Algebra univers. **51** (2004) 177–206
0002-5240/04/030177 – 30
DOI 10.1007/s00012-004-1856-2

Algebra Universalis

Varieties of two-dimensional cylindric algebras II

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Abstract. In [2] we investigated the lattice $\Lambda(\mathbf{Df}_2)$ of all subvarieties of the variety $\mathbf{Df}_2$ of two-dimensional diagonal free cylindric algebras. In the present paper we investigate the lattice $\Lambda(\mathbf{CA}_2)$ of all subvarieties of the variety $\mathbf{CA}_2$ of two-dimensional cylindric algebras. We prove that the cardinality of $\Lambda(\mathbf{CA}_2)$ is that of the continuum, give a criterion for a subvariety of $\mathbf{CA}_2$ to be locally finite, and describe the only pre locally finite subvariety of $\mathbf{CA}_2$. We also characterize finitely generated subvarieties of $\mathbf{CA}_2$ by describing all fifteen pre finitely generated subvarieties of $\mathbf{CA}_2$. Finally, we give a rough picture of $\Lambda(\mathbf{CA}_2)$, and investigate algebraic properties preserved and reflected by the reduct functors $\mathbb{F}\colon \mathbf{CA}_2 \to \mathbf{Df}_2$ and $\Phi\colon \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$.

1. Introduction

This paper is a sequel to [2] and in it we investigate the lattice $\Lambda(\mathbf{CA}_2)$ of all subvarieties of the variety $\mathbf{CA}_2$ of two-dimensional cylindric algebras. The variety $\mathbf{CA}_2$ is widely studied in the literature. One of the main references is the fundamental work by Henkin, Monk, and Tarski [8]. Among many other things it is well known that

- Unlike the variety $\mathbf{Df}_2$ of two-dimensional diagonal free cylindric algebras, not every member of $\mathbf{CA}_2$ is representable;
- The representable members of $\mathbf{CA}_2$ form a proper subvariety of $\mathbf{CA}_2$, usually denoted by $\mathbf{RCA}_2$;
- Both $\mathbf{CA}_2$ and $\mathbf{RCA}_2$ are finitely axiomatizable and their equational theories are decidable;
- Both $\mathbf{CA}_2$ and $\mathbf{RCA}_2$ are finitely approximable, that is, generated by their finite members. However, neither of them is locally finite.

To these results we add a criterion for a variety of two-dimensional cylindric algebras to be locally finite, a characterization of finitely generated and pre finitely generated varieties of two-dimensional cylindric algebras, and a rough description of the lattice $\Lambda(\mathbf{CA}_2)$.

Presented by B. Jónsson.
Received November 29, 2002; accepted in final form February 26, 2004.
2000 *Mathematics Subject Classification*: 03G15, 08B15.
Key words and phrases: Two-dimensional cylindric algebras, representable algebras, locally finite varieties, pre finitely generated varieties, reduct functors.

The paper is organized as follows. Section 2 has a preliminary purpose and it contains all the information about $\mathbf{Df}_2$ and $\mathbf{CA}_2$ needed in subsequent sections. In Section 3 we characterize representable two-dimensional cylindric algebras. In Section 4 we show that there exists a continuum of subvarieties of $\mathbf{RCA}_2$, and that there exists a continuum of varieties in between $\mathbf{RCA}_2$ and $\mathbf{CA}_2$. In Section 5 we describe the only pre locally finite subvariety of $\mathbf{CA}_2$, and characterize locally finite varieties of two-dimensional cylindric algebras. In Section 6 we characterize finitely generated subvarieties of $\mathbf{CA}_2$ by describing all fifteen pre finitely generated subvarieties of $\mathbf{CA}_2$. Finally, in Section 7 we give a rough picture of the lattice structure of $\Lambda(\mathbf{CA}_2)$, define the reduct functors $\mathbb{F}\colon \mathbf{CA}_2 \to \mathbf{Df}_2$ and $\Phi\colon \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$, and investigate algebraic properties preserved and reflected by $\mathbb{F}$ and Φ.

Acknowledgements. Special thanks go to my brother Guram. I would also like to thank Leo Esakia, Revaz Grigolia, and Yde Venema for helpful discussions, as well as the referee for valuable suggestions including many pointers to [8].

2. Preliminaries

2.1. $\mathbf{Df}_2$. In this subsection we review the results about two-dimensional diagonal free cylindric algebras which will be used subsequently.

Definition 2.1 ([7, p.40])**.** Suppose $(B, \wedge, \vee, -, 0, 1)$ is a Boolean algebra. A unary operation $\exists\colon B \to B$ is called a *monadic operator* on B if the following three conditions are satisfied for all $a, b \in B$:

- $\exists 0 = 0$;
- $a \leq \exists a$;
- $\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

Definition 2.2 ([8, Definition 1.1.2])**.** A triple $\mathcal{B} = (B, \exists_1, \exists_2)$ is called a *two-dimensional diagonal-free cylindric algebra*, or a $\mathbf{Df}_2$*-algebra* for short, if B is a Boolean algebra, and $\exists_1, \exists_2$ are monadic operators on B satisfying the following condition for all $a \in B$:

$$\exists_1\exists_2 a = \exists_2\exists_1 a.$$

The variety of two-dimensional diagonal-free cylindric algebras is denoted by $\mathbf{Df}_2$.

Suppose X is a nonempty set, R is a binary relation on X, $x \in X$ and $A \subseteq X$. Let

- $R(x) = \{y \in X : xRy\}$,
- $R^{-1}(x) = \{y \in X : yRx\}$,
- $R(A) = \bigcup_{x \in A} R(x)$,
- $R^{-1}(A) = \bigcup_{x \in A} R^{-1}(x)$.

We call $R(x)$ the *R-saturation* of x, and $R(A)$ the *R-saturation* of A. Note that if R is an equivalence relation, then $R(x) = R^{-1}(x)$ and $R(A) = R^{-1}(A)$.

Recall that a subset A of a topological space X is called a *clopen* subset of X if it is simultaneously closed and open. Also recall that a topological space X is called a *Stone space* if X is 0-dimensional (that is clopen subsets of X form a basis for the topology), compact, and Hausdorff. Denote by $CP(X)$ the Boolean algebra of all clopen subsets of a Stone space X. A relation R on a Stone space X is said to be a *clopen* relation if $A \in CP(X)$ implies $R^{-1}(A) \in CP(X)$. We call R *point-closed* if $R(x)$ is a closed subset of X for every $x \in X$.

Definition 2.3 ([2, p.15])**.** A triple (X, E_1, E_2) is said to be a $\mathbf{Df}_2$*-space* if X is a Stone space and E_1 and E_2 are point-closed and clopen equivalence relations on X with $E_1E_2(x) = E_2E_1(x)$ for every $x \in X$.

Given two $\mathbf{Df}_2$-spaces (X, E_1, E_2) and (X', E_1', E_2'), a function $f\colon X \to X'$ is said to be a $\mathbf{Df}_2$*-morphism* if f is continuous and $fE_i(x) = E_i'f(x)$ for every $x \in X$, $i = 1, 2$. We denote the category of $\mathbf{Df}_2$-spaces and $\mathbf{Df}_2$-morphisms by $\mathbf{DS}$. Then we have the following representation of $\mathbf{Df}_2$-algebras:

Theorem 2.4. [2, Theorem 2.4] *$\mathbf{Df}_2$ is dual to $\mathbf{DS}$. In particular, every $\mathbf{Df}_2$-algebra can be represented as $(CP(X), E_1, E_2)$ for the corresponding $\mathbf{Df}_2$-space (X, E_1, E_2).*

For a $\mathbf{Df}_2$-space (X, E_1, E_2), let $E_0 = E_1 \cap E_2$. It is routine to check that E_0 is an equivalence relation on X. Call the E_i-equivalence classes, that is the sets of the form $E_i(x)$, E_i*-clusters* $(i = 0, 1, 2)$. A subset A of X is called *saturated* if $E_1E_2(A) = A$. A $\mathbf{Df}_2$-space (X, E_1, E_2) is called a *component* if $E_1E_2(x) = X$ for each $x \in X$. A partition R of X is called *correct* if

(1) From $\neg(xRy)$ it follows that there exists an R-saturated clopen A such that $x \in A$ and $y \notin A$,
(2) $RE_i(x) \subseteq E_iR(x)$ for every $x \in X$ and $i = 1, 2$.

Then we have the following dual characterization of congruences and subalgebras of $\mathbf{Df}_2$-algebras, as well as subdirectly irreducible and simple $\mathbf{Df}_2$-algebras.

Theorem 2.5. [2, Theorems 2.3, 2.5, 2.8]

(1) *The lattice of congruences of a $\mathbf{Df}_2$-algebra $(B, \exists_1, \exists_2)$ is isomorphic to the lattice of open saturated subsets of its dual (X, E_1, E_2).*
(2) *The lattice of subalgebras of $(B, \exists_1, \exists_2) \in \mathbf{Df}_2$ is dually isomorphic to the lattice of correct partitions of its dual (X, E_1, E_2).*
(3) *$(B, \exists_1, \exists_2) \in \mathbf{Df}_2$ is subdirectly irreducible iff $(B, \exists_1, \exists_2)$ is simple iff its dual (X, E_1, E_2) is a component.*

2.2. $\mathbf{CA}_2$.

Definition 2.6 ([8, Definition 1.1.1]). A quadruple $\mathfrak{B} = (B, \exists_1, \exists_2, d)$ is said to be a *two-dimensional cylindric algebra*, or a $\mathbf{CA}_2$*-algebra* for short, if $(B, \exists_1, \exists_2)$ is a $\mathbf{Df}_2$-algebra and $d \in B$ is a constant satisfying the following conditions for all $a \in B$ and $i = 1, 2$.

(1) $\exists_i(d) = 1$;
(2) $\exists_i(d \wedge a) = -\exists_i(d \wedge -a)$.

Denote the variety of all two-dimensional cylindric algebras by $\mathbf{CA}_2$.

Since in this paper we only deal with two-dimensional cylindric algebras, we will simply call them cylindric algebras. Below we will generalize the duality for $\mathbf{Df}_2$-algebras to $\mathbf{CA}_2$-algebras.

Definition 2.7. A quadruple (X, E_1, E_2, D) is said to be a *cylindric space* if the triple (X, E_1, E_2) is a $\mathbf{Df}_2$-space and D is a clopen subset of X such that every E_i-cluster of X contains a unique point from D for $i = 1, 2$.

A routine consequence of this definition is the following proposition.

Proposition 2.8 (For an algebraic version see [8, Theorem 1.5.3]). *Suppose $\mathcal{X}$ is a cylindric space. Then the cardinality of the set of all E_1-clusters of $\mathcal{X}$ is equal to the cardinality of the set of all E_2-clusters of $\mathcal{X}$.*

Proof. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the sets of all E_1 and E_2-clusters of $\mathcal{X}$, respectively. Define $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ by putting $f(C) = E_2(C \cap D)$. Suppose $C_1, C_2 \in \mathcal{E}_1$, $C_1 \neq C_2$, $C_1 \cap D = \{x\}$, and $C_2 \cap D = \{y\}$. Since every E_i-cluster of $\mathcal{X}$ contains a unique point from D, it follows that $f(C_1) = E_2(x) \neq E_2(y) = f(C_2)$. Therefore, f is an injection. Now suppose $C' \in \mathcal{E}_2$ and $C' \cap D = \{x\}$. If we let $C = E_1(x)$, then $f(C) = E_2(x) = C'$. Thus, f is a surjection. Hence, we obtain that f is a bijection. □

Given two cylindric spaces (X, E_1, E_2, D) and (X', E_1', E_2', D'), a function $f\colon X \to X'$ is said to be a *cylindric morphism* if f is a $\mathbf{Df}_2$-morphism and $f^{-1}(D') = D$. We denote the category of cylindric spaces and cylindric morphisms by $\mathbf{CS}$. Then we have the following representation of cylindric algebras:[1]

Theorem 2.9. $\mathbf{CA}_2$ *is dual to* $\mathbf{CS}$. *In particular, every cylindric algebra $\mathfrak{B} = (B, \exists_1, \exists_2, d)$ can be represented as $(CP(X), E_1, E_2, D)$ for the corresponding cylindric space $\mathcal{X} = (X, E_1, E_2, D)$.*

Proof. A routine adaptation of Theorem 2.4 to cylindric algebras. □

[1] With regard to the extent of this being a true representation theorem see the discussion in [8, Remarks 2.7.45, 2.7.46].

Remark 2.10. We would like to point out a close connection between cylindric spaces and cylindric atom structures defined in [8]. We recall from [8, Definition 2.7.32] that if $\mathfrak{B} = (B, \exists_1, \exists_2, d)$ is a cylindric algebra, where B is a complete and atomic Boolean algebra, then the cylindric atom structure of B is defined as the quadruple $\mathfrak{At}(\mathfrak{B}) = (At(B), E_1, E_2, D)$, where $At(B)$ is the set of all atoms of B; E_i is defined by putting xE_iy iff $\exists_i x = \exists_i y$, for $x, y \in At(B)$, $i = 1, 2$; and $D = \{x \in At(B) : x \leq d\}$.

