a \le \Box\Diamond a$. Similar versions of these results can be stated in terms of MDV-algebras.

For an MKH-space $\mathfrak{X} = (X, R)$, the above results show $\mathfrak{X}$ and $\mathcal{M}\mathfrak{X}$ satisfy the same Sahlqvist sequents. As $\Omega\mathfrak{X}$ is a subalgebra of $\mathcal{M}\mathfrak{X}$, any sequent satisfied by $\mathcal{M}\mathfrak{X}$ is satisfied by $\mathfrak{X}$, but for sequents that are not Sahlqvist, we do not know if the converse holds.

8. Concluding remarks

There are a number of avenues for further consideration. The dualities here were based on Isbell and de Vries dualities for compact Hausdorff spaces. When modalities were incorporated, interesting algebraic structures arose. It may be worthwhile to see if incorporation of modalities into the other dualities mentioned in the introduction, such as Gelfand-Stone or Kakutani-Yosida duality, would yield interesting results.

The Vietoris functor naturally generalizes to categories of spaces more general than compact Hausdorff spaces. One might consider algebraic counterparts of coalgebras in this setting, much as we have done here for the Vietoris functor on KHaus.

Finally, there are many logical questions to further the work begun in the previous section. Among many questions are ones related to Sahlqvist completeness, finite model property, and decidability. One might also develop matters in more general languages than our positive fragment of modal logic.

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