Coalgebraic Semantics for Fischer Servi Intuitionistic Modal Logic *

Rodrigo Nicolau Almeida[1] [0000−0002−1326−502X], Nick Bezhanishvili[1] [0009−0005−6692−5051], and Sarah Maja Dukic[2] [0009−0002−8711−9161]

[1] Institute for Logic Language and Computation (ILLC) - University of Amsterdam, Science Park 107, 1098 XG Amsterdam, Netherlands
r.dacruzsilvapinadealmeida@uva.nl
n.bezhanishvili@uva.nl

[2] Leiden Institute of Advanced Computer Science (LIACS) - Leiden University, Gorleaus Building, Einsteinweg 55, 2333 CC, Leiden, Netherlands
dukicsm@liacs.leidenuniv.nl

Abstract. We present a new coalgebraic semantics for the intuitionistic modal logic known as **IK** or *Fischer Servi logic*. Recently, a method was introduced that provides a functorial way of turning coalgebras for a positive modal logic into coalgebras for its least intuitionistic extension, allowing for a coalgebraic representation of topological models and image-finite Kripke models for intuitionistic modal logics [1]. Such an approach does not suffice on its own to treat Fischer Servi logic, as it is not the least intuitionistic extension of a positive modal logic. In this paper we fill this gap, by providing a modified approach, which yields coalgebraic completeness for **IK** and other related logics. As an application of these results, we study bisimulations for Fischer Servi logic, describe the dual spaces of free **IK** algebras, and show how our approach can be used to capture rank-1 extensions of **IK**.

Keywords: Coalgebraic Modal Logic, Intuitionistic Modal Logic, Fischer Servi Logic, IK, Modal Heyting Algebras.

1 Introduction

Relational semantics of modal logics admit natural coalgebraic representations. Recall that a coalgebra is a pair $(X, \alpha : X \to FX)$, where F is an endofunctor on a category $\mathbb{C}$, and X and α are respectively an object and a morphism in $\mathbb{C}$. A map $f : (X, \alpha) \to (Y, \gamma)$ is a homomorphism between F-coalgebras if $f : X \to Y$ is a map in $\mathbb{C}$ such that $\gamma \circ f = Ff \circ \alpha$. These definitions correspond in a very natural way to relational structures and bounded morphisms between them. At the level of objects, a relational structure (X, R) can be seen as a coalgebra

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$(X, R[-] : X \to FX)$, where $R[-]$ is the map sending a point in X to its set of successors, the "shape" of which is restricted by the functor F. For example, in the case of a classical Kripke frame, where for $x \in X$, $R[x]$ can be any subset of X, F is the powerset functor [35]. In case of descriptive frames, i.e., "topological Kripke frames", where $R[-]$ must meet additional conditions, one considers the Vietoris functor on the category of Stone spaces [35], [36], which captures said conditions. At the level of arrows, the nature of a bounded morphism between frames corresponds naturally to the commutativity of its corresponding diagram (see [35, Example 9.10]). We say that the category of Kripke frames and bounded morphisms is equivalent to the category $\mathbf{Coalg}(\mathcal{P})$ of powerset-coalgebras and coalgebra morphisms and the category of descriptive frames and continuous bounded morphisms is equivalent to the category $\mathbf{Coalg}(\mathcal{V})$ of Vietoris coalgebras and coalgebra morphisms. Under this perspective, the basic step in providing the coalgebraic semantics for a given modal logic amounts to determining the correct functor. If we have this representation, then studying the properties of the logic amounts to studying the coalgebraic behaviour – which naturally generates notions such as bisimulation and free algebras.

Besides classical modal logic, intuitionistic modal logic is increasingly relevant for computer science, with applications in, e.g., formal verification, type systems, and knowledge representation, often modeling "growing information" in Kripke frames [21,28,4]. Thus, there is strong motivation to situate intuitionistic modal logic within the uniform framework provided by the theory of coalgebras. However, while classical and positive modal logics have well-known coalgebraic semantics [36,29,8], the intuitionistic case is much more elusive [27,15]. The issue is that, in a coalgebraic representation, the morphisms which constitute the coalgebras and the homomorphisms between the coalgebras should belong to the same base category. Unlike the classical and positive cases, intuitionistic propositional logic is already modal – with intuitionistic implication treated semantically as a modality governed by the partial order. Thus, to preserve truth of intuitionistic implication, morphisms between intuitionistic frames must respect the partial order, and thus the corresponding coalgebra homomorphisms must be p-morphisms. If we then add another relation R to govern the modalities, it is not necessarily the case that $R[-]$ is itself a p-morphism. Thus, in general, representing intuitionistic modal frames coalgebraically requires a way to impose different conditions on the coalgebra morphisms and the homomorphisms between coalgebras. Such an approach, going beyond coalgebras and into *dialgebras* was given by de Groot and Pattinson [15].

It was only recently that a technique was introduced in [1] and [2] to derive coalgebraic semantics for intuitionistic modal logics, by turning coalgebras for a positive modal logic L into coalgebras for its least intuitionistic extension – the logic obtained by adding to L the axioms of IPC. This method springs from the construction, due to [2] (generalizing the work of Ghilardi [25]), of the right adjoint $\mathcal{V}_G$ to the inclusion of Esakia spaces into Priestley spaces. Such a construction is rather complex, corresponding to the idea of layering the intuitionistic frame as an infinite sequence of positive frames; technically, it involves infinitely many iterations of a powerset-like construction, and yields a dual representation of free Heyting algebras. Once the correspondence between positive and intuitionistic coalgebras is made, one can then lift these coalgebras to a category whose maps are additionally p-morphisms, thereby removing all unwanted coalgebra homomorphisms.

However, a central limitation of such an approach lies in the fact that one needs to work with logics which are least intuitionistic extensions of some positive modal logic. This makes such a method ill-suited to handle some well-known interesting logics, like Fischer-Servi's **IK**. In this paper, we show how the approach of [1] can be modified to yield coalgebraic completeness for **IK** and other related logics. Schematically, this is achieved by choosing a fragment of intuitionistic logic, richer than the pure positive one, which contains enough implications to express the axioms of **IK**, modifying it in steps by adding single layers of implications, constructing a smaller functor FS_2, and using variations of the functor $\mathcal{V}_G$.

Most of our results will be given for categories of ordered topological spaces with additional relations, which are dual to the usual categories of distributive lattices and Heyting algebras with operations. Such structures are slightly more involved than their topology-free reducts, though they appear more natural in the algebraic side, and allow us to see a richer array of categorical structure. Nevertheless, we will throughout state most of our results also for categories of image-finite posets with additional relations (see Section 2 for a discussion of the restriction to image-finiteness).

This paper is structured as follows: in Section 2, we introduce the relevant logics together with their algebraic and topological semantics. In Section 3, we introduce in detail the step-by-step construction of $\mathcal{V}_G$ due to [2]. In Section 4, we present our main results, providing a coalgebraic representation for Fischer Servi logic. In Section 5, we derive several consequences of our construction. In particular, we derive a notion of bisimulation between Fischer Servi frames, provide an explicit construction of the dual spaces of free **IK**-algebras, and show how our approach subsumes rank-1 extensions of Fischer Servi logic.