Suppose $\mathfrak{B} = (B, \exists_1, \exists_2, d)$ is a cylindric algebra, $\mathfrak{B}^+ = (B^+, \exists_1^+, \exists_2^+, d^+)$ is the canonical extension of $\mathfrak{B}$, and $i\colon \mathfrak{B} \to \mathfrak{B}^+$ is the canonical embedding, [8, Definition 2.7.4]. Then it is well known that B^+ is complete and atomic. Let $\mathfrak{At}(\mathfrak{B}^+)$ be the cylindric atom structure of $\mathfrak{B}^+$. For $a \in B$ let $O_a = \{x \in At(B^+) : x \leq i(a)\}$. We make $\mathfrak{At}(\mathfrak{B}^+)$ into a topological space by letting $\{O_a\}_{a\in B}$ to be a bases for the topology τ. Then it can be shown that $(\mathfrak{At}(\mathfrak{B}^+), \tau)$ is a cylindric space, and that $(\mathfrak{At}(\mathfrak{B}^+), \tau)$ is isomorphic to the dual cylindric space of $\mathfrak{B}$.

As an easy corollary of Theorem 2.9 we obtain that the category Fin$\mathbf{CA}_2$ of finite cylindric algebras is dual to the category Fin**CS** of finite cylindric spaces with the discrete topology. In particular, every finite cylindric algebra is represented as the algebra $(P(X), E_1, E_2, D)$ for the corresponding finite cylindric space (X, E_1, E_2, D) (see, e.g., [8, Theorem 2.7.34]).

To obtain the dual description of homomorphic images and subalgebras of cylindric algebras, as well as subdirectly irreducible and simple cylindric algebras, we need the following two definitions. Suppose $\mathcal{X}$ is a cylindric space. A correct partition R of X is called a *cylindric partition* if $R(D) = D$. A cylindric space $\mathcal{X}$ is called a *quasi-square* if $E_1E_2(x) = X$ for every $x \in X$.

Theorem 2.11. (1) *The lattice of congruences of a cylindric algebra $\mathfrak{B}$ is isomorphic to the lattice of open saturated subsets of its dual $\mathcal{X}$.*

(2) *The lattice of subalgebras of a cylindric algebra $\mathfrak{B}$ is dually isomorphic to the lattice of cylindric partitions of its dual $\mathcal{X}$.*

(3) *A cylindric algebra $\mathfrak{B}$ is subdirectly irreducible iff it is simple iff its dual $\mathcal{X}$ is a quasi-square.*

Proof. A routine adaptation of Theorem 2.5 to cylindric algebras. For (3) also see [8, Theorems 2.4.43, 2.4.14]. □

Then we have the following corollary of Theorem 2.11.

Corollary 2.12. (1) $\mathbf{CA}_2$ *is semi-simple.*

(2) $\mathbf{CA}_2$ *is congruence-distributive.*

(3) $\mathbf{CA}_2$ *has the congruence extension property.*

Proof. Follows immediately from Theorem 2.11. □

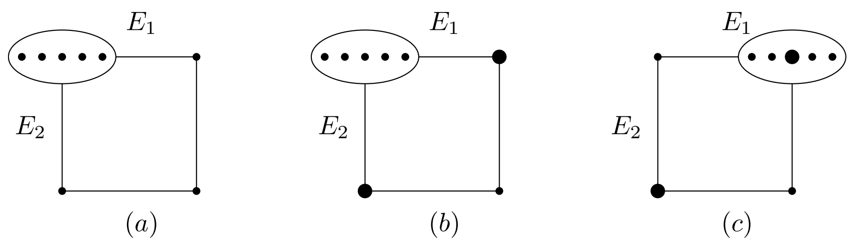

FIGURE 1. Some cylindric spaces and their reducts

Now define the reduct functor $\mathbb{F}\colon \mathbf{CA}_2 \to \mathbf{Df}_2$ by putting

$$\mathbb{F}(B, \exists_1, \exists_2, d) = (B, \exists_1, \exists_2).$$

Thus, $\mathbb{F}$ forgets the diagonal element d from the signature of cylindric algebras.

Remark 2.13. Note that it follows from Theorems 2.5(1) and 2.11(1) that for any cylindric algebra $\mathfrak{B}$, the lattice of congruences of $\mathfrak{B}$ is isomorphic to the lattice of congruences of $\mathbb{F}(\mathfrak{B})$.

Now we show that $\mathbb{F}$ is not onto. In fact, the set $\mathbf{Df}_2 - \mathbb{F}(\mathbf{CA}_2)$ is infinite. For this, define the reduct functor $\mathbb{R}\colon \mathbf{CS} \to \mathbf{DS}$ by putting

$$\mathbb{R}(X, E_1, E_2, D) = (X, E_1, E_2).$$

Suppose $(Y, E_1, E_2) \in \mathbf{DS}$ is a component. Call (Y, E_1, E_2) a *quasi-square* if the cardinality of the sets of all E_1 and E_2-clusters coincide with each other. It follows immediately from Proposition 2.8 that a component (Y, E_1, E_2) is a reduct of some cylindric space iff it is a quasi-square. Note that not every component from $\mathbf{DS}$ is a quasi-square. The simplest examples of components which are not quasi-squares are finite rectangle $\mathbf{Df}_2$-spaces.[2] Since there are infinitely many finite rectangle $\mathbf{Df}_2$-spaces, the set $\mathbf{DS} - \mathbb{R}(\mathbf{CS})$ is infinite.

Now call a $\mathbf{Df}_2$-algebra a *quasi-square algebra* if its dual space is a quasi-square. As follows from the above and Theorem 2.11, for every simple cylindric algebra $\mathfrak{B}$, its $\mathbf{Df}_2$-reduct is a quasi-square algebra. Therefore, the set $\mathbf{Df}_2 - \mathbb{F}(\mathbf{CA}_2)$ is infinite. Moreover, one $\mathbf{Df}_2$-algebra can be the reduct of many non-isomorphic cylindric algebras. For instance, a $\mathbf{Df}_2$-algebra whose dual space is shown in Figure 1(a) is the reduct of the cylindric algebras whose dual cylindric spaces are shown in Figures 1(b) and 1(c), where dots represent points of the spaces, while big dots represent the points belonging to the (diagonal) set D.

[2]We recall from [2, Definition 3.1] that a finite $\mathbf{Df}_2$-space $(n \times m, E_1, E_2)$ is called a *rectangle* if $n, m < \omega$ and E_1 and E_2 are defined in the following way: $(i_1, i_2)E_1(j_1, j_2)$ iff $i_2 = j_2$, and $(i_1, i_2)E_2(j_1, j_2)$ iff $i_1 = j_1$, for $i_1, i_2 < n$ and $j_1, j_2 < m$. Note that the concept of a "rectangle $\mathbf{Df}_2$-space" is different from the one of a "rectangular element" defined in [8, Definition 1.10.6].

3. Representable cylindric algebras

For any cardinal κ, define on the cartesian square $\kappa \times \kappa$ two equivalence relations E_1 and E_2 by putting

$$(i_1, i_2)E_1(j_1, j_2) \text{ iff } i_2 = j_2,$$
$$(i_1, i_2)E_2(j_1, j_2) \text{ iff } i_1 = j_1,$$

for $i_1, i_2, j_1, j_2 \in \kappa$. Let also $D = \{(i, i) : i \in \kappa\}$ and call $(\kappa \times \kappa, E_1, E_2, D)$ a *square*. Obviously $(P(\kappa \times \kappa), E_1, E_2, D)$ is a cylindric algebra, which we call a *square algebra*.[3] Denote the class of all square algebras by $\mathbf{Sq}$.

Definition 3.1 ([8, Remark 1.1.13, Definition 3.1.1(vii)])**.** A cylindric algebra $\mathfrak{B}$ is called *representable* if $\mathfrak{B} \in \mathbf{SP}(\mathbf{Sq})$, where $\mathbf{S}$ and $\mathbf{P}$ denote the operations of taking subalgebras and direct products, respectively.[4]

It is known that the class of representable cylindric algebras is also closed under homomorphic images, and so forms a variety which is usually denoted by $\mathbf{RCA}_2$. It is known that $\mathbf{RCA}_2$ is a proper subvariety of $\mathbf{CA}_2$, that $\mathbf{RCA}_2$ is generated by finite square algebras, and that $\mathbf{RCA}_2$ can be axiomatized by adding the following *Henkin axioms* to the axiom system of $\mathbf{CA}_2$ (see [8, Theorem 3.2.65(ii)]):

$$\text{(H)} \qquad \exists_i(a \wedge -b \wedge \exists_j(a \wedge b)) \leq \exists_j(-d \wedge \exists_i a), \quad i \neq j, \quad i, j = 1, 2.$$

In [12, §3.5.2] Venema has simplified these equations to the following ones:

$$\text{(V)} \qquad d \wedge \exists_i(-a \wedge \exists_j a) \leq \exists_j(-d \wedge \exists_i a), \quad i \neq j, \quad i, j = 1, 2.$$

Below we will recall the dual characterization of representable cylindric algebras, and construct rather simple finite non-representable cylindric algebras.

Suppose (X, E_1, E_2, D) is a cylindric space. Call $x \in D$ a *diagonal point*, and $x \in X - D$ a *non-diagonal point*. Also call an E_0-cluster C a *diagonal E_0-cluster* if it contains a diagonal point. Otherwise call C a *non-diagonal E_0-cluster*.

Lemma 3.2 (For an algebraic version of Lemma 3.2 we refer to [8, Theorem 1.10.13(ii)])**.** *Let $\mathcal{X}$ be a cylindric space. If a diagonal point $x \in D$ is not an isolated point, then $E_0(x) \neq \{x\}$.*

Proof. Suppose $x \in D$ is not an isolated point. Then x is a limit point, and so there exists a net $\{x_i\}_{i \in I}$ converging to x.[5] Since D is a clopen, we can assume that each

[3] The square algebras are defined in [8, Definition 1.1.5(iv)], where they are called "full cylindric set algebras of dimension 2 with base κ". However, since we work only with two-dimensional cylindric algebras the term "square algebra" is more convenient.

[4] The definition of representability is not quite the same as the original one from [8] but is equivalent to it.

[5] Recall that a *net* is a map from a directed set $(I, \leq)$ to X. If X is a Hausdorff space, then every converging net has a unique limit (see, e.g., [6, §1.6] for details).

x_i belongs to D. Moreover, since $\{x_i\}_{i\in I}$ converges to x, without loss of generality we can assume that $E_1(x_i) \cap E_2(x) \neq \varnothing$ for every x_i. Let $y_i \in E_1(x_i) \cap E_2(x)$. Since X is compact, $\{y_i\}_{i\in I}$ converges to some point $y \in X$. Moreover, $y \in E_2(x)$ because $\{y_i\}_{i\in I} \subseteq E_2(x)$ and $E_2(x)$ is closed. Since D contains a unique point from every E_i-cluster, we have that $\{y_i\}_{i\in I} \subseteq -D$. But then $y \in -D$ because $-D$ is a clopen. Therefore, $y \neq x$. Let $E_1(y) \cap D = \{z\}$. If $z \neq x$, then z is not a limit of $\{x_i\}_{i\in I}$, hence there exists a clopen $A \subseteq D$ such that $z \in A$ and for every $j \in I$ there is $j' \geq j$ with $x_{j'} \notin A$. But then $y_{j'} \notin E_1(A)$, which is impossible since $E_1(A)$ is a clopen, $y \in E_1(A)$ and y is a limit of $\{y_i\}_{i\in I}$. Thus, $z = x$, implying that yE_1x. Therefore, $y \in E_0(x)$, and so $E_0(x) \neq \{x\}$. □

Definition 3.3. A cylindric space $\mathcal{X}$ is said to satisfy $(*)$ if there exists a diagonal point $x_0 \in D$ such that $E_0(x_0) = \{x_0\}$ and there exists a non-singleton E_0-cluster C which is either E_1 or E_2-related to x_0.

In the terminology of [8] a cylindric space satisfies the condition $(*)$ of Definition 3.3 iff the corresponding cylindric algebra has at least one defective atom (for details see [8, Lemma 3.2.59]).

Now we will give a dual characterization of representable cylindric algebras. A similar characterization can also be found in [8, Lemma 3.2.59, Theorem 3.2.65]. However, our characterization uses Venema's axioms, while the one in [8] uses Henkin's axioms. Moreover, our proof below appears to be simpler than the original one in [8].

Theorem 3.4. *A cylindric algebra $\mathfrak{B}$ is representable iff its dual cylindric space $\mathcal{X}$ does not satisfy* $(*)$.

Proof. Suppose $\mathcal{X}$ satisfies $(*)$. We show that (V) does not hold in $\mathfrak{B}$, implying that $\mathfrak{B}$ is not representable. Let x_0 be a diagonal point with $E_0(x_0) = \{x_0\}$ and C be a non-singleton E_0-cluster say E_1-related to x_0 (the case when C is E_2-related to x_0 is proved similarly). It follows from Lemma 3.2 that x_0 is an isolated point. Therefore, $E_1(x_0)$ is a clopen. Choose two different points y and z from C, and consider an open set $E_1(x_0) - \{x_0, y\}$. Let $A \subseteq E_1(x_0) - \{x_0, y\}$ be a clopen containing z. Then $y \in -A \cap E_2(A)$, and so $x_0 \in D \cap E_1(-A \cap E_2(A))$. On the other hand, $E_1(A) = E_1(x_0)$. Therefore, $x_0 \notin E_2(-D \cap E_1(A))$, implying that (V) does not hold in $\mathfrak{B}$. Thus, $\mathfrak{B}$ is not representable.