We note that this paper is based on one of the authors' Master's thesis [16], written under the supervision of the other authors. A preliminary version of some of these results was presented in an extended abstract [3], though the current paper significantly expands on that material and includes new results.

2 Preliminaries

We assume familiarity with the theory of coalgebras, and point the reader to [26] and [35] for an overview of their application to modal logic. We also assume some familiarity with Priestley and Esakia duality [19], as well as with intuitionistic propositional logic (IPL) and its Kripke-style semantics [14]. We will also work with its implication- and negation-free fragment, known as positive propositional logic (PPL).

2.1 Priestley spaces, Esakia spaces, and image-finite posets

The algebraic semantics of PPL and IPL are given, respectively, over distributive lattices and Heyting algebras. Their dual relational and topological semantics are given by Priestley and Esakia spaces and (image-finite) posets, which we define in this subsection.

Definition 1. *A Priestley space is a pair* $(X, \leq)$ *where* X *is a compact topological space and* $\leq$ *is a partial order, satisfying the* Priestley Separation Axiom*:* $\forall x, y \in X.\ x \not\leq y \implies \exists U \in \mathsf{ClopUp}(X). x \in U$ *and* $y \notin U$*.*

We will denote a Priestley space $(X, \leq)$ by X wherever no ambiguity arises. We will also not make notational distinctions between the orders of two different Priestley Spaces or posets.

Definition 2. *Let* $(X, \leq), (Y, \leq)$ *be two posets, and* $f : X \to Y$ *be a map between them. The map* f *is said to be* monotone *if whenever* $x, y \in X$*, and* $x \leq y$ *then* $f(x) \leq f(y)$*. We say that* f *is a* p-morphism *if it is monotone and in addition, whenever* $x \in X$*,* $y \in Y$ *and* $f(x) \leq y$*, then there is some* x' *such that* $x \leq x'$ *and* $f(x') = y$*.*

We let **Pries** denote the category of Priestley spaces and continuous monotone maps. It is well known (see [32]) that **Pries** is dually equivalent to the category **DL** of distributive lattices and bounded lattice homomorphisms.

Definition 3. *An Esakia space is a Priestley space such that whenever* $U \subseteq X$ *is clopen, then* $\downarrow U$ *is clopen.*

We denote by **Esa** the category of Esakia spaces and continuous p-morphisms. It is well-known (see e.g. [19]) that the category **Esa** is dual to the category **HA** of Heyting algebras and Heyting homomorphisms.

We denote by **Pos** the category of posets with monotone maps, and by $\mathbf{Pos}_p$ the category of posets with p-morphisms. We will work with a specific subcategory of this:

Definition 4. *A poset* P *is called* image-finite *if for each* $x \in P$*, the set* $\uparrow x := \{y \in P \mid y \geq x\}$ *is finite.*

We will denote by $\mathbf{ImFinPos}_p$ the category of image-finite posets and p-morphisms. Our results on coalgebraic representations are restricted to this category, where we see these posets as intuitionistic Kripke frames[3]; this arises because whilst $\mathbf{ImFinPos}_p$ enjoys good general categorical properties (e.g., it is complete and cocomplete, a regular category, monadic over **Pos**) the category $\mathbf{Pos}_p$ is very ill-behaved, lacking, amongst other properties, binary products [5]. It is also notable that $\mathbf{ImFinPos}_p$ is dually equivalent to the category **ProHA** of profinite Heyting algebras [7], similarly to how **Pos** is dually equivalent to the category of profinite distributive lattices [33], **ProDL**.

[3] This is justified as IPL has the finite model property, so is also complete with respect to image-finite posets.

2.2 Bimodal logics over intuitionistic and positive bases

We now consider modal logics resulting from adding the $\Box$ and $\Diamond$ operators to a positive or intuitionistic base. We note here that intuitionistic and positive modal logics differ from the classical case in that the $\Box$ and $\Diamond$ operators are not each other's duals, and therefore not interdefinable. Thus, one can add only one of the operators to either a positive or intuitionistic base. We begin by presenting the cases where the operators do not interact and are each governed by their own independent relation. This will serve as a basis for treating Fischer Servi logic, which does impose interactions between the modalities.

Definition 5. *A* positive $\Box\Diamond$-algebra *is a triple* $(D, \Box, \Diamond)$ *such that* $D \in \mathbf{DL}$ *and* $\Box$ *and* $\Diamond$ *are unary functions on* D *satisfying the following normality axioms:*

Normality axioms for $\Box$	**Normality axioms for $\Diamond$**
1. $\Box\top = \top$	3. $\Diamond\bot = \bot$
2. $\Box(a \wedge b) = \Box a \wedge \Box b$	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$

If D *is a Heyting algebra, then we call* $(D, \Box, \Diamond)$ *a* $\Box\Diamond$-Heyting algebra.

We denote the following categories:

1. $\mathbf{PMA}^{\Box\Diamond}$ the category of positive $\Box\Diamond$-algebras and bounded lattice homomorphisms preserving $\Box$ and $\Diamond$;
2. $\mathbf{PMProA}^{\Box\Diamond}$ the full subcategory of $\mathbf{PMA}^{\Box\Diamond}$ consisting of profinite distributive lattices;
3. $\mathbf{HA}^{\Box\Diamond}$ the category of $\Box\Diamond$-Heyting algebras and Heyting homomorphisms preserving $\Box$ and $\Diamond$;
4. $\mathbf{ProHA}^{\Box\Diamond}$ the full subcategory of $\mathbf{HA}^{\Box\Diamond}$ consisting of profinite Heyting algebras.

Throughout the paper, by a *frame* we will mean a poset or a Priestley space; our definitions will incorporate this, with some clauses addressing the additional topological structure.

Recall that given a set X and a relation $R \subseteq X^2$, we write $\Diamond_R U = \{x \in X \mid \exists y.xRy \text{ and } y \in U\}$ and $\Box_R U = X - \Diamond_R(X - U)$.

Definition 6. *A* $\Box\Diamond$*-frame is a triple* $(X, R_\Box, R_\Diamond)$ *where* X *is a frame, such that the following conditions hold:*

1. $R_\Box[x]$ *is a (closed) upset;*
2. $R_\Diamond[x]$ *is a (closed) downset;*
3. *If* X *is Priestley and* U *is a clopen upset, then* $\Diamond_{R_\Diamond} U$ *and* $\Box_{R_\Box} U$ *are clopen upsets;*
4. $R_\Box = \leq \circ R_\Box \circ \leq$ *and* $R_\Diamond = \geq \circ R_\Diamond \circ \geq$.

We call $(X, R_\Box, R_\Diamond)$ *a* $\Box\Diamond$*-Priestley/Esakia space or a* $\Box\Diamond$*-(image-finite) poset, respectively.*

It is well known that, for a $\Box\Diamond$-frame $(X, R_\Box, R_\Diamond)$, $R_\Box$ is monotone with respect to reverse inclusion and $R_\Diamond$ is monotone with respect to inclusion. That is, if $x \leq y$ then $R_\Box[x] \supseteq R_\Box[y]$ and $R_\Diamond[x] \subseteq R_\Diamond[y]$.