Conversely, suppose $\mathfrak{B}$ is not representable. We show that $(*)$ holds in $\mathcal{X}$. We know that (V) does not hold in $\mathfrak{B}$. Therefore, there exist a point $x \in X$ and a clopen $A \subseteq X$ such that $x \in D \cap E_i(-A \cap E_j(A))$ but $x \notin E_j(-D \cap E_i(A))$ for $i, j = 1, 2$ and $i \neq j$. Since $x \in D \cap E_i(-A \cap E_j(A))$, then $x \in D$ and there exist points $y, z \in X$ such that xE_iy, yE_jz, $y \notin A$ and $z \in A$. From $y \notin A$ and $z \in A$ it follows that y and z are different. Also xE_iy and yE_jz imply that there exists a point $u \in X$ such that xE_ju and uE_iz. If $u \neq x$, then u is a non-diagonal point, and so $u \in -D \cap E_i(A)$. But then $x \in E_j(-D \cap E_i(A))$, which contradicts our assumption. Thus, $u = x$ and xE_iz. Therefore, yE_0z and both y and z are E_i-related to x. Moreover, if $E_0(x) \neq \{x\}$, then by choosing a point $u \in E_0(x)$ different from x we obtain again that $u \in -D \cap E_i(A)$, and so $x \in E_j(-D \cap E_i(A))$, which is impossible. Therefore, $E_0(x) = \{x\}$ and $E_0(y)$ is a non-singleton E_0-cluster E_i-related to x_0. Thus, $(*)$ holds in $\mathcal{X}$. □

Using this criterion it is easy to see that the cylindric algebras corresponding to the cylindric spaces shown in Figure 1(c) are representable, while the cylindric algebras corresponding to the cylindric spaces shown in Figure 1(b) are not. Moreover, the smallest non-representable cylindric algebra is the algebra corresponding to the cylindric space shown in Figure 1(b), where the non-singleton E_0-cluster contains only two points.

4. Cardinality of $\Lambda(\mathbf{CA}_2)$

Denote the lattice of subvarieties of $\mathbf{CA}_2$ by $\Lambda(\mathbf{CA}_2)$ and the lattice of subvarieties of $\mathbf{RCA}_2$ by $\Lambda(\mathbf{RCA}_2)$. We want to show that the cardinality of $\Lambda(\mathbf{RCA}_2)$ as well as the cardinality of $\Lambda(\mathbf{CA}_2) - \Lambda(\mathbf{RCA}_2)$ is that of continuum. For this define a partial order on the class of all non-isomorphic finite simple cylindric algebras by putting

$$\mathfrak{A} \leq \mathfrak{B} \text{ iff } \mathfrak{A} \in \mathbf{S}(\mathfrak{B}).$$

Lemma 4.1. *Every two non-isomorphic finite square algebras are $\leq$-incomparable.*

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two non-isomorphic finite square algebras and let $\mathcal{X}_{\mathfrak{A}}$ and $\mathcal{X}_{\mathfrak{B}}$ be their dual spaces. Then $\mathcal{X}_{\mathfrak{A}}$ is isomorphic to $(n \times n, E_1, E_2, D)$ and $\mathcal{X}_{\mathfrak{B}}$ is isomorphic to $(m \times m, E_1', E_2', D')$ where $n \neq m$. Without loss of generality we can assume that $n > m$. Then obviously $\mathfrak{A}$ can not be a subalgebra of $\mathfrak{B}$. Suppose $\mathfrak{B}$ is a proper subalgebra of $\mathfrak{A}$. Then there exists a cylindric partition R of $\mathcal{X}_{\mathfrak{A}}$ such that $\mathcal{X}_{\mathfrak{A}}/R$ is isomorphic to $\mathcal{X}_{\mathfrak{B}}$. Therefore, R must identify points from different E_1 or E_2-clusters of $\mathcal{X}_{\mathfrak{A}}$. Without loss of generality we can assume that R identifies points from different E_1-clusters C_1 and C_2. Let $x_1 \in C_1$ be the diagonal point of C_1 and $x_2 \in C_2$ be the diagonal point of C_2. Since $R(D) = D$, we have that x_1Rx_2. Let $E_1(x_1) \cap E_2(x_2) = \{y_1\}$. Since x_2Rx_1 and $x_1E_1y_1$, there exists $y_2 \in \mathcal{X}_{\mathfrak{A}}$ such that y_1Ry_2 and $y_2E_1x_2$. Consider $R(x_1)$ and $R(y_1)$. It is obvious that $R(x_1)E_0R(y_1)$. Also $R(x_1) \neq R(y_1)$ since $R(D) = D$. Therefore, there exists a non-singleton E_0-cluster of $\mathcal{X}_{\mathfrak{B}}$, which is impossible since $\mathcal{X}_{\mathfrak{B}}$ is a square. Thus, $\mathfrak{B}$ is not a proper subalgebra of $\mathfrak{A}$, and so every two non-isomorphic finite square algebras are $\leq$-incomparable. □

As an immediate consequence of Lemma 4.1 we obtain the following theorem.

Theorem 4.2. *The cardinality of* $\Lambda(\mathbf{RCA}_2)$ *is that of continuum.*

Proof. Let $\mathcal{X}_n$ be the square $(n \times n, E_1, E_2, D)$ and $\mathfrak{B}_n$ be the square algebra $(P(n \times n), E_1, E_2, D)$. Consider the family $\Delta = \{\mathfrak{B}_n\}_{n\in\omega}$. From Lemma 4.1 it follows that Δ forms a $\leq$-anti-chain. For any subset Γ of Δ, let $\mathbf{V}_\Gamma$ denote the variety generated by Γ, that is, $\mathbf{V}_\Gamma = \mathbf{HSP}(\Gamma)$. Using the standard splitting technique, we can easily show that $\mathbf{V}_\Gamma \neq \mathbf{V}_{\Gamma'}$ whenever $\Gamma \neq \Gamma'$ (the fact we use here is that every finite simple cylindric algebra is a splitting algebra; see, e.g., Kracht [9, Corollary 7.3.12]). Therefore, there exist $2^{\aleph_0}$-many subvarieties of $\mathbf{RCA}_2$. $\square$

For $n > 1$ let $\mathcal{Y}_n$ denote the finite cylindric space obtained from the $n \times n$ square by substituting a singleton non-diagonal E_0-cluster by a two-element E_0-cluster. For example, $\mathcal{Y}_2$ is shown in Figure 1(b), where the non-diagonal E_0-cluster contains two points. Denote by $\mathfrak{A}_n$ the cylindric algebra corresponding to $\mathcal{Y}_n$. Obviously $\mathcal{Y}_n$ satisfies $(*)$, and so $\mathfrak{A}_n$ is not representable. Similarly to Lemma 4.1, we can prove the following lemma.

Lemma 4.3. *The family* $\{\mathfrak{A}_n\}_{n\in\omega}$ *forms a* $\leq$*-anti-chain.*

As an immediate consequence of Lemma 4.3 and the fact that $\{\mathfrak{A}_n\}_{n\in\omega} \subseteq \mathbf{CA}_2 - \mathbf{RCA}_2$ we obtain the following theorem.

Theorem 4.4. *The cardinality of* $\Lambda(\mathbf{CA}_2) - \Lambda(\mathbf{RCA}_2)$ *is that of continuum.*

Finally, for $\Gamma, \Gamma' \subseteq \{\mathfrak{A}_n\}_{n\in\omega}$ it is obvious that $\Gamma \neq \Gamma'$ implies $\mathbf{RCA} \vee \mathbf{V}_\Gamma \neq \mathbf{RCA} \vee \mathbf{V}_{\Gamma'}$. Therefore, we obtain the following corollary.

Corollary 4.5. *There exist continuum many varieties in between* $\mathbf{RCA}_2$ *and* $\mathbf{CA}_2$*.*

5. Locally finite subvarieties of $\mathbf{CA}_2$

Recall that a variety $\mathbf{V}$ of universal algebras is said to be *locally finite* if every finitely generated $\mathbf{V}$-algebra is finite. It is called *pre locally finite* if it is not locally finite but all its proper subvarieties are. It is known (see, e.g., [8, Theorem 2.1.11]) that $\mathbf{RCA}_2$, and hence any variety in the interval $[\mathbf{RCA}_2, \mathbf{CA}_2]$, is not locally finite. In this section, we present a criterion for a variety of cylindric algebras to be locally finite, and show that there exists exactly one pre locally finite subvariety of $\mathbf{CA}_2$.

Let $\mathfrak{B}$ be a cylindric algebra and $\mathcal{X}$ be its corresponding dual cylindric space. We have that $\mathfrak{B}$ is simple iff $\mathcal{X}$ is a quasi-square. We also have that the cardinalities of the sets of E_1 and E_2-clusters of $\mathcal{X}$ coincide.

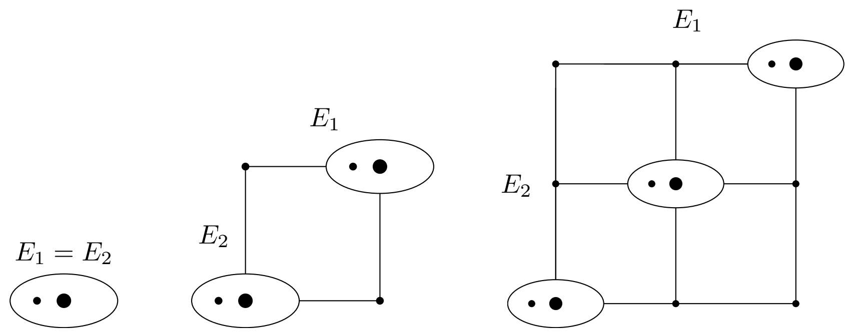

FIGURE 2. Uniform quasi-squares

Definition 5.1. (1) A quasi-square $\mathcal{X}$ is said to be of *depth* n $(0 < n < \omega)$ if the cardinality of the set of E_1-clusters (E_2-clusters) of $\mathcal{X}$ is equal to n.

(2) A quasi-square $\mathcal{X}$ is said to be of an *infinite depth* if the cardinality of the set of E_1-clusters (E_2-clusters) of $\mathcal{X}$ is infinite.

(3) A simple cylindric algebra $\mathfrak{B}$ is said to be of *depth* n if its dual quasi-square $\mathcal{X}$ is of depth n.

(4) A simple cylindric algebra $\mathfrak{B}$ is said to be of an *infinite depth* if its dual quasi-square $\mathcal{X}$ is of an infinite depth.

(5) A variety $\mathbf{V}$ of cylindric algebras is said to be of *depth* n if there is a simple $\mathbf{V}$-algebra of depth n and the depth of every other simple $\mathbf{V}$-algebra is less than or equal to n.

(6) A variety $\mathbf{V}$ is said to be of *depth* ω if the depth of simple members of $\mathbf{V}$ is not bounded by any natural number.

We note that there exists a formula measuring the depth of a variety of cylindric algebras (see [2, Theorem 4.2]). Let $d(\mathbf{V})$ denote the depth of the variety $\mathbf{V}$. Our goal is to show that a variety $\mathbf{V}$ of cylindric algebras is locally finite iff $d(\mathbf{V}) < \omega$. For this we need the following definition.

Definition 5.2. (1) Call a quasi-square $\mathcal{X}$ *uniform* if every non-diagonal E_0-cluster of $\mathcal{X}$ is a singleton set, and every diagonal E_0-cluster of $\mathcal{X}$ contains only two points.

(2) Call a simple cylindric algebra $\mathfrak{B}$ *uniform* if its dual quasi-square $\mathcal{X}$ is uniform.

Finite uniform quasi-squares are shown in Figure 2, where big dots denote the diagonal points. Denote by $\mathcal{X}_n$ the uniform quasi-square of depth n. Also let $\mathfrak{B}_n$ denote the uniform cylindric algebra of depth n. It is obvious that $\mathcal{X}_n$ is (isomorphic to) the dual cylindric space of $\mathfrak{B}_n$. Let $\mathbf{U}$ denote the variety generated by all finite uniform cylindric algebras, that is $\mathbf{U} = \mathbf{HSP}(\{\mathfrak{B}_n\}_{n\in\omega})$.

Proposition 5.3. $\mathbf{U} \subseteq \mathbf{RCA}_2$.

Proof. Since none of the diagonal E_0-clusters of $\mathcal{X}_n$ is a singleton set, $\mathcal{X}_n$ does not satisfy $(*)$. Therefore, each $\mathfrak{B}_n$ is representable by Theorem 3.4. Thus, $\{\mathfrak{B}_n\}_{n\in\omega} \subseteq \mathbf{RCA}_2$, implying that $\mathbf{U} \subseteq \mathbf{RCA}_2$. □

Lemma 5.4. (1) *If $\mathfrak{B}$ is a simple cylindric algebra of an infinite depth, then each $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$.*
(2) *If $\mathfrak{B}$ is a simple cylindric algebra of depth $2n$, then $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$.*

Proof. (1) Suppose $\mathfrak{B}$ is a simple cylindric algebra of an infinite depth and $\mathcal{X}$ is its dual cylindric space. Then $\mathcal{X}$ is a quasi-square with infinitely many E_1 and E_2-clusters. In the same way as in the proof of Claim 4.7 of [2], for every n we can divide $\mathcal{X}$ into n-many E_1-saturated disjoint clopen sets $G_1, \ldots, G_n$. We let $D_i = D \cap G_i$ and $F_i = E_2(D_i)$ for $i = 1, \ldots, n$. Obviously each of the D_i's and F_i's is clopen. Define a partition R of $\mathcal{X}$ by putting

- xRy if $x, y \in D$ and there exists $i = 1, \ldots, n$ such that $x, y \in D_i$;
- xRy if $x, y \in X - D$ and there exist $1 \leq j, k \leq n$ such that $x, y \in G_j \cap F_k$.

It is easy to check, either directly or by transforming the proof of Claim 4.7 of [2], that R is a cylindric partition of $\mathcal{X}$, and that $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n$. Therefore, by Theorem 2.11(2), each $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$.

(2) Suppose $\mathfrak{B}$ is a simple cylindric algebra of depth $2n$ and $\mathcal{X}$ is its dual cylindric space. Then $\mathcal{X}$ is a quasi-square. Moreover, there are exactly $2n$-many E_1 and E_2-clusters of $\mathcal{X}$. Obviously all of them are clopens. Let $C_1, \ldots, C_{2n}$ be E_1-clusters of $\mathcal{X}$ and let $G_i = C_{2i-1} \cup C_{2i}$ for $i = 1, \ldots, n$. Obviously every G_i is E_1-saturated clopen. Now applying the same technique as in (1) shows that $\mathfrak{B}_n$ is a subalgebra of $\mathfrak{B}$. □

Theorem 5.5. *For a variety $\mathbf{V}$ of cylindric algebras, $d(\mathbf{V}) = \omega$ iff $\mathbf{U} \subseteq \mathbf{V}$.*

Proof. It is obvious that $d(\mathbf{U}) = \omega$. So, if $\mathbf{U} \subseteq \mathbf{V}$, then obviously $d(\mathbf{V}) = \omega$. Conversely, suppose $d(\mathbf{V}) = \omega$. We want to show that every finite uniform cylindric algebra belongs to $\mathbf{V}$. Since $d(\mathbf{V}) = \omega$, the depth of simple members of $\mathbf{V}$ is not restricted to any natural number. So, either there exists a family of simple $\mathbf{V}$-algebras of increasing finite depth, or there exists a simple $\mathbf{V}$-algebra of an infinite depth. In either case, it follows from Lemma 5.4 that $\{\mathfrak{B}_n\}_{n\in\omega} \subseteq \mathbf{V}$. Therefore, $\mathbf{U} \subseteq \mathbf{V}$ since $\{\mathfrak{B}_n\}_{n\in\omega}$ generates $\mathbf{U}$. □

Our next task is to show that $\mathbf{U}$ is not a locally finite variety. For this we will need the following lemma.

Lemma 5.6. (1) *Every finite square algebra is 1-generated.*
(2) *Every finite uniform algebra is 1-generated.*

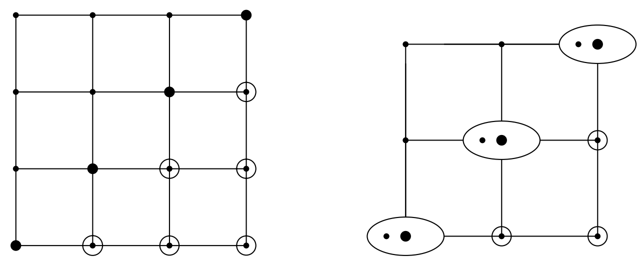

Figure 3. Generators of square and uniform quasi-square algebras

Proof. (1) For a finite square $\mathcal{X} = (n \times n, E_1, E_2, D)$, consider the set $g = \{(k, m) : k < m\}$. It is well known (see e.g., [8, p.253] or [2, p.24]) that a cylindric algebra generated by g contains all singleton subsets of $n \times n$. Hence, $(P(n \times n), E_1, E_2, D)$ is generated by g.

(2) is proved analogously to (1). If $\mathfrak{B}$ is a finite uniform algebra and $\mathcal{X}$ is its dual cylindric space, then $\mathcal{X}$ is obtained from a finite square by replacing every diagonal point by the two point E_0-cluster containing one diagonal point. The same arguments as above show that every E_0-cluster of $\mathcal{X}$ belongs to the algebra generated by the lower triangle g' (see Figure 3, where big dots represent the diagonal points and points in circles represent the points belonging to g and g', respectively). Hence it is left to be shown that for every diagonal E_0-cluster C and $x \in C$, the singleton set $\{x\}$ belongs to the algebra generated by g'. But for any $x \in C$, either $x \in D$ and hence $\{x\} = C \cap D$ or $x \notin D$ and $\{x\} = C \cap -D$. Hence every singleton set belongs to the cylindric algebra generated by g' and therefore g' generates $\mathfrak{B}$. □

Remark 5.7. Note that the $\mathbf{Df}_2$-reducts of finite uniform algebras are not generated by g'. Indeed, the $\mathbf{Df}_2$-algebra generated by g' does not contain the singleton sets from non-singleton E_0-clusters. We point out here that no finite uniform algebra is a 1-generated $\mathbf{Df}_2$-algebra since we can show that the following theorem holds true: A quasi-square $\mathbf{Df}_2$-algebra is 1-generated iff either it is a square algebra, or every E_0-cluster of its dual space is a singleton set except one E_0-cluster that contains exactly two points. Since this fact is not important from the point of view of this paper we skip the details.

Now in order to conclude that $\mathbf{U}$ is not locally finite all we need is to remember the following characterization of locally finite varieties from G. Bezhanishvili [1].

Theorem 5.8. *A variety* $\mathbf{V}$ *of a finite signature is locally finite iff for every natural number* n *there exists a natural number* $M(n)$ *such that the cardinality of every* n*-generated subdirectly irreducible* $\mathbf{V}$*-algebra is less than or equal to* $M(n)$*.*

Corollary 5.9. $\mathbf{U}$ *is not locally finite.*

Proof. Follows from Lemma 5.6 and Theorem 5.8. □

Next we show that if a variety of cylindric algebras is of finite depth, then it is locally finite.

Theorem 5.10. *If* $d(\mathbf{V}) < \omega$*, then* $\mathbf{V}$ *is locally finite.*

Proof. The proof is analogous to that for the diagonal-free case (see [2, Lemma 4.4]): To show $\mathbf{V}$ is locally finite, by Theorem 5.8, it is sufficient to prove that the cardinality of every n-generated simple $\mathbf{V}$-algebra is bounded by some natural number $M(n)$. Let $\mathfrak{B}$ be an n-generated simple $\mathbf{V}$-algebra. Let also $B_i = \{\exists_i b : b \in B\}$, for $i = 1, 2$. Since $d(\mathbf{V}) < \omega$, we have $|B_1| = |B_2| < \omega$. Suppose $\mathfrak{B}$ is generated by $G = \{g_1, \ldots, g_n\}$. Then as a Boolean algebra $\mathfrak{B}$ is generated by $G \cup B_1 \cup B_2 \cup \{d\}$. Since the variety of Boolean algebras is locally finite, there exists $M(n) < \omega$ such that $|\mathfrak{B}| \le M(n)$ (in fact $|\mathfrak{B}| \le 2^{2^{n+2|B_1|+1}}$). Hence $\mathbf{V}$ is locally finite. □

Finally, combining Theorem 5.5 with Corollary 5.9 and Theorem 5.10 we obtain the following characterization of locally finite varieties of cylindric algebras.

Theorem 5.11. (1) *For* $\mathbf{V} \subseteq \mathbf{CA}_2$ *the following conditions are equivalent:*

(a) $\mathbf{V}$ *is locally finite;*

(b) $d(\mathbf{V}) < \omega$*;*

(c) $\mathbf{U} \not\subseteq \mathbf{V}$*.*

(2) $\mathbf{U}$ *is the only pre locally finite subvariety of* $\mathbf{CA}_2$*.*

Therefore, in contrast to the diagonal-free case, there exist uncountably many subvarieties of $\mathbf{CA}_2$ ($\mathbf{RCA}_2$) which are not locally finite. Since every locally finite variety is obviously generated by its finite members, we obtain from Theorem 5.11 that every subvariety of $\mathbf{CA}_2$ of a finite depth is generated by its finite members. We conjecture that every subvariety of $\mathbf{CA}_2$ is in fact generated by its finite members.

6. Finitely generated and pre finitely generated subvarieties of $\mathbf{CA}_2$

Recall that a variety of universal algebras is said to be *finitely generated* if it is generated by a finite universal algebra. We call a variety *pre finitely generated* if it is not finitely generated but all its proper subvarieties are. It was shown in [2, Theorem 5.4] that there exist exactly six pre finitely generated varieties in $\Lambda(\mathbf{Df}_2)$. The situation is more complex in $\Lambda(\mathbf{CA}_2)$. In this section, we show that there exist exactly fifteen pre finitely generated varieties in $\Lambda(\mathbf{CA}_2)$, and that six of them belong to $\Lambda(\mathbf{RCA}_2)$. It trivially implies a characterization of finitely generated subvarieties of $\Lambda(\mathbf{CA}_2)$.

Consider the finite quasi-squares $\mathcal{X}_n^i$ shown in Figure 4, where $i = 1, \ldots, 15$ and $n \geq 2$. Again big dots represent the diagonal points. The pattern according to which the quasi-squares are depicted is the following: First come the spaces with depth 1, then the spaces with depth 2, and finally the spaces with depth 3; quasi-squares with more clusters come later in the list; the first and last quasi-squares (of the same depth) do not satisfy $(*)$, i.e., the corresponding algebras are representable. As it can be seen from the figure, each E_0-cluster of $\mathcal{X}_n^i$ consists of either one, two or n points. Let $\mathfrak{B}_n^i$ denote the cylindric algebra corresponding to $\mathcal{X}_n^i$. For fixed $i = 1, \ldots, 15$ let $\mathbf{V}_i$ denote the variety generated by the family $\{\mathfrak{B}_n^i : n \geq 2\}$. From Theorem 3.4 it follows that only $\mathfrak{B}_n^1$ $\mathfrak{B}_n^2$, $\mathfrak{B}_n^3$, $\mathfrak{B}_n^7$, $\mathfrak{B}_n^{14}$ and $\mathfrak{B}_n^{15}$ are representable algebras, and so only $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_7$, $\mathbf{V}_{14}$ and $\mathbf{V}_{15}$ belong to $\Lambda(\mathbf{RCA}_2)$.

Now we are in a position to prove that $\mathbf{V}_1 - \mathbf{V}_{15}$ are the only pre finitely generated subvarieties of $\mathbf{CA}_2$. For this we need to show that $\mathbf{V}_1 - \mathbf{V}_{15}$ are not finitely generated, which follows from their definition, and that every variety of cylindric algebras which is not finitely generated contains exactly one of $\mathbf{V}_1 - \mathbf{V}_{15}$.

Lemma 6.1. $\mathbf{V}_3 \subset \mathbf{U}$.

Proof. Suppose $\mathfrak{B}_n$ is the finite uniform algebra of depth n. We show that $\mathfrak{B}_n^3$ is a subalgebra of $\mathfrak{B}_n$. Consider the uniform square $\mathcal{X}_n$ of depth n, fix a diagonal E_0-cluster, say C, and let $D \cap C = \{x_0\}$. Define an equivalence relation R on X by putting

- xRy if $x = y$ for all $x, y \in C$;
- xRy for all $x, y \in E_1(C) - C$;
- xRy for all $x, y \in E_2(C) - C$;
- xRy for all $x, y \in D - \{x_0\}$;
- Let each of the remaining $n-1$ R-equivalence classes consist of $n-1$ points chosen so that each class contains exactly one point from each E_i-cluster of $X - (E_1(C) \cup E_2(C) \cup D)$ for $i = 1, 2$.