Definition 7. *Let* $(X, R^X_\Box, R^X_\Diamond)$ *and* $(Y, R^Y_\Box, R^Y_\Diamond)$ *be two* $\Box\Diamond$*-frames. A map* $f : X \to Y$ *is a* $\Box\Diamond$*-morphism provided*

1. f *is monotone;*
2. $xR^X_\Diamond z$ *implies* $f(x)R^Y_\Diamond f(z)$*;*
3. $xR^X_\Box z$ *implies* $f(x)R^Y_\Box f(z)$*;*
4. $f(x)R^Y_\Diamond y$ *implies there exists* $z \in X$ *such that* $xR^X_\Diamond z$ *and* $y \leq f(z)$*.*
5. $f(x)R^Y_\Box y$ *implies there exists* $z \in X$ *such that* $xR^X_\Box z$ *and* $f(z) \leq y$ [4]*.*

We denote the following categories:

1. $\mathbf{Pries}_{\Box\Diamond}$ the category of $\Box\Diamond$-Priestley spaces and continuous $\Box\Diamond$-morphisms;
2. $\mathbf{Pos}_{\Box\Diamond}$ the category of $\Box\Diamond$-posets and $\Box\Diamond$-morphisms;
3. $\mathbf{Esa}_{\Box\Diamond}$ the category of $\Box\Diamond$-Esakia spaces and continuous p-morphisms which are $\Box\Diamond$-morphisms;
4. $\mathbf{ImFinPos}^{\Box\Diamond}_p$ the $\Box\Diamond$-image finite posets with p-morphisms which are $\Box\Diamond$-morphisms.

The following is then a general consequence of [30, Theorem 6.1.11]:

Theorem 1. *The following categories are dually equivalent:*

- *The category* $\mathbf{Pries}_{\Box\Diamond}$ *and the category* $\mathbf{PMA}_{\Box\Diamond}$*.*
- *The category* $\mathbf{Pos}_{\Box\Diamond}$ *and the category* $\mathbf{PMProA}^{\Box\Diamond}$*.*
- *The category* $\mathbf{Esa}_{\Box\Diamond}$ *and the category* $\mathbf{HA}_{\Box\Diamond}$*.*
- *The category* $\mathbf{ImFinPos}^{\Box\Diamond}_p$ *and the category* $\mathbf{ProHA}_{\Box\Diamond}$*.*

2.3 Fischer Servi logic (IK)

So far, the $\Box$ and $\Diamond$ operators are completely independent, governed by separate relations which do not interact. This will serve as a basis for our construction, but in order to achieve an intuitionistic or positive analogue for classical modal logic, some interaction is necessary. In the positive case, the axioms $\Diamond x \wedge \Box y \leq \Diamond(x \wedge y)$ and $\Box(x \vee y) \leq \Box x \vee \Diamond y$ are added to yield Dunn-style positive modal logic [17], whose coalgebraic semantics, given by the *convex Vietoris functor* [11], [8], [36] are well-known. There exist multiple approaches to intuitionistic modal logic, though there are compelling reasons to prefer the logic **IK** or *Fischer Servi logic*[5] [22], [23], [24], which will be the focus of this paper. We refer the reader to Simpson's thesis [34] for an in-depth discussion and evaluation of intuitionistic

Axioms:

0. All substitution instances of theorems of IPL
1. $\Box(A \to B) \to (\Box A \to \Box B)$
2. $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
3. $\neg\Diamond\bot$
4. $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$
5. $(\Diamond A \to \Box B) \to \Box(A \to B)$

Rules:

(MP) From A and $A \to B$, deduce B.
(Nec) From A, deduce $\Box A$.

Fig. 1: Axiomatization of **IK** [31]

modal logics. We introduce the logic **IK** and present its Kripke-style, algebraic, and topological semantics. The axiomatization of **IK** is provided in Figure 1.

Definition 8. *Let $F = (X, \leq, R)$ be a tuple where $(X, \leq)$ is a poset and R a binary relation. We say that F is an* **IK**-triple *provided the following conditions are satisfied:*

– ***F1***: $x' \geq xRy \implies \exists y'.x'Ry' \geq y$;
– ***F2***: $xRy \leq y' \implies \exists x'.x \leq x'Ry'$.

Models for **IK***, sometimes referred to as* birelation models*, are constructed by equipping* **IK***-triples with monotone valuations.*

Definition 9. *Let $M = (X, \leq, R, V)$ be an* **IK***-model. The semantics of the connectives $\wedge, \vee, \neg$ are given by the same inductive clauses as in the classical case, and satisfaction for the modal connectives is given as follows:*

– $M, w \models \Box A \iff \forall w' \geq w.\forall v'.\ w'Rv' \implies M, v' \models A$;
– $M, w \models \Diamond A \iff \exists v.\ wRv\ \&\ M, v \models A$;
– $M, w \models A \to B \iff \forall v \geq w.\ M, v \models A \implies M, v \models B$.

A proof of completeness for **IK** with respect to these semantics can be found in [24].

Definition 10. *An* **IK**-algebra *is a $\Box\Diamond$-Heyting algebra $(H, \Box, \Diamond)$ which additionally satisfies axioms* ***A*** *and* ***B***:

[4] Note that if we require f to be a p-morphism, condition (v) turns into the usual back condition with $f(z) = y$.
[5] We note that **IK** was also introduced independently by Ewald [20] and Plotkin and Stirling [31].

$$\textbf{\textit{A}}\textbf{.}\ \Diamond(a \to b) \leq \Box a \to \Diamond b \qquad \textbf{\textit{B}}.\ \Diamond a \to \Box b \leq \Box(a \to b)$$

We denote by **IKHA** the category of **IK**-algebras and Heyting algebra homomorphisms preserving $\Box$ and $\Diamond$, and by **IKProHA** the full subcategory of those **IK**-algebras which are profinite Heyting algebras.

Definition 11. *An* **IK**-frame *is a pair* (X, R), *where* X *is an Esakia space or image-finite poset, such that the following conditions hold:*

- *If* X *is Esakia,* $R[x]$ *is closed;*
- $R[\uparrow x]$ *is a (closed) upset;*
- *If* X *is Esakia, and* U *is a clopen upset, then* $\Diamond_R U$ *and* $\Box_{(\leq \circ R)} U$ *are clopen upsets;*
- $R[x] = R[\uparrow x] \cap \downarrow R[x]$*;*
- $(R \circ \leq) \subseteq (\leq \circ R)$ *and* $(\geq \circ R) \subseteq (R \circ \geq)$*.*

We call it an **IK***-Esakia space or* **IK***-image-finite poset respectively.*

Definition 12. *Let* (X, R) *and* (Y, S) *be* **IK***-frames. A map* $f : X \to Y$ *is an* **IK** p-morphism *iff for every* $x, x', y \in X, z \in Y$,

1. *if* $x \leq_X y$ *then* $f(x) \leq_Y f(y)$.
2. *If* $f(x) \leq_Y z$ *then* $f(x') = z$ *for some* $x' \in \uparrow x$.
3. *If* xRy *then* $f(x)Sf(y)$.
4. *If* $f(x)Sz$ *then* $z \leq_Y f(x')$ *for some* $x' \in R[x]$.
5. *If* $f(x)(\leq_Y \circ S)z$ *then* $f(x') = z$ *for some* $x' \in R[\uparrow x]$.