It is a matter of routine verification that R is a cylindric partition, and that $\mathcal{X}_n/R$ is isomorphic to $\mathcal{X}_n^3$. Therefore, $\mathfrak{B}_n^3$ is a subalgebra of $\mathfrak{B}_n$ for every n, implying that $\mathbf{V}_3 \subset \mathbf{U}$. □

Therefore, we obtain that if $d(\mathbf{V}) = \omega$, then $\mathbf{V}_3 \subseteq \mathbf{V}$. Suppose $d(\mathbf{V}) < \omega$. Then $\mathbf{V}$ is locally finite by Theorem 5.11. Let $\mathrm{Fin}\mathbf{V}_S$ denote the class of all finite simple $\mathbf{V}$-algebras. Since $\mathbf{V}$ is locally finite, $\mathbf{V}$ is generated by $\mathrm{Fin}\mathbf{V}_S$. Suppose

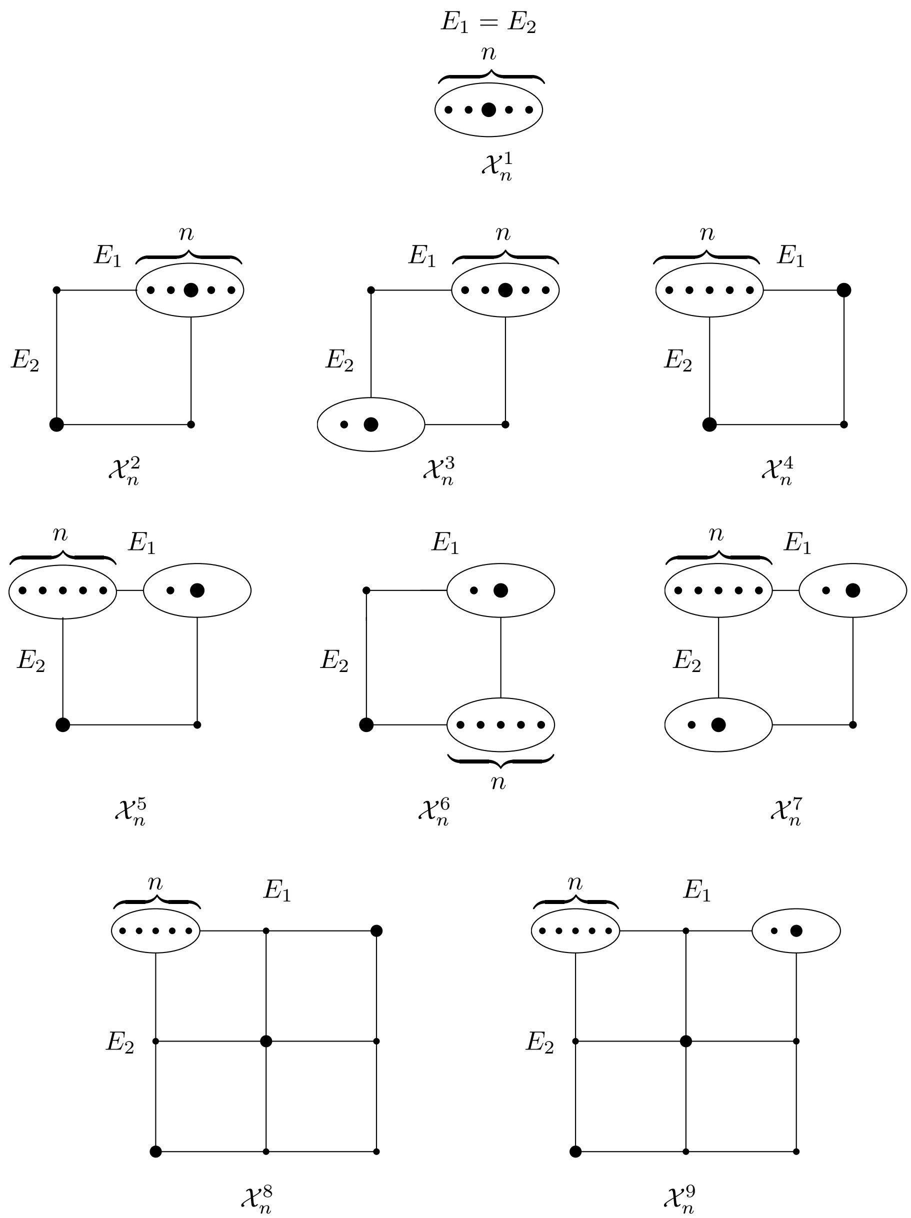

FIGURE 4. Quasi-squares $\mathcal{X}_n^1 - \mathcal{X}_n^{15}$

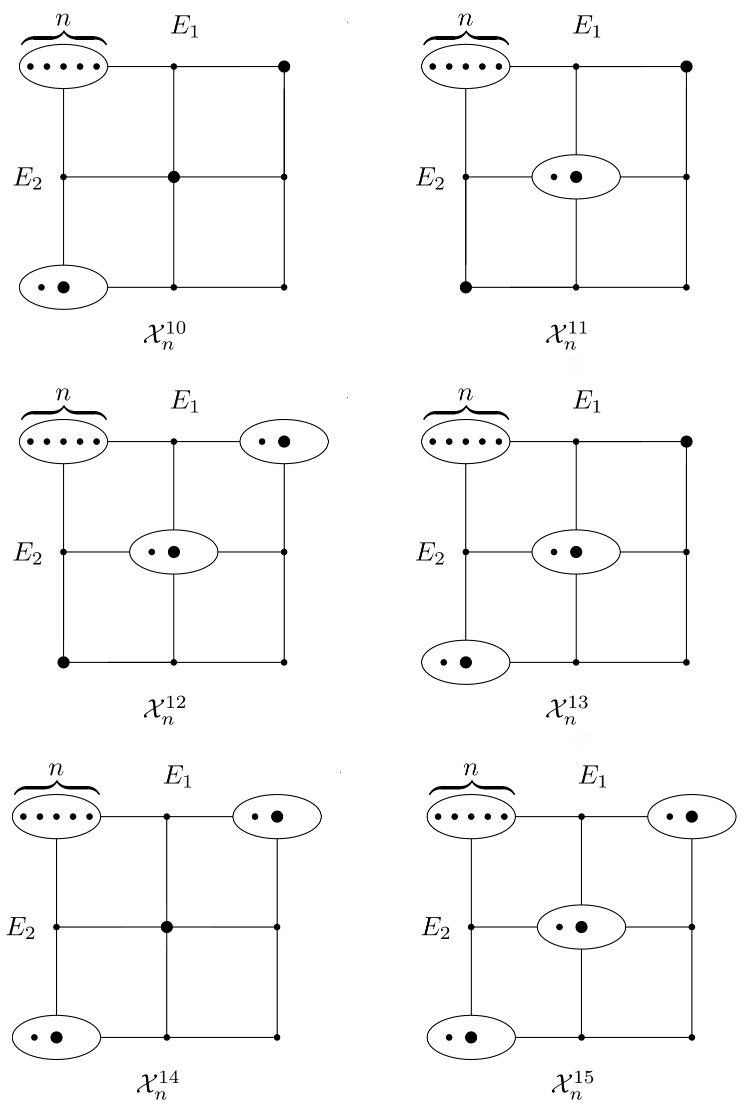

FIGURE 4. (Continued) Quasi-squares $\mathcal{X}_n^1 - \mathcal{X}_n^{15}$

$\mathfrak{B} \in \mathrm{Fin}\mathbf{V}_S$ and $\mathcal{X}$ is its dual cylindric space. Then $\mathcal{X}$ is a finite quasi-square. Fix $x \in X$.

Definition 6.2. (1) Call the number of elements of $E_0(x)$ the *girth* of x.
(2) The maximum of the girths of all $x \in E_0(D)$ is called the *diagonal girth* of $\mathcal{X}$.
(3) The maximum of the girths of all $x \in X - E_0(D)$ is called the *non-diagonal girth* of $\mathcal{X}$.
(4) The *diagonal girth* of $\mathfrak{B}$ is the diagonal girth of $\mathcal{X}$.
(5) The *non-diagonal girth* of $\mathfrak{B}$ is the non-diagonal girth of $\mathcal{X}$.
(6) The diagonal girth of $\mathbf{V}$ is said to be n if there is $\mathfrak{B} \in \mathrm{Fin}\mathbf{V}_S$ whose diagonal girth is n, and the diagonal girth of every other member of $\mathrm{Fin}\mathbf{V}_S$ is less than or equal to n.
(7) The diagonal girth of $\mathbf{V}$ is said to be ω if the diagonal girths of the members of $\mathrm{Fin}\mathbf{V}_S$ are not bounded by any integer.
(8) The non-diagonal girth of $\mathbf{V}$ is said to be n if there is $\mathfrak{B} \in \mathrm{Fin}\mathbf{V}_S$ whose non-diagonal girth is n, and the non-diagonal girth of every other member of $\mathrm{Fin}\mathbf{V}_S$ is less than or equal to n.
(9) The non-diagonal girth of $\mathbf{V}$ is said to be ω if the non-diagonal girths of the members of $\mathrm{Fin}\mathbf{V}_S$ are not bounded by any integer.

Lemma 6.3. *If $\mathbf{V}$ is a variety of cylindric algebras of finite depth whose diagonal and non-diagonal girths are bounded by some integer, then $\mathbf{V}$ is a finitely generated variety.*

Proof. There exist only finitely many non-isomorphic finite simple cylindric algebras whose depth, the diagonal girth and the non-diagonal girth are bounded by some integer. Therefore, there are only finitely many non-isomorphic finite simple $\mathbf{V}$-algebras, implying that $\mathbf{V}$ is finitely generated. $\square$

It follows that if a variety $\mathbf{V}$ of a finite depth is not finitely generated, then either the diagonal girth or the non-diagonal girth of $\mathbf{V}$ must be ω.

Lemma 6.4. *If $\mathbf{V}$ is a variety of finite depth whose diagonal girth is ω, then one of $\mathbf{V}_1 - \mathbf{V}_3$ is contained in $\mathbf{V}$.*

Proof. Since the diagonal girth of $\mathbf{V}$ is ω, for each n there is $\mathfrak{B} \in \mathrm{Fin}\mathbf{V}_S$ whose diagonal girth is n. Let $\mathcal{X}$ be the dual cylindric space of $\mathfrak{B}$. Then $\mathcal{X}$ is a quasi-square. Denote by C the diagonal E_0-cluster of $\mathcal{X}$ containing n points. Then two cases are possible. Either $d(\mathcal{X}) = 1$ or $d(\mathcal{X}) \geq 2$ for infinitely many n. In the former case, it is obvious that $\mathcal{X}$ is isomorphic to $\mathcal{X}_n^1$, and so $\mathbf{V}_1 \subseteq \mathbf{V}$. In the latter case, define an equivalence relation R on $\mathcal{X}$ by putting

- xRy if $x = y$ for any $x, y \in C \cup D$;
- xRy if xE_0y for any $x, y \in X - (C \cup D)$.

Clearly R is a cylindric partition. Denote $\mathcal{X}/R$ by $\mathcal{Y}$. Then every non-diagonal E_0-cluster of $\mathcal{Y}$ is a singleton set and every diagonal E_0-cluster different from C contains either one or two points. Again there are two cases possible. Either $d(\mathcal{Y}) = 2$ or $d(\mathcal{Y}) > 2$ for infinitely many n. In the former case, $\mathcal{Y}$ is isomorphic to either $\mathcal{X}_n^2$ or $\mathcal{X}_n^3$ for infinitely many n. Therefore, either $\mathbf{V}_2 \subset \mathbf{V}$ or $\mathbf{V}_3 \subset \mathbf{V}$. And in the latter case, define an equivalence relation Q on Y by putting

- xQy if $x = y$ for any $x, y \in C$;
- xQy for any $x, y \in E_1(C) - C$;
- xQy for any $x, y \in E_2(C) - C$;
- xQy for any $x, y \in D - C$;
- xQy for any $x, y \in Y - (E_1(C) \cup E_2(C) \cup D)$.

It is a matter of routine verification that Q is a cylindric partition, and that $\mathcal{Y}/Q$ is isomorphic to $\mathcal{X}_n^3$. Thus, $\mathbf{V}_3 \subset \mathbf{V}$. □

Lemma 6.5. *If $\mathbf{V}$ is a variety of finite depth whose non-diagonal girth is ω, then one of $\mathbf{V}_4 - \mathbf{V}_{15}$ is contained in $\mathbf{V}$.*

Proof. Since the non-diagonal girth of $\mathbf{V}$ is ω, for each n there is $\mathfrak{B} \in \mathrm{Fin}\mathbf{V}_S$ whose non-diagonal girth is n. Let $\mathcal{X}$ be the dual cylindric space of $\mathfrak{B}$. Then $\mathcal{X}$ is a quasi-square. Denote by C the non-diagonal E_0-cluster of $\mathcal{X}$ containing n points. Since the non-diagonal E_0-clusters exist only in cylindric spaces of depth > 1, we have $d(\mathcal{X}) > 1$. Define an equivalence relation R on $\mathcal{X}$ by putting

- xRy if $x = y$ for any $x, y \in C \cup D$;
- xRy if xE_0y for any $x, y \in X - (C \cup D)$.

Clearly R is a cylindric partition. Since $d(\mathcal{X}) > 1$, there are three cases possible. Either $d(\mathcal{X}) = 2$, $d(\mathcal{X}) = 3$, or $d(\mathcal{X}) > 3$ for infinitely many n.

If $d(\mathcal{X}) = 2$ for infinitely many n, then $\mathcal{X}/R$ is isomorphic to one of $\mathcal{X}_n^4 - \mathcal{X}_n^7$ for infinitely many n, implying that one of $\mathbf{V}_4 - \mathbf{V}_7$ is contained in $\mathbf{V}$.

If $d(\mathcal{X}) = 3$ for infinitely many n, then $\mathcal{X}/R$ is isomorphic to one of $\mathcal{X}_n^8 - \mathcal{X}_n^{15}$ for infinitely many n, implying that one of $\mathbf{V}_8 - \mathbf{V}_{15}$ is contained in $\mathbf{V}$.

Finally, let $3 < d(\mathcal{X}) < \omega$ for infinitely many n. Denote by C' the diagonal E_0-cluster E_1-related to C, and by C'' - the diagonal E_0-cluster E_2-related to C. Define an equivalence relation R on $\mathcal{X}$ by putting

- xRy if $x = y$ for any $x, y \in C \cup ((C' \cup C'') \cap D)$;
- xRy for any $x, y \in D - (C' \cup C'')$;
- xRy for any $x, y \in X - (D \cup E_1(C') \cup E_2(C') \cup E_1(C'') \cup E_2(C''))$;
- xRy if xE_0y for any $x, y \in (E_2(C') \cap E_1(C'')) \cup ((C' \cup C'') - D)$;
- xRy for any $x, y \in E_2(C) - (C \cup C'')$;
- xRy for any $x, y \in E_1(C) - (C \cup C')$;
- xRy for any $x, y \in E_2(C') - (E_1(C'') \cup C')$;

- xRy for any $x, y \in E_1(C'') - (E_2(C') \cup C'')$.