We call f *a* continuous **IK** p-morphism *if* X, Y *are* **IK***-Esakia spaces, and for every* $A \in \mathsf{ClopUp}(Y)$, $f^{-1}[A] \in \mathsf{ClopUp}(X)$.

We denote by **IKE** the category of **IK**-Esakia spaces and continuous **IK** p-morphisms, and by **IKP** the category of **IK**-image-finite posets and **IK** p-morphisms. The following is due to Palmigiano [30][6]:

Theorem 2. *The categories* **IKE** *and* **IKHA** *are dually equivalent; the categories* **IKP** *and* **IKProHA** *are dually equivalent.*

The following is proved for example in [13]:

Lemma 1. *An* **IK***-frame is a* $\Box\Diamond$*-frame where* $R := R_\Box \cap R_\Diamond$ *and the following conditions hold:*

(I) $R_\Diamond = \downarrow(R_\Box \cap R_\Diamond)$ *and*
(II) $R_\Box = \leq \circ (R_\Box \cap R_\Diamond)$

Or, in other words, $R_\Box[x] = R[\uparrow x]$ *and* $R_\Diamond[x] = \downarrow R[x]$.

Hence, we will often treat **IK**-frames as $\Box\Diamond$-frames satisfying (I) and (II).

[6] Palmigiano obtained the duality results for the categories of Esakia spaces and algebras; the restriction to profinite Heyting algebras follows by general considerations from the correspondence given in [7].

3 The step-by-step construction

In this section, we recall the step-by-step construction of the functor $\mathcal{V}_G$ given in [2], which will enable us to turn coalgebras for positive modal logic into coalgebras for its smallest intuitionistic extension.

As mentioned in the introduction, defining coalgebraic semantics for intuitionistic modal logics is complicated due to the presence of implication, as it is treated semantically as a modality governed by the $\leq$ relation (see Definition 9). Thus, in order to preserve the truth of implication, maps between intuitionistic frames must be p-morphisms with respect to the order. However, when introducing an additional modality, its corresponding relation (seen as a map) is only required to be monotone. Consequently, coalgebras over a base category with monotone maps would result in more coalgebra homomorphisms than frame homomorphisms, and restricting to a category with p-morphisms would result in more frames than coalgebras.

The solution pursued here is the following: first ensure a 1-1 correspondence between the intuitionistic modal spaces and coalgebras for a **Pries** endofunctor F, then turn F into an **Esa** endofunctor, thereby excluding the coalgebra homomorphisms that are not Esakia. The key to ensuring a unique lifting lies in using a universal construction. This will be achieved by the functor $\mathcal{V}_G$, dually corresponding to the free functor from **DL** to **HA**, which we present in this section.

Definition 13. *Let X be a Stone space. The* Vietoris hyperspace *of X is the set $K(X)$ of closed subsets of X equipped with the topology generated by the subbasis consisting of*

$$\langle U \rangle = \{F \in K(X) | F \cap U \neq \emptyset\} \text{ and } [V] = \{F \in K(X) | F \subseteq V\},$$

for $U, V \in \mathsf{Clop}(X)$ [7].

We say that a subset $S \subseteq X$ is *rooted* if there is a point $x \in X$ such that $\forall y \in S.\ x \leq y$.

Definition 14. *The endofunctor $\mathcal{V}_r$ on* **Pries**[8] *is defined as follows:*

- ***On objects:*** *Let X be a Priestley space. Then $\mathcal{V}_r(X) = \{C \subseteq X \mid C$ is closed and rooted$\}$, equipped with the Vietoris topology and ordered by reverse inclusion.*
- ***On morphisms:*** *Given a Priestley map $f : X \to Y$, $\mathcal{V}_r(f) : \mathcal{V}_r(X) \to \mathcal{V}_r(Y)$ is defined by $\mathcal{V}_r(f)(A) = f[A]$.*

[7] Note that the given subbasis is not the usual one of the Vietoris topology as found for example in [18], but yields the same topology.

[8] See [2, Lemma 13] for a proof that $\mathcal{V}_r$ defines an endofunctor on **Pries**. We note that we are referring to Version 3 of [2], and numbering has changed in the most recent version.

Definition 15. *Let X, Y, and Z be Priestley spaces, and let $f : X \to Y$ and $g : Y \to Z$ be Priestley morphisms. We say f is* g-open, *or* open relative to g, *if f^{-1} preserves relative pseudocomplements of the form $g^{-1}[U] \to g^{-1}[V]$ for $U, V \in \mathsf{ClopUp}(Z)$. That is,*

$$f^{-1}(g^{-1}[U] \to g^{-1}[V]) = f^{-1}(g^{-1}[U]) \to f^{-1}(g^{-1}[V]).$$

This is equivalent to the following condition [2, Lemma 12]:

$$(*) \qquad \forall a \in X, \forall b \in Y, (f(a) \leq b \implies \exists a' \in X, (a \leq a' \ \&\ g(f(a')) = g(b))).$$

We say that a closed subset $S \subseteq X$ is g-open *(seen as a Priestley space with the induced order and topology) if the inclusion $\iota : S \hookrightarrow X$ is g-open.*

Definition 16. *Let $g : X \to Y$ be a Priestley morphism. Define $\mathcal{V}_g(X) \subseteq \mathcal{V}_r(X)$ as the Priestley space*[9]

$$\mathcal{V}_g(X) := \{C \subseteq X \mid C \text{ is closed, rooted, and } g\text{-open}\}$$

with the subspace topology and order inherited from $\mathcal{V}_r(X)$.

Additionally, $\mathcal{V}_g(X)$ is equipped with the g-open, surjective Priestley morphism $r_g : \mathcal{V}_g(X) \to X$ sending each rooted subset to its root ([2, Lemma 15]).

Lemma 2. *Let X, Y, Z be Priestley spaces, $g : X \to Y$ be a Priestley morphism, and $h : Z \to X$ be a g-open Priestley map. Then the map h' defined by $h'(x) = h[\uparrow x]$ is the unique r_g-open Priestley map such that $r_g \circ h' = h$.*

This provides the following key construction:

Definition 17. *Let $g : X \to Y$ be a Priestley morphism. The* g-Ghilardi complex $(V^g_\bullet(X), \leq_\bullet)$ *over X is the sequence*

$$(V_0(X), V_1(X), ..., V_n(X), ...)$$

connected by morphisms $r_{i+1} : V_{i+1}(X) \to V_i(X)$ such that:

- $V_0(X) = X$
- $r_0 = g$
- *For* $i \geq 0$, $V_{i+1}(X) := \mathcal{V}_{r_i}(V_i(X))$
- $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ *is the root map.*

The projective limit *of this family is called $\mathcal{V}^g_G(X)$. When g is the terminal map to the one-element poset, we will omit it and simply write $\mathcal{V}_G$.*

As $\mathcal{V}_G(X)$ is a projective limit, it comes equipped with projections $\lambda_i : \mathcal{V}_G(X) \to V_i(X)$. Furthermore, the projection $\lambda_0 : \mathcal{V}_G(X) \to X$ is surjective.