It is a matter of routine verification that R is a cylindric partition. Moreover, there are four cases possible. Either both C' and C'' are singleton sets, C' is a singleton set and C'' is not, C'' is a singleton set and C' is not, or neither C' nor C'' are singleton sets, for infinitely many n. In the first case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{11}$, in the second case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{13}$, in the third case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{12}$, and finally in the fourth case $\mathcal{X}/R$ is isomorphic to $\mathcal{X}_n^{15}$. Therefore, one of $\mathbf{V}_{12} - \mathbf{V}_{15}$ is contained in $\mathbf{V}$.

Thus, going through all the cases we obtain that one of $\mathbf{V}_4 - \mathbf{V}_{15}$ is contained in $\mathbf{V}$. □

Corollary 6.6. (1) *The only pre finitely generated varieties in* $\Lambda(\mathbf{CA}_2)$ *are* $\mathbf{V}_1$–$\mathbf{V}_{15}$.

(2) *The only pre finitely generated varieties in* $\Lambda(\mathbf{RCA}_2)$ *are* $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$, $\mathbf{V}_7$, $\mathbf{V}_{14}$ *and* $\mathbf{V}_{15}$.

Proof. This is an immediate consequence of Lemmas 6.1, 6.3 – 6.5, and the fact that all the fifteen varieties are non-comparable. □

7. Lattice structure of $\Lambda(\mathbf{CA}_2)$

In order to obtain a rough picture of the lattice structure of subvarieties of $\mathbf{CA}_2$, we need the following notation:

- $\mathbf{FG} = \{\mathbf{V} \in \Lambda(\mathbf{CA}_2) : \mathbf{V}$ is finitely generated$\}$;
- $\mathbf{D}_F = \{\mathbf{V} \in \Lambda(\mathbf{CA}_2) : d(\mathbf{V}) < \omega$ and $\mathbf{V} \notin \mathbf{FG}\}$;
- $\mathbf{D}_\omega = \{\mathbf{V} \in \Lambda(\mathbf{CA}_2) : d(\mathbf{V}) = \omega\}$.

Let also $\mathbf{V}_\perp$ denote the trivial variety.

Theorem 7.1. (1) $\{\mathbf{FG}, \mathbf{D}_F, \mathbf{D}_\omega\}$ *is a partition of* $\Lambda(\mathbf{CA}_2)$.

(2) $\mathbf{V}_\perp$ *is a least element of* $\mathbf{FG}$.

(3) $\mathbf{FG}$ *does not have maximal elements.*

(4) $\mathbf{D}_F$ *has precisely fifteen minimal elements.*

(5) $\mathbf{D}_F$ *does not have maximal elements.*

(6) $\mathbf{U}$ *and* $\mathbf{CA}_2$ *are the least and the greatest elements of* $\mathbf{D}_\omega$, *respectively.*

Proof. This follows immediately from Theorem 5.5 and Corollary 6.6. □

The lattice $\Lambda(\mathbf{CA}_2)$ can be roughly depicted as shown in Figure 5. Now we will investigate the lower part of $\Lambda(\mathbf{CA}_2)$ in a greater detail. It follows from Corollary 6.6 that a variety $\mathbf{V} \subseteq \mathbf{CA}_2$ ($\mathbf{RCA}_2$) is finitely generated iff $\mathbf{V}$ does not contain one of the fifteen (six) pre finitely generated varieties. Another criterion is given by the following theorem.

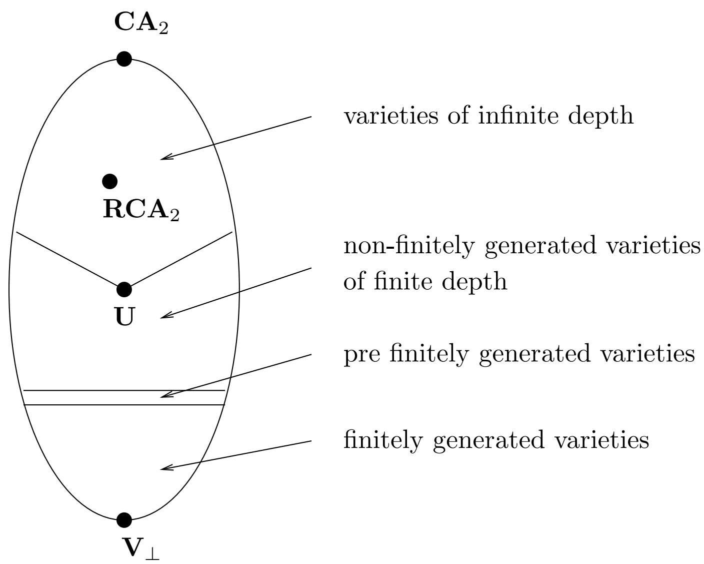

FIGURE 5. Rough picture of $\Lambda(\mathbf{CA}_2)$

Theorem 7.2. *For a variety* $\mathbf{V} \subseteq \mathbf{CA}_2$ *the following conditions are equivalent:*

(1) $\mathbf{V}$ *is finitely generated.*
(2) $\mathbf{V}$ *has only finitely many subvarieties.*
(3) $\mathbf{V}$ *contains only finitely many non-isomorphic simple algebras (and all of them are finite).*

Proof. (1) $\Rightarrow$ (2) is straightforward since $\mathbf{CA}_2$ is congruence-distributive.

(2) $\Rightarrow$ (3). First note that if $\mathbf{V}$ contains an infinite simple algebra, then it contains infinitely many non-isomorphic finite simple algebras. To see this, let $\mathfrak{B}$ be an infinite simple $\mathbf{V}$-algebra. If $d(\mathfrak{B}) \geq \omega$, then by Lemma 5.4 $\mathfrak{B}$ has infinitely many non-isomorphic simple subalgebras. If $d(\mathfrak{B}) < \omega$, then either the diagonal or non-diagonal girth of $\mathfrak{B}$ is infinite and the same arguments as in the proofs of Lemmas 6.4 and 6.5 show that there are infinitely many non-isomorphic simple subalgebras of $\mathfrak{B}$. Now suppose that $\mathbf{V}$ contains an infinite family $\{\mathfrak{B}_i\}_{i \in I}$ of non-isomorphic simple algebras. Then $\mathbf{V}$ contains an infinite family $\{\mathfrak{B}'_i\}_{i \in I}$ of finite non-isomorphic simple algebras. By Jónsson's Lemma $\{\mathbf{HSP}(\mathfrak{B}'_i)\}_{i \in I}$ is an infinite family of distinct subvarieties of $\mathbf{V}$, which is a contradiction.

(3) $\Rightarrow$ (1). Let $\{\mathfrak{B}_i\}_{i=1}^n$ be the family of all simple non-isomorphic $\mathbf{V}$-algebras. It follows from the above that they are all finite. Then $\prod_{i=1}^n \mathfrak{B}_i$ generates $\mathbf{V}$ and therefore $\mathbf{V}$ is finitely generated. $\square$

By an *immediate successor* of a variety $\mathbf{V} \subseteq \mathbf{CA}_2$ we mean an immediate successor in the lattice $\Lambda(\mathbf{CA}_2)$.

Corollary 7.3. (1) *Every immediate successor of a finitely generated variety of cylindric algebras is finitely generated.*
(2) *A finitely generated variety of cylindric algebras has only finitely many immediate successors.*

Proof. (1) Let $\mathbf{V}'$ be an immediate successor of a finitely generated variety $\mathbf{V}$. Since $\mathbf{V}$ is finitely generated $\mathbf{V} = \mathbf{HSP}(\mathfrak{B})$ for a finite cylindric algebra $\mathfrak{B}$. Since $\mathbf{V} \subset \mathbf{V}'$, there is a simple cylindric algebra $\mathfrak{B}' \in \mathbf{V}'$ with $\mathfrak{B}' \notin \mathbf{V}$. Then $\mathbf{V} \subset \mathbf{HSP}(\mathfrak{B} \times \mathfrak{B}')$ and because $\mathbf{V}'$ is an immediate successor of $\mathbf{V}$ we have that $\mathbf{V}' = \mathbf{HSP}(\mathfrak{B} \times \mathfrak{B}')$. Moreover, if $\mathfrak{B}'$ is infinite, then the same arguments as in the proof of Theorem 7.2 show that $\mathfrak{B}'$ has infinitely many non-isomorphic subalgebras, which is impossible since $\mathbf{V}'$ is an immediate successor of $\mathbf{V}$ and $\mathbf{V}$ is finitely generated. Hence $\mathfrak{B}'$ is finite and therefore $\mathbf{V}'$ is finitely generated.

(2) The proof is analogous to the standard proof that a finitely generated variety of interior algebras has only finitely many immediate successors (see, e.g., Blok [4, Theorem 7.5]). □

7.1. Varieties of cylindric algebras of depth one. In this subsection we give a complete characterization of the lattice structure of the varieties of cylindric algebras of depth one. In the diagonal free case, the lattice of varieties of $\mathbf{Df}_2$-algebras of E_1 and E_2-depth one is relatively simple. It is isomorphic to the lattice of varieties of monadic algebras and is an $(\omega+1)$-chain (see [10, Theorem 4] and [8, Theorem 4.1.22]). As we will see below, the structure of the lattice of varieties of cylindric algebras of depth one is more complex.

Let $\mathbf{2}^n$ denote the 2^n-element $\mathbf{Df}_2$-algebra, where $n \geq 1$ and

$$\exists_i(a) = \begin{cases} 0 & \text{if } a{=}0, \\ 1 & \text{otherwise}, \end{cases}$$

for $i = 1, 2$. Let also d be an atom of $\mathbf{2}^n$. Then $(\mathbf{2}^n, d)$ is a cylindric algebra. It is obvious that $(\mathbf{2}^n, d)$ is simple and has depth one. Observe that the dual space of $(\mathbf{2}^n, d)$ is isomorphic to $\mathcal{X}_n^1$ defined in Section 6. Hence $(\mathbf{2}^n, d) \in \mathbf{RCA}_2$ for every $n \in \omega$. It is also clear that up to isomorphism $\mathcal{X}_n^1$, $n \in \omega$, are the only finite quasi-squares of depth one. Thus $(\mathbf{2}^n, d)$ are the only finite simple cylindric algebras of depth one.

We recall that in the diagonal-free case the two-element $\mathbf{Df}_2$-algebra $\mathbf{2}$ is a subalgebra of every non-trivial $\mathbf{Df}_2$-algebra. For $\mathbf{CA}_2$ the situation is different.

Proposition 7.4. *Suppose $\mathfrak{B}$ is a non-trivial simple cylindric algebra.*
(1) $(\mathbf{2}, 1)$ *is a subalgebra of $\mathfrak{B}$ iff $\mathfrak{B}$ is isomorphic to* $(\mathbf{2}, 1)$.

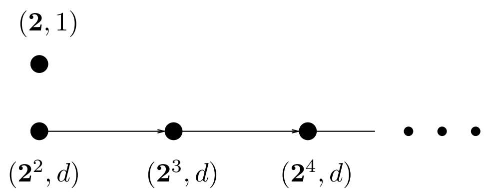

FIGURE 6. The poset of simple cylindric algebras of depth one

(2) *If* $\mathfrak{B}$ *is not isomorphic to* $(\mathbf{2},1)$*, then* $(\mathbf{2}^2,d)$ *is a subalgebra of* $\mathfrak{B}$*.*

Proof. (1) If $(\mathbf{2},1)$ is a subalgebra of $\mathfrak{B}=(\mathcal{B},d)$, then $d=1$. Suppose there is an element $a\in B$ with $0<a<1$. Then, by Definition 2.6(2), $\exists_i -a \leq -a$ for $i=1,2$. Hence the ideal $(-a]$ corresponds to a non-trivial proper congruence, i.e., $\mathfrak{B}/_{(-a]}$ is a proper non-trivial homomorphic image of $\mathfrak{B}$, which is impossible since $\mathfrak{B}$ is simple. Therefore, $\mathcal{B}=\mathbf{2}$ and $\mathfrak{B}$ is isomorphic to $(\mathbf{2},1)$.

(2) It is known that $\exists_1\exists_2 -d = \exists_1 -d = \exists_2 -d$ (see, e.g., [8, Theorem 1.3.18(ii)]). Since $\mathfrak{B}$ is not isomorphic to $(\mathbf{2},1)$, we have $d\neq 1$. Hence, $-d\neq 0$. So, $\exists_1\exists_2 -d = 1$ since $\mathfrak{B}$ is simple. Thus, $\{1,0,d,-d\}$ is a cylindric subalgebra of $\mathfrak{B}$.

□

Let $\mathbf{Var}(\mathbf{2},1)$ and $\mathbf{Var}(\mathbf{2}^2,d)$ denote the varieties generated by $(\mathbf{2},1)$ and $(\mathbf{2}^2,d)$, respectively.

Corollary 7.5. (1) *The varieties* $\mathbf{Var}(\mathbf{2},1)$ *and* $\mathbf{Var}(\mathbf{2}^2,d)$ *are the only atoms in* $\Lambda(\mathbf{CA}_2)$*.*

(2) *If a variety* $\mathbf{V}$ *of cylindric algebras contains the two-element cylindric algebra* $(\mathbf{2},1)$*, then* $\mathbf{V}$ *is generated by a simple algebra iff* $\mathbf{V}=\mathbf{Var}(\mathbf{2},1)$*.*

Proof. (1) It follows from Proposition 7.4(1) that $\mathbf{Var}(\mathbf{2},1)$ and $\mathbf{Var}(\mathbf{2}^2,d)$ are incomparable. Now let $\mathbf{V}$ be a non-trivial subvariety of $\mathbf{CA}_2$, and $\mathfrak{B}$ a simple $\mathbf{V}$-algebra. By Proposition 7.4 either $(\mathbf{2},1)$ or $(\mathbf{2}^2,d)$ is a subalgebra of $\mathfrak{B}$. Thus $\mathbf{Var}(\mathbf{2},1)\subseteq\mathbf{V}$ or $\mathbf{Var}(\mathbf{2}^2,d)\subseteq\mathbf{V}$.

(2) Suppose $(\mathbf{2},1)\in\mathbf{V}$ and $\mathbf{V}$ is generated by a simple $\mathbf{V}$-algebra $\mathfrak{B}$. Using the standard splitting technique (see, e.g., Kracht [9, §7.3]) we obtain that $(\mathbf{2},1)\in\mathbf{S}(\mathfrak{B})$, and applying Proposition 7.4 we get that $\mathfrak{B}$ is isomorphic to $(\mathbf{2},1)$.

□

Let $\mathbf{V}^1\subseteq\mathbf{CA}_2$ be the variety of all cylindric algebras of depth one. It is known from [2, Theorem 4.2] that

$$\mathbf{V}^1=\mathbf{CA}_2+(\exists_2\exists_1 a=\exists_1 a)=\mathbf{RCA}_2+(\exists_2\exists_1 a=\exists_1 a).$$

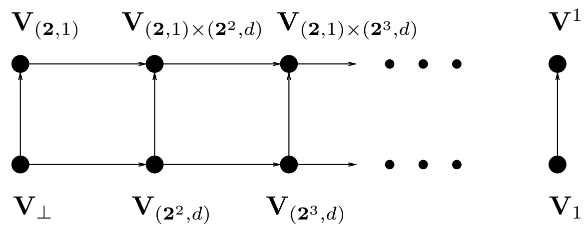

FIGURE 7. The lattice of varieties of cylindric algebras of depth one

Let $(\mathcal{F}, \leq)$ denote the partially ordered set of all non-isomorphic finite cylindric algebras of depth one. We recall from Section 3 that $\leq$ is defined on $\mathcal{F}$ by putting $\mathfrak{B} \leq \mathfrak{B}'$ iff $\mathfrak{B} \in \mathbf{S}(\mathfrak{B}')$. As we pointed out above, $\mathcal{F} = \{(\mathbf{2}^n, d) : n \in \omega\}$. It follows from Proposition 7.4 that $(\mathcal{F}, \leq)$ is isomorphic to the disjoint union of the set of natural numbers $(\mathbf{N}, \leq)$ with the set consisting of one reflexive point (see Figure 6).

Recall that $\mathcal{G} \subseteq \mathcal{F}$ is called a *downset* of $\mathcal{F}$ if $\mathfrak{A} \in \mathcal{G}$ and $\mathfrak{B} \leq \mathfrak{A}$ imply $\mathfrak{B} \in \mathcal{G}$. Since every variety of cylindric algebras of finite depth is locally finite, applying [5, Theorem 3.3] we obtain the following representation of the lattice of varieties of cylindric algebras of depth one.

Theorem 7.6. *The lattice of varieties of cylindric algebras of depth one is isomorphic to the lattice of downsets of* $(\mathcal{F}, \leq)$.

The lattice of varieties of cylindric algebras of depth one is shown in Figure 7. To explain the labeling, with each downset of $(\mathcal{F}, \leq)$ of the form $\downarrow(\mathbf{2}^n, d) = \{(\mathbf{2}^k, d) : 1 < k < n\}$ we associated the variety $\mathbf{V}_{(\mathbf{2}^n,d)}$ generated by $(\mathbf{2}^n, d)$; and with each downset of the form $\downarrow(\mathbf{2}^n, d) \cup \{(\mathbf{2}, 1)\}$ we associated the variety $\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ generated by $(\mathbf{2}, 1) \times (\mathbf{2}^n, d)$; furthermore, $\mathbf{V}_1 = \mathbf{HSP}(\{(\mathbf{2}^n, d) : n > 1\})$.

Theorem 7.7. *Every subvariety of* $\mathbf{V}^1$ *is finitely axiomatizable.*

Proof. A proof similar to the one in Scroggs [10, p.119] shows that the inequality

$$(\mathrm{S}_n) \qquad \bigwedge_{k=1}^{n+1} \exists_1 a_k \leq \bigvee_{\substack{1 \leq k,j \leq n+1 \\ k \neq j}} \exists_1 (a_k \wedge a_j)$$

holds true in a simple cylindric algebra of depth one iff the corresponding quasi-square contains $\leq n$ points. Therefore, the varieties $\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ are axiomatized by the identities of $\mathbf{V}^1$ plus (S_n). On the other hand, the identity $\exists_1 - d = 1$ holds true in $(\mathbf{2}^n, d)$ iff $n > 1$. Therefore, the variety $\mathbf{V}_1$ is axiomatized by the identities of $\mathbf{V}^1$ plus $\exists_1 - d = 1$, while the varieties $\mathbf{V}_{(\mathbf{2}^n,d)}$ are axiomatized by adding $\exists_1 - d = 1$ to the identities of $\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$. $\square$

Remark 7.8. In fact, using the Jankov-Fine type formulas, the technique analogous to [3] shows that every subvariety of $\mathbf{CA}_2$ of finite depth is finitely axiomatizable. (Note that the proof of this fact is much simpler than the original one from [3] for the diagonal free case since, in contrast with $\mathbf{Df}_2$-algebras, every cylindric algebra has the same E_1 and E_2-depth.) Nevertheless, the cardinality of $\Lambda(\mathbf{CA}_2)$ is that of continuum, which means that there exist uncountably many non-finitely axiomatizable subvarieties of $\mathbf{CA}_2$ of infinite depth.

7.2. Reduct functors. Suppose $\mathfrak{B} = (B, \exists_1, \exists_2, d)$ is a cylindric algebra. In Section 2 we denoted its $\mathbf{Df}_2$-reduct by $\mathbb{F}(\mathfrak{B}) = (B, \exists_1, \exists_2) \in \mathbf{Df}_2$. If K is a subclass of $\mathbf{CA}_2$, let $\mathbb{F}(K) = \{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in K\}$. Also if M is a subclass of $\mathbf{Df}_2$, let $\mathbb{F}^{-1}(M) = \{\mathfrak{B} \in \mathbf{CA}_2 : \mathbb{F}(\mathfrak{B}) \in M\}$.

Lemma 7.9. *Suppose $K \subseteq \mathbf{CA}_2$ and $M \subseteq \mathbf{Df}_2$. Then the following hold.*

(1) $\mathbf{H}\mathbb{F}(K) = \mathbb{F}\mathbf{H}(K)$.
(2) $\mathbf{S}\mathbb{F}(K) \supset \mathbb{F}\mathbf{S}(K)$.
(3) $\mathbf{P}\mathbb{F}(K) = \mathbb{F}\mathbf{P}(K)$.
(4) $\mathbf{H}\mathbb{F}^{-1}(M) \subset \mathbb{F}^{-1}\mathbf{H}(M)$.
(5) $\mathbf{S}\mathbb{F}^{-1}(M) \subset \mathbb{F}^{-1}\mathbf{S}(M)$.
(6) $\mathbf{P}\mathbb{F}^{-1}(M) = \mathbb{F}^{-1}\mathbf{P}(M)$.

Proof. (1) This claim follows immediately from Theorems 2.5(1) and 2.11(1) (see Remark 2.13).

(2) It is obvious that if $\mathfrak{B}$ is a cylindric subalgebra of $\mathfrak{A}$, then $\mathbb{F}(\mathfrak{B})$ is a $\mathbf{Df}_2$-subalgebra of $\mathbb{F}(\mathfrak{A})$. Hence, $\mathbb{F}\mathbf{S}(K) \subseteq \mathbf{S}\mathbb{F}(K)$. To see that the converse inclusion does not hold, let $d(K) \geq 2$ and consider $\mathfrak{B} \in K$ with $d(\mathfrak{B}) \geq 2$. Denote by $\mathcal{X} = (X, E_1, E_2, D)$ the dual cylindric space of $\mathfrak{B}$. Define a partition R on X by putting xRy if xE_2y. Then R is a correct $\mathbf{Df}_2$-partition and the $\mathbf{Df}_2$-algebra $\mathcal{A}$ corresponding to the $\mathbf{Df}_2$-space X/R belongs to $\mathbf{S}\mathbb{F}(K)$. On the other hand, the E_1-depth of X/R is 1 and the E_2-depth of X/R is ≥ 2. Therefore, X/R has different E_1 and E_2 depths, which by Proposition 2.8 implies that $\mathcal{A}$ can not be the reduct of any cylindric algebra. Thus, $\mathbf{S}\mathbb{F}(K) \not\subseteq \mathbb{F}\mathbf{S}(K)$.

(3) Follows from the fact that for any family $\{\mathfrak{B}_i\}_{i\in I}$ of cylindric algebras we have $\mathbb{F}(\prod_{i\in I} \mathfrak{B}_i) = \prod_{i\in I} \mathbb{F}(\mathfrak{B}_i)$.

(4) That $\mathbf{H}\mathbb{F}^{-1}(M) \subseteq \mathbb{F}^{-1}\mathbf{H}(M)$ follows from the fact that every cylindric homomorphism is a also a $\mathbf{Df}_2$-homomorphism. To show that this inclusion is proper, consider a cylindric algebra $\mathfrak{B}$ and let $\mathcal{A}$ be a $\mathbf{Df}_2$-algebra such that $d_1(\mathcal{A}) \neq d_2(\mathcal{A})$. Then $\mathbb{F}(\mathfrak{B})$ is a homomorphic image of $\mathbb{F}(\mathfrak{B})\times\mathcal{A}$, but since $d_1(\mathcal{A}) \neq d_2(\mathcal{A})$, $\mathbb{F}(\mathfrak{B})\times\mathcal{A}$ is not the reduct of any cylindric algebra. Hence, $\mathfrak{B} \in \mathbb{F}^{-1}\mathbf{H}(\{\mathbb{F}(\mathfrak{B}) \times \mathcal{A}\})$, but $\mathbf{H}\mathbb{F}^{-1}(\{\mathbb{F}(\mathfrak{B}) \times \mathcal{A}\})$ is empty.

(5) That $\mathbf{S}\mathbb{F}^{-1}(M) \subseteq \mathbb{F}^{-1}\mathbf{S}(M)$ follows from the fact that if $\mathfrak{B}$ is a cylindric subalgebra of $\mathfrak{A}$, then $\mathbb{F}(\mathfrak{B})$ is a $\mathbf{Df}_2$-subalgebra of $\mathbb{F}(\mathfrak{A})$. To see that this inclusion is proper, suppose the two-element $\mathbf{Df}_2$-algebra $\mathbf{2}$ does not belong to M. Then the two-element cylindric algebra, $(\mathbf{2}, 1)$ does not belong to $\mathbb{F}^{-1}(M)$. By Proposition 7.4 $(\mathbf{2}, 1) \notin \mathbf{S}\mathbb{F}^{-1}(M)$. On the other hand, $\mathbf{2}$ is a $\mathbf{Df}_2$-subalgebra of every $\mathbf{Df}_2$-algebra. Therefore, $\mathbf{2} \in \mathbf{S}(M)$ and $(\mathbf{2}, 1) \in \mathbb{F}^{-1}(\mathbf{S}(M))$.

(6) That $\mathbf{P}\mathbb{F}^{-1}(M) \subseteq \mathbb{F}^{-1}\mathbf{P}(M)$ follows from the definition of the product of cylindric algebras. To see the converse, suppose $\mathfrak{B} \in \mathbb{F}^{-1}\mathbf{P}(M)$. Then $\mathfrak{B} = (\mathcal{B}, d)$, where $\mathcal{B} = \prod_{i\in I} \mathcal{B}_i$ for some $\mathbf{Df}_2$-algebras $\mathcal{B}_i \in M$. Let $(\mathcal{B}_i, d_i)$ be the i-th projection of $\mathfrak{B}$. Since the i-th projection is an onto $\mathbf{Df}_2$-homomorphism, by Remark 2.13 it is also a cylindric homomorphism. Hence each $(\mathcal{B}_i, d_i)$ is a cylindric algebra and $d = \langle d_i \rangle_{i\in I}$. Then $\mathfrak{B}$ is isomorphic to $\prod_{i\in I}(\mathcal{B}_i, d_i)$. Now every $(\mathcal{B}_i, d_i)$ belongs to $\mathbb{F}^{-1}(M)$. Hence, $\mathbb{F}^{-1}\mathbf{P}(M) \subseteq \mathbf{P}\mathbb{F}^{-1}(M)$. □

Theorem 7.10. (1) *If* $\mathbf{K}$ *is a subvariety of* $\mathbf{Df}_2$*, then* $\mathbb{F}^{-1}(\mathbf{K})$ *is a subvariety of* $\mathbf{CA}_2$*.*

(2) *For a non-trivial subvariety* $\mathbf{V}$ *of* $\mathbf{CA}_2$*,* $\mathbb{F}(\mathbf{V})$ *is a subvariety of* $\mathbf{Df}_2$ *iff* $\mathbf{V} = \mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ *for some* $n \in \omega$*.*

Proof. (1) By Lemma 7.9 we have $\mathbf{HSP}\mathbb{F}^{-1}(\mathbf{K}) \subseteq \mathbb{F}^{-1}(\mathbf{HSP}(\mathbf{K})) = \mathbb{F}^{-1}(\mathbf{K})$. Hence, $\mathbb{F}^{-1}(\mathbf{K})$ is a variety of cylindric algebras.