[9] The proof that $(\mathcal{V}_g(X), \supseteq)$ is a Priestley space can be found in [2, Lemma 14].

Proposition 1. *Let Y and Z be Priestley spaces with $g : Y \to Z$ a Priestley map such that g-indexed relative pseudocomplements exist. Suppose that X is an Esakia space and $f : X \to Y$ is a g-open Priestley morphism. Then there is a unique Esakia morphism $\overline{f} : X \to \mathcal{V}_G^g(Y)$ extending f, given by*

$$\overline{f}(x) = (f_0(x), f_1(x), ...)$$

for the family $f_n : X \to V_n(Y)$ given by

- $f_0 = f$
- $f_{n+1}(x) = f_n[\uparrow x]$

$\mathcal{V}_G$: **Pries** $\to$ **Esa** can now be defined as a functor, sending a Priestley space to the Esakia space $\mathcal{V}_G(X)$ and a Priestley map to the Esakia morphism $\overline{p}$. The following is due to [2, Theorem 22]:

Theorem 3. *$\mathcal{V}_G$ is an endofunctor on* **Pries***, and is the right adjoint to the inclusion of* **Esa** *into* **Pries***.*

We note that, as stated in [2, Theorem 23], dualizing the above construction yields the free functor from **DL** to **HA**. In general, given an endofunctor on **Pries**, one can simply lift it to Esakia spaces through composition with $\mathcal{V}_G$, which allows for coalgebras for a positive modal logic to be turned into coalgebras for its least intuitionistic extension. This was exemplified in [1, Theorem 4.2] for the case of intuitionistic modal logic with only the $\Box$ operator.

To finish this section, we briefly outline how the above can be adapted to the connection between **Pos** and **ImFinPos**$_p$. For a longer discussion see [2,1].

1. Given image-finite posets P, Q and $g : P \to Q$ a monotone map, $\mathcal{P}_g(P) = \{C \subseteq P : C$ is rooted, finite, and g-open $\}$.
2. One defines the complex $(P_0(P), P_1(P), \ldots, .P_n(P), ...)$ by similarly iterating, the operation: $P_0(P) = P$, $r_0 = g$, for $i \geq 0$, $P_{i+1}(P) = \mathcal{P}_{r_i}(V_i(P))$, $r_{i+1} = r_{r_i}$, the root map.
3. We denote by $\mathcal{P}_G^g(P)$ the collection of image-finite points of the projective limit of the above complex.
4. The functor $\mathcal{P}_G$ is given as above by providing a unique lift from monotone maps to p-morphisms.

All of this provides a cofree functor from **Pos** to **ImFinPos**$_p$, similarly allowing us to lift endofunctors on **Pos** to endofunctors on **ImFinPos**$_p$.

4 Coalgebras for Fischer Servi logic

In this section, we present our main results: we develop a coalgebraic representation for **IK**-spaces – the descriptive general frames for Fischer Servi logic. Given the interaction between modalities and implications in axioms **A**. $\Diamond(a \to b) \leq \Box a \to \Diamond b$ and **B**. $\Diamond a \to \Box b \leq \Box(a \to b)$, there is no ostensible positive reduct for which **IK** is the smallest intuitionistic extension. We must therefore begin with the fragment consisting of only the normality axioms, which can all be expressed in positive modal logic, then apply the $\mathcal{V}_G$ construction in steps whilst quotienting over the remaining axioms appropriately. We preface this section by providing a sketch of what will be the algebraic side at each step, as an intuition for the algebraically-minded reader.

1. We begin with an Esakia space X, denoting its dual Heyting algebra by D_X.
2. Generate $D_{\Box\Diamond}$, the free distributive lattice over $\{\Box a | a \in D_X\}$ and $\{\Diamond a | a \in D_X\}$, quotiented over the normality axioms for $\Box$ and $\Diamond$. $D_{\Box\Diamond}$ is the dual distributive lattice of the product $\mathcal{V}^{\uparrow}(X) \times \mathcal{V}^{\downarrow}(X)$, which we will define shortly (Definitions 18 and 19).
3. As D_X is a Heyting algebra, we have all relative pseudocomplements $a \to b \in D_X$ for $a, b \in D_X$, so $D_{\Box\Diamond}$ has all elements $\Box(a \to b)$ and $\Diamond(a \to b)$. Then as axiom **A** can be residuated to $\Diamond(a \to b) \wedge \Box a \leq \Diamond b$, we can quotient $D_{\Box\Diamond}$ over **A**, yielding the distributive lattice $D^{\mathbf{A}}_{\Box\Diamond}$.
4. Axiom **B** on the other hand involves implications between modal formulas, so before quotienting, we must freely add relative pseudocomlements of the form $\{a \to b \mid a, b \in D^{\mathbf{A}}_{\Box\Diamond}\}$. This is dually achieved by applying the functor $\mathcal{V}_r$ (Definition 14). With this one layer of implications added, we may quotient over axiom **B** to yield the distributive lattice $D^{\mathbf{AB}}_{\Box\Diamond}$, which now satisfies all of the **IK** axioms.
5. Finally, completing the $\mathcal{V}_G$ construction, we (dually) freely add relative pseudocomplements to $D^{\mathbf{AB}}_{\Box\Diamond}$, yielding the desired Heyting algebra.

Accordingly, we begin by presenting the coalgebraic representation for $\Box\Diamond$ spaces, which correspond to the fragment of **IK** consisting of only the normality axioms.

Definition 18. *Let $(X, \leq, \tau)$ be a Priestley space, and let $\mathsf{ClUp}(X)$ be the set of* closed upsets *of X. The functor $\mathcal{V}^{\uparrow}$ (called the* upper Vietoris functor*) is defined as follows:*

- *On **Objects:** $\mathcal{V}^{\uparrow}(X, \leq, \tau) = (\mathsf{ClUp}(X), \supseteq, \tau^*)$, with the topology τ^* given by sets of the form $[U]$ and $\langle X - V\rangle$ for U, V clopen upsets of X. We call this the* upper Vietoris space *of X.*
- *On **morphisms:** $\mathcal{V}^{\uparrow}$ sends a Priestley morphism $f : X \to Y$ to the map $V^{\uparrow}(f)$ defined by $V^{\uparrow}(f)(A) = {\uparrow} f[A]$.*

Definition 19. *Let $(X, \leq, \tau)$ be a Priestley space, and $\mathsf{ClDown}(X)$ be the set of closed downsets of X. The functor $\mathcal{V}^{\downarrow}$ (called the* lower Vietoris functor*) is defined as follows:*

- *On **objects:** $\mathcal{V}^{\downarrow}(X, \leq, \tau) = (\mathsf{ClDown}(X), \subseteq, \tau_*)$, with the topology τ_* given by sets of the form $[U]$ and $\langle X - V\rangle$ for U, V clopen downsets of X. We call this the* lower Vietoris space *of X.*
- *On **morphisms:** $\mathcal{V}^{\downarrow}$ sends a Priestley morphism $f : X \to Y$ to the map $\mathcal{V}^{\downarrow}(f) : \mathcal{V}^{\downarrow}(X) \to \mathcal{V}^{\downarrow}(Y)$ defined by $V^{\downarrow}(f)(A) = {\downarrow} f[A]$.*

When X is just a poset, we denote by Up and Down respectively the functors with the same essential definitions but ignoring the topology. We will denote the product $\mathcal{V}^{\uparrow} \times \mathcal{V}^{\downarrow}$ by the shorthand $\mathcal{V}^{\uparrow\downarrow}$, and $\mathcal{P}^{\uparrow\downarrow}$ the topology-free version. The following theorem is easy to prove, and a direct consequence of the results in e.g. [8]:

Theorem 4. *The category* $\mathbf{Pries}_{\Box\Diamond}$ *is equivalent to* $\mathbf{Coalg}(\mathcal{V}^{\uparrow\downarrow})$*. Similarly, the category* $\mathbf{Pos}_{\Box\Diamond}$ *is equivalent to* $\mathbf{Coalg}(\mathcal{P}^{\uparrow\downarrow})$*.*

We also have the following results, proved in [16, Theorem 4.11 and 4.29]:

Theorem 5. *The following are in 1-1 correspondence:*

1. *$\Box\Diamond$-Esakia spaces and coalgebras for the Priestley endofunctor $\mathcal{V}^{\uparrow\downarrow}$.*
2. *$\Box\Diamond$-image finite posets and coalgebras for the endofunctor $\mathcal{P}^{\uparrow\downarrow}$.*

Despite this correspondence on objects, we do not have an equivalence of the categories $\mathbf{Esa}_{\Box\Diamond}$ and $\mathbf{Coalg}(\mathcal{V}^{\uparrow\downarrow})$, as morphisms between $\Box\Diamond$-Esakia spaces must be p-morphisms, but homomorphisms between $\mathcal{V}^{\uparrow\downarrow}$-coalgebras need only be monotone. However, by applying the functor $\mathcal{V}_G$, we arrive at the following theorem, the proof of which can be found in [16, Theorem 4.12]:

Theorem 6. *The categories* $\mathbf{Esa}_{\Box\Diamond}$ *and* $\mathbf{Coalg}(\mathcal{V}_G(\mathcal{V}^{\uparrow\downarrow}))$ *are equivalent*[10]*.*

Thus, we may now capture spaces which dually satisfy the normality axioms for $\Box$ and $\Diamond$. To achieve a coalgebraic representation for **IK**-spaces, we will need to look at subspaces which, in addition, dually satisfy axioms **A** and **B**.

Note that in the space $\mathcal{V}^{\uparrow}(X)$, elements $\Box a$ correspond to the clopen upsets $[U]$, and in the space $\mathcal{V}^{\downarrow}(X)$, elements $\Diamond a$ correspond to the clopen upsets $\langle U\rangle$ for $U \in \mathsf{ClopUp}(X)$. We start by identifying the following subspace of $\mathcal{V}^{\uparrow\downarrow}(X)$, corresponding to condition (I) of Lemma 1:

Definition 20. *Given a Priestley space X, we let*

$$FS_1(X) = \{(D,C) \in \mathcal{V}^{\uparrow\downarrow}(X) : C = {\downarrow}(D \cap C)\}.$$

Similarly, given a poset P, let $FS_1(P) = \{(D,C) \in \mathcal{P}^{\uparrow\downarrow}(P) : C = {\downarrow}(D \cap C)\}$.

Proposition 2. *$FS_1(X)$ is the Priestley subspace of $\mathcal{V}^{\uparrow\downarrow}(X)$ for which axiom **A** dually holds, i.e. $FS_1(X) = \{(D,C) \in \mathcal{V}^{\uparrow\downarrow}(X) \mid \forall U, V \in \mathsf{ClopUp}(X).\ (D,C) \in (\mathcal{V}^{\uparrow}(X) \times \langle U \to V\rangle) \cap ([U] \times \mathcal{V}^{\downarrow}(X)) \implies (D,C) \in \mathcal{V}^{\uparrow}(X) \times \langle V\rangle\}$.*

That is, $R_\Diamond = {\downarrow}(R_\Box \cap R_\Diamond)$ if and only if axiom **A** is dually satisfied. Notice that if X is an Esakia space, then we have implications $U \to V \in \mathsf{ClopUp}(X)$ for all $U, V \in \mathsf{ClopUp}(X)$ between clopen upsets (recall that $U \to V = -{\downarrow}(U - V)$), so for axiom **A** we only need to add the modal operators via the functor $\mathcal{V}^{\uparrow\downarrow}(X)$, and dually quotient over **A** by taking the subspace $FS_1(X)$. However, axiom **B** involves implications between modal formulas, so in order to quotient over **B** we

[10] The proof of this theorem is a special case of the proof of [1, Theorem 4.2].

need a space that has implications between clopen upsets of $\mathcal{V}^{\uparrow\downarrow}(X)$ (or, dually, add elements of the form $\{a \to b | a, b \in D_{FS_1(X)}\}$). To this end, we now take $\mathcal{V}_r(FS_1(X))$. Recall from Definition 14 that this yields the Priestley space of the closed, rooted subsets of $FS_1(X)$, ordered by reverse inclusion, whose basis is given by subsets $[U], \langle V \rangle$ for $U, V \in \mathsf{Clop}(FS_1(X))$.

Now we may look at a subspace satisfying (II) of Lemma 1, and show that this dually corresponds to a distributive lattice satisfying axiom **B**.

Definition 21. *Given a Priestley space X, let*

$$\begin{aligned} FS_2(X) = &\{C \in \mathcal{V}_r(FS_1(X)) \mid \forall (D,E) \in C, y \in D \text{ and } y \leq z, \\ &\text{there exists } (D', E') \geq (D,E), (D,E) \in C, z \in D' \cap E'\}. \end{aligned}$$

Given a poset P, $FS_2(P)$ is similarly defined, except taking $C \in \mathcal{P}_r(FS_1(P))$ satisfying the same condition.

Proposition 3. *$FS_2(X)$ is the Priestley subspace of $\mathcal{V}_r(FS_1(X))$ for which axiom **B** dually holds, i.e. $FS_2(X) = \{C \in \mathcal{V}_r(FS_1(X)) \mid \forall U, V \in \mathsf{ClopUp}(X) \,.\, C \in [-(\mathcal{V}^{\uparrow}(X) \times \langle U \rangle) \cup ([V] \times \mathcal{V}^{\downarrow}(X))] \implies C \in [[U \to V] \times \mathcal{V}^{\downarrow}(X)]$.*

The proofs of Propositions 2 and 3 can be found in [16, Proposition 4.14 and 4.16].