(2) Suppose $\mathbf{V}$ is a subvariety of $\mathbf{CA}_2$. If $d(\mathbf{V}) > 1$, then it follows from the proof of Lemma 7.9(2) that $\mathbb{F}(\mathbf{V})$ is not closed under subalgebras, hence is not a variety. Thus, if $\mathbb{F}(\mathbf{V})$ is a variety, then $\mathbf{V} \subseteq \mathbf{V}^1$. If $(\mathbf{2}, 1) \notin \mathbf{V}$, then $\mathbb{F}(\mathbf{2}, 1) \notin \mathbb{F}(\mathbf{V})$ and again $\mathbb{F}(\mathbf{V})$ is not a variety since every nontrivial variety of diagonal-free cylindric algebras contains $\mathbf{2} = \mathbb{F}(\mathbf{2}, 1)$. We now show that $\mathbb{F}(\mathbf{V}^1)$ is not a variety. Let $\mathbb{C}$ denote the Cantor space. Then $\mathcal{X} = (\mathbb{C}, E_1, E_2)$ is a $\mathbf{Df}_2$-space, where $E_1(c) = E_2(c) = \mathbb{C}$ for any $c \in \mathbb{C}$. If $\mathcal{X}$ were the reduct of a cylindric space, then $\mathcal{X}$ would contain an isolated point. Since $\mathbb{C}$ is dense in itself, it follows that $\mathcal{X}$ is not the reduct of any cylindric space. Let $\{y\}$ be a singleton topological space. Then $\mathcal{Y} = (\mathbb{C} \oplus \{y\}, E_1, E_2, \{y\})$ is a cylindric space, where $E_1(x) = E_2(x) = \mathbb{C} \oplus \{y\}$ for any $x \in \mathbb{C} \oplus \{y\}$. Moreover, $\mathfrak{B} = (CP(\mathcal{Y}), E_1, E_2, \{y\})$ is an infinite simple cylindric algebra of depth 1, and so $\mathfrak{B} \in \mathbf{V}^1$. Now consider $\mathbb{R}(\mathcal{Y}) = (\mathbb{C} \oplus \{y\}, E_1, E_2)$. Fix any point $c \in \mathbb{C}$ and let R be the smallest equivalence relation which identifies y and c. It is easy to check that R is a correct $\mathbf{Df}_2$-partition, and that $\mathbb{R}(\mathcal{Y})/R$ is isomorphic to $\mathcal{X}$. So, $\mathcal{A} = (CP(\mathcal{X}), E_1, E_2)$ is isomorphic to a $\mathbf{Df}_2$-subalgebra of $\mathbb{F}(\mathfrak{B})$, but it is not the reduct of any cylindric algebra. Hence, $\mathcal{A}$ does not belong to $\mathbb{F}(\mathbf{V}^1)$, and so $\mathbb{F}(\mathbf{V}^1)$ is not a variety. Therefore, if $\mathbf{V} \neq \mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}$ for any $n \in \omega$, then $\mathbb{F}(\mathbf{V})$ is not a variety. Finally, one can easily verify that for any $n \in \omega$, $\mathbf{HS}(\{(\mathbf{2}, 1) \times (\mathbf{2}^n, d)\}) = \{(\mathbf{2}^m, d) : m \leq n\}$. This implies that $\mathbb{F}(\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)}) =$

$\mathbf{V}_{\mathbf{2}^n}$, where $\mathbf{2}^n = \mathbb{F}((\mathbf{2}^n, d))$. Therefore, we obtained that $\mathbb{F}(\mathbf{V}_{(\mathbf{2},1)\times(\mathbf{2}^n,d)})$ is a variety for any $n \in \omega$, which finishes the proof of the theorem. $\square$

We define a map $\Phi\colon \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$ from the lattice $\Lambda(\mathbf{CA}_2)$ of subvarieties of $\mathbf{CA}_2$ to the lattice $\Lambda(\mathbf{Df}_2)$ of subvarieties of $\mathbf{Df}_2$ by putting $\Phi(\mathbf{V}) = \mathbf{S}(\mathbb{F}(\mathbf{V}))$. It follows from Lemma 7.9 that Φ is well defined. The following theorem establishes basic properties of Φ.

Theorem 7.11. (1) Φ *is order preserving.*
(2) *For* $\mathbf{K} \in \Lambda(\mathbf{Df}_2)$, *if* $d_1(\mathbf{K}) \neq d_2(\mathbf{K})$, *then* $\Phi^{-1}(\mathbf{K}) = \emptyset$.
(3) $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$.
(4) Φ *is neither onto nor* 1–1 *and does not preserve* $\wedge$.
(5) Φ *preserves top, bottom, and* $\vee$.

Proof. (1) directly follows from the definition of Φ.

(2) First we show that

$$d(\mathbf{V}) = d_1(\Phi(\mathbf{V})) = d_2(\Phi(\mathbf{V})) \tag{D}$$

for every $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$. It is obvious that $d(\mathbf{V}) \leq d_1(\Phi(\mathbf{V})), d_2(\Phi(\mathbf{V}))$. Conversely, for each finite simple algebra $\mathcal{A} \in \Phi(\mathbf{V})$, there exists $\mathcal{B} \in \mathbb{F}(\mathbf{V})$ such that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. Hence, $d_i(\mathcal{A}) \leq d_i(\mathcal{B}) \leq d(\mathbf{V})$, and therefore, $d(\mathbf{V}) \geq d_1(\Phi(\mathbf{V})), d_2(\Phi(\mathbf{V}))$. Now suppose $\mathbf{K} \in \Lambda(\mathbf{Df}_2)$ and $d_1(\mathbf{K}) \neq d_2(\mathbf{K})$. If there exists $\mathbf{V} \in \Phi^{-1}(\mathbf{K})$, then it follows from (D) that $d(\mathbf{V}) = d_1(\mathbf{K}) = d_2(\mathbf{K})$, which is a contradiction.

(3) First we show that $\Phi(\mathbf{U}) = \mathbf{Df}_2$. Let $\mathfrak{B}_n$ be a finite uniform cylindric algebra and $\mathcal{X}_n$ its dual uniform space. Then the quotient-space $\mathbb{R}(\mathcal{X}_n)/E_0$ is isomorphic to $n \times n$. Hence, every finite square $\mathbf{Df}_2$-algebra is a subalgebra of $\mathbb{F}(\mathfrak{B}_n)$ for some $n \in \omega$. Therefore, every finite square $\mathbf{Df}_2$-algebra belongs to $\Phi(\mathbf{U})$. Since $\mathbf{Df}_2$ is generated by its finite square algebras (see, e.g., [11] or [2, p.23]), then $\Phi(\mathbf{U}) = \mathbf{Df}_2$. Now since Φ is order preserving, we get that $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$.

(4) That Φ is not onto follows from (2). To see that it is not 1–1 consider the varieties $\mathbf{V}_{\mathfrak{B}_2^2}$ and $\mathbf{V}_{\mathfrak{B}_2^4}$, where $\mathfrak{B}_2^2$ and $\mathfrak{B}_2^4$ denote the cylindric algebras of the power sets of the cylindric spaces $\mathcal{X}_2^2$ and $\mathcal{X}_2^4$ shown in Figure 4 above (see Section 6). Since $\mathfrak{B}_2^2$ is representable and $\mathfrak{B}_2^4$ is not, $\mathfrak{B}_2^2$ is not isomorphic to $\mathfrak{B}_2^4$. Therefore, $\mathbf{V}_{\mathfrak{B}_2^2} \neq \mathbf{V}_{\mathfrak{B}_2^4}$. However, $\mathbb{F}(\mathfrak{B}_2^2)$ is isomorphic to $\mathbb{F}(\mathfrak{B}_2^4)$. Hence, $\Phi(\mathbf{V}_{\mathfrak{B}_2^2}) = \Phi(\mathbf{V}_{\mathfrak{B}_2^4})$, and so $\Phi\colon \Lambda(\mathbf{CA}_2) \to \Lambda(\mathbf{Df}_2)$ is not 1–1.

To show that Φ does not preserve $\wedge$ we consider again the varieties $\mathbf{V}_{\mathfrak{B}_2^2}$ and $\mathbf{V}_{\mathfrak{B}_2^4}$. It is easy to check that $(\mathbf{2}^2, d)$ is the only simple member of the variety $\mathbf{V}_{\mathfrak{B}_2^2} \cap \mathbf{V}_{\mathfrak{B}_2^4}$. Therefore, $\mathbf{V}_{\mathfrak{B}_2^2} \cap \mathbf{V}_{\mathfrak{B}_2^4} = \mathbf{V}_{(\mathbf{2}^2,d)}$. However, since $\mathbb{F}(\mathfrak{B}_2^2)$ is isomorphic to $\mathbb{F}(\mathfrak{B}_2^4)$, $\mathbb{F}(\mathfrak{B}_2^2)$ belongs to both $\Phi(\mathbf{V}_{\mathfrak{B}_2^2})$ and $\Phi(\mathbf{V}_{\mathfrak{B}_2^4})$. Hence, it also belongs to their intersection. By (D) we know that $d_1(\Phi(\mathbf{V}_{(\mathbf{2}^2,d)})) = d_2(\Phi(\mathbf{V}_{(\mathbf{2}^2,d)})) = 1$. On the other hand, $d_i(\Phi(\mathbf{V}_{\mathfrak{B}_2^2}) \cap \Phi(\mathbf{V}_{\mathfrak{B}_2^4})) = 2$ for $i = 1, 2$. Therefore, $\Phi(\mathbf{V}_{(\mathbf{2}^2,d)}) \neq \Phi(\mathbf{V}_{\mathfrak{B}_2^2}) \cap \Phi(\mathbf{V}_{\mathfrak{B}_2^4})$, and so Φ does not preserve $\wedge$.

(5) That $\Phi(\mathbf{CA}_2) = \mathbf{Df}_2$ follows from (3). Hence, Φ preserves top. Obviously the Φ-reduct of the trivial variety of cylindric algebras is the trivial variety of $\mathbf{Df}_2$-algebras. Therefore, Φ preserves bottom. Finally, we show that Φ preserves $\vee$, that is $\Phi(\mathbf{V}_1 \vee \mathbf{V}_2) = \Phi(\mathbf{V}_1) \vee \Phi(\mathbf{V}_2)$. Indeed,

$$\begin{aligned}\Phi(\mathbf{V}_1 \vee \mathbf{V}_2) &= \mathbf{HSP}(\{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in \mathbf{V}_1 \vee \mathbf{V}_2\}) \\ &= \mathbf{HSP}(\{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in (\mathbf{V}_1 \vee \mathbf{V}_2)_S\}).\end{aligned}$$

By Jónsson's lemma $(\mathbf{V}_1 \vee \mathbf{V}_2)_S = (\mathbf{V}_1)_S \cup (\mathbf{V}_2)_S$. Also recall that for arbitrary classes of universal algebras Γ and Δ, we have $\mathbf{HSP}(\Gamma \cup \Delta) = \mathbf{HSP}(\mathbf{HSP}(\Gamma) \cup \mathbf{HSP}(\Delta))$. Hence,

$$\begin{aligned}\Phi(\mathbf{V}_1 \vee \mathbf{V}_2) &= \mathbf{HSP}\big(\{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in (\mathbf{V}_1)_S \cup (\mathbf{V}_2)_S\}\big) \\ &= \mathbf{HSP}\big(\{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in (\mathbf{V}_1)_S\} \cup \{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in (\mathbf{V}_2)_S\}\big) \\ &= \mathbf{HSP}\big(\mathbf{HSP}(\{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in (\mathbf{V}_1)_S\}) \cup \mathbf{HSP}(\{\mathbb{F}(\mathfrak{B}) : \mathfrak{B} \in (\mathbf{V}_2)_S\})\big) \\ &= \mathbf{HSP}(\Phi(\mathbf{V}_1) \cup \Phi(\mathbf{V}_2)) \\ &= \Phi(\mathbf{V}_1) \vee \Phi(\mathbf{V}_2).\end{aligned}$$

□

Note that there are subvarieties $\mathbf{K}$ of $\mathbf{Df}_2$ such that $d_1(\mathbf{K}) = d_2(\mathbf{K})$ and still $\Phi^{-1}(\mathbf{K}) = \emptyset$. For example, let $\mathbf{K}$ be a proper subvariety of $\mathbf{Df}_2$ with $d_1(\mathbf{K}) = d_2(\mathbf{K}) = \omega$. We know from [2, Theorem 4.10] that such varieties exist