We now have a correspondence between **IK**-spaces (X, R) and $\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow})$ coalgebras $(X, \alpha : X \to FS_2(X))$ such that α is open relative to the root map $r : FS_2(X) \to FS_1(X)$. The condition that the coalgebra morphisms be r-open is required in order to preserve the layer of relative pseudocomplements that were previously added. Now, to turn these into coalgebras for an appropriate endofunctor on Esakia spaces, we look at the composition $\mathcal{V}_G^r \circ FS_2$. Here, recall that the r superscript specifies that we take the r-Ghilardi complex (Definition 17). In other words, we have already completed the first step of the construction, while incorporating the necessary additional quotienting. This leads us to our key result, a proof of which can be found in [16, Theorem 4.17]:

Theorem 7. *Let X be an Esakia space. The following are in 1-1 correspondence:*

(i) ***IK**-spaces (X, R) over X,*
(ii) r-open Priestley maps $f : X \to FS_2(X)$, and
(iii) Esakia morphisms $f' : X \to \mathcal{V}_G^r(FS_2(X))$

The key consequence of this result are the following coalgebraic representations, yielding coalgebraic semantics for Fischer Servi modal logic **IK**:

Theorem 8. *The category* **IKE** *is equivalent to the category* $\mathbf{Coalg}(\mathcal{V}_G^r(FS_2(-)))$.

In a similar way, using the functors handling the discrete case, one gets the following coalgebraic representation for **IKP**:

Theorem 9. *The category* **IKP** *is equivalent to the category* $\mathbf{Coalg}(\mathcal{P}_G^r(FS_2(-)))$.

The proofs of Theorems 8 and 9 can be found in [16, Theorems 4.18 and 4.35].

5 Consequences

In this section, we show some of the consequences that can be derived from coalgebraic completeness for Fischer Servi logic. In particular, we discuss notions of bisimulation for **IK**-frames, give a construction for the free **IK**-algebra on X generators, and show how our method can be applied for extensions of Fischer Servi logic with rank-1 axioms.

5.1 Bisimulation for IK-frames

To our knowledge, there has not yet been a characterisation of bisimulation between Fischer Servi frames. In the case of *image-finite posets* P, Q, the following seems like the most natural candidate:

Definition 22. *Let* (P, R) *and* (Q, S) *be two* ***IK****-image finite posets. We say that a relation* $\sim \subseteq X \times Y$ *is an* **IK**-bisimulation *if the following conditions are met. Throughout, let* $x, x' \in X$ *and* $y, y' \in Y$*.*

- *(Forth$_\leq$) If* $x \leq x'$ *and* $x \sim y$*, there is some* $y' \geq y$ *such that* $x' \sim y'$*;*
- *(Forth$_R$) If* xRx' *and* $x \sim y$*, there is some* $y' \in S[y]$ *such that* $x' \sim y'$*.*
- *(Back$_\leq$) If* $y \leq y'$ *and* $x \sim y$*, there is some* $x' \geq x$ *such that* $x' \sim y'$*;*
- *(Back$_R$) If* ySy' *and* $x \sim y$*, there is some* $x' \in R[x]$ *such that* $x' \sim y'$*.*

A proof that this corresponds to truth-invariance for **IKE** models can be found in [16, Proposition 5.2].

An Aczel-Mendler bisimulation between F-coalgebras (X, α) and (Y, γ) is a relation $B \subseteq X \times Y$, equipped with a coalgebra structure $\beta : B \to FB$ such that $F\pi_Y \circ \beta = \gamma \circ \pi_Y$ and $F\pi_X \circ \beta = \alpha \circ \pi_X$, where π_X and π_Y are the projections from $X \times Y$. We refer the reader to [35, Chapter 11] for a detailed overview of bisimulations in coalgebra.

The following can then be shown:

Theorem 10. *Let* $(P, R_\Box, R_\Diamond)$ *and* $(Q, S_\Box, S_\Diamond)$ *be two* ***IK****-image-finite posets. Then the following are in one-to-one correspondence:*

(i) **IK**-*bisimulations between* P *and* Q*;*
(ii) Aczel-Mendler bisimulations for the endofunctor $\mathcal{P}^r_G(FS_2(-))$ *on* $\mathbf{ImFinPos}_p$*.*

The proof of this theorem is similar to that of [1, Theorem 5.2].

Integral to the above is the fact that given image-finite posets P, Q, the product $P \times Q$ is also image-finite, and that a subposet $B \subseteq P \times Q$ will likewise be image-finite. This reveals a lot of categorical fine-structure yielding a wide variety of other notions of bisimulation which deserve to be explored. The most basic for image-finite posets is the following: if one takes B not as a subset of $X \times Y$, but as an *upset* of $X \otimes Y$ – the categorical product, which *does not coincide* with $P \times Q$ – it is not clear whether the obtained notion is equivalent.

This becomes a bigger issue if one wishes to obtain bisimulations for **IKE**. There, one would need to take an approach similar to [10] defining bisimulations via predicate liftings, but to ensure objects live in the right categories, it will be necessary to ensure that bisimulations are themselves Esakia spaces. A characterization of such bisimulations is left as an interesting direction of further research.

5.2 Free IK-algebras

In this section, we show how our construction can be used to generate free **IK** algebras. The construction we use is due to [12], and is analogous to that in [1] to construct the free $\Box$-intuitionistic algebra.

Definition 23. *Let X be an Esakia space. Define the following sequence:*

$$(M_0(X), M_1(X), ..., M_n(X), ...)$$

and a sequence of morphisms $\pi_k : M_k(X) \to M_{k-1}(X)$ for $k > 0$ and $\pi_0 : M_0(X) \to M_0(X)$ defined as follows:

- $M_0(X) = X;$
- $M_{n+1}(X) := X \times \mathcal{V}_G^r(FS_2(M_n(X)));$
- $\pi_0 = id_{M_0}$ *and* $\pi_1(x, C) = x;$
- $\pi_{n+1}(x, C) = (x, (\mathcal{V}_G^r(\mathcal{V}_r(\mathcal{V}^{\uparrow\downarrow}(\pi_n)))(C)))$.

*We denote the inverse limit (in **Pries**) of this system by $M_\infty(X)$.*

Note that each π_k is a p-morphism, given that π_0 and π_1 are clearly p-morphisms, and the action of $\mathcal{V}_G$ on a Priestley map (from an Esakia space to a Priestley space) yields a p-morphism given Proposition 1, which is crucial to show the resulting space to be an Esakia space.

Let us fix some notation. Given $C \in \mathcal{V}_G^r(FS_2(M_k(X)))$ is a sequence $(C_0, C_1, ...)$, define $\lambda_0[C] = C_0$. The projection of $x \in M_\infty(X)$ to its n'th coordinate will be denoted $x(n)$.

For each k, we now define the relations $R_k^\Box \subseteq (X \times \mathcal{V}_G^r(FS_2(M_k(X)) \times M_k(X)$ and $R_k^\Diamond \subseteq (X \times \mathcal{V}_G^r(FS_2(M_k(X)) \times M_k(X)$, given as:

$$(x, C)R_k^\Box y \iff y \in p_0(r(\lambda_0[C])),$$
$$(x, C)R_k^\Diamond y \iff y \in p_1(r(\lambda_0[C])).$$

Where $r : FS_2(M_k(X)) \to FS_1(M_k(X))$ is the root map. We can furthermore define the relation $R_k \subseteq (X \times \mathcal{V}_G^r(FS_2(M_k(X)) \times M_k(X)$, given canonically as the intersection of the $R_k^\Box$ and $R_k^\Diamond$:

$$(x, C)R_k y \iff y \in R_k^\Box[(x, C)] \cap R_k^\Diamond[(x, C)].$$

This very quickly becomes difficult to keep track of, so for the sake of intuition, the idea is that the desired relations always live in $FS_1(-) \subseteq \mathcal{V}^{\uparrow\downarrow}(-)$. So we first take $\lambda_0[C] \in FS_2(M_k(X))$, where the relation does not yet arise transparently as this is a closed upset of pairs in $FS_1(M_k(X))$. Thus, we take $r(\lambda_0[C]) \in FS_1(M_k(X))$, which gives us the pair $(R_k^\Box[(x, C)], R_k^\Diamond[(x, C)]$, and then we may take the intersection of these relations to get $R_k[(x, C)]$. With this in place, we may now define relations on $M_\infty(X)$.

Definition 24. *Let $x, y \in M_\infty(X)$. We define the relations $R^\Box_\omega$, $R^\Diamond_\omega$, and R_ω as follows:*

$$\begin{aligned} xR^\Box_\omega y &\iff \forall k \in \omega, x(k+1)R^\Box_k y(k), \\ xR^\Diamond_\omega y &\iff \forall k \in \omega, x(k+1)R^\Diamond_k y(k), \\ xR_\omega y &\iff \forall k \in \omega, x(k+1)R_k y(k). \end{aligned}$$

We clearly have $R_\omega[x] = R^\Box_\omega[x] \cap R^\Diamond_\omega[x]$, as desired.

Proposition 4. *Given any Esakia space X, $M_\infty(X)$ is an* **IK**-*Esakia space. Moreover, given X an Esakia space, Y an* **IK**-*Esakia space, and $f : Y \to X$ a p-morphism, there exists a unique lifting $f_\infty : Y \to M_\infty(X)$ which is an* **IKE** *p-morphism from $(Y, S_\Box, S_\Diamond)$ to $(M_\infty(X), R^\Box_\omega, R^\Diamond_\omega)$.*

A proof of Proposition 4 can be found in [16, Propositions 5.9 and 5.10].

Theorem 11. *Let X be a set of generators, and let $\mathbb{X}_{F_D(X)}$ denote the Priestley dual of the free distributive lattice $F_D(X)$ over X. Then $M_\infty(\mathbb{X}_{F_D(X)})$ is the dual to the free* **IK**-*algebra on X many generators.*

Thus, using our coalgebraic treatment of Fischer Servi logic, we have provided an explicit characterization of the dual space to the free **IK**-algebra. We remark that our construction does not obviously yield normal forms for Fischer Servi logic, as it requires infinitely many applications of the $\mathcal{V}_G$ construction. However, it may do so for related logics for which the construction terminates after finitely many steps (see [2, Section 5.3]). One could thus hope to use these results to obtain normal forms for semilinear Fischer Servi logic, as well as other related systems.

5.3 Rank-1 extensions of IK

We now show how our general technique provides a template for studying intuitionistic modal logics. We exemplify this by considering the additional rank-1 axiom of seriality, providing the dual space to the resulting algebra. We will do this by exploiting our coalgebraic representation, with modifications made only at the quotienting steps.

Recall that an algebra satisfies seriality if it satisfies the following modal axiom:

$$\textbf{(D)}\ \Box p \to \Diamond p,$$

where the corresponding frame condition is $\forall w. \exists v. wRv$.

Definition 25. *Define $\mathbf{IKHA_D}$ as the subcategory of* ***IK***-*algebras that satisfy the* (**D**) *axiom. We denote by $\mathbf{IKE_D}$ the subcategory of Esakia spaces dual to $\mathbf{IKHA_D}$.*

Definition 26. *Let $(X, R_\Box, R_\Diamond)$ be a $\Box\Diamond$-frame, and define $S(X) = \{C \in FS_2(X) \mid \forall (D, E) \in C.\ D \cap E \neq \emptyset\}$*

Intuitively, **(D)** dually corresponds to the condition $\forall x(R_\Box \cap R_\Diamond)[x] \neq \emptyset$. That is, every point has a successor which it sees through both relations.

Proposition 5. *$S(X)$ is the Priestley subspace of $FS_2(X)$ which dually satisfies **(D)**, i.e.* $S(X) = \{C \in FS_2(X) \mid \forall U \in \mathsf{ClopUp}(X).\ C \in [[U] \times \mathcal{V}^{\downarrow}(X)] \implies C \in [\mathcal{V}^{\uparrow}(X) \times \langle U \rangle]\}$.

A proof of Proposition 5 can be found in [16, Proposition 5.14]. This leads to the following result, analogous to Theorem 7:

Theorem 12. *The following are in 1-1 correspondence:*

(i) $\mathbf{IKE_D}$*-frames* (X, R) *over* X*,*
(ii) r-open Priestley maps $f : X \to S(X)$*,*
(iii) Esakia morphisms $f' : X \to \mathcal{V}_G^r(S(X))$.

Theorem 13. *The category* $\mathbf{IKE_D}$ *is equivalent to the category* $\mathbf{Coalg}(\mathcal{V}_G^r(S(-)))$.

Thus, our method of quotienting in stages while applying the $\mathcal{V}_G$ construction subsumes rank-1 extensions of Fischer Servi logic. This furthermore means that our dual characterization of the free **IK**-algebra can also be applied to these logics, providing a uniform way of treating them.

6 Conclusions and future work

In this paper, we presented a new coalgebraic semantics for Fischer Servi logic, thereby situating intuitionistic modal logic more firmly within the uniform coalgebraic framework for modal logic. Building on the constructions from [2], we developed a method of treating intuitionistic modal logics which are not the least intuitionistic extension of a positive reduct, by performing additional quotienting within the step-by-step approach. Following the work in [1], we derived coalgebraic representations both for modal spaces and image-finite Kripke frames for Fischer Servi logic.

We highlighted the contribution of our construction by deriving results that follow from coalgebraic completeness. Our basic theory lead naturally to a notion of bisimulation for Fischer Servi logic, which to our knowledge is novel. Furthermore, our construction of the free **IK**-algebra lays a foundation for several interesting lines of research. For instance, having shown that our approach subsumes rank-1 extensions of **IK**, one could investigate whether the free algebras for these logics are intuitionistic tense algebras, as is often the case with free modal algebras (see e.g. [2]). It would be interesting to investigate in the future whether our approach can be modified to accommodate axioms of higher rank. Furthermore, one might expect that normal forms can be derived if one restricts to a locally tabular logic, where the $\mathcal{V}_G$ construction will terminate after finitely-many steps. We also expect that our approach can be extended to frame conditions of special interest, such as monadic intuitionistic propositional calculus (MIPC) (see e.g. [6], [9]) and intuitionistic S4 (see [24], [31], [34]). Overall, it is worth investigating whether the method presented in this paper provides a general recipe for dealing with such logics.

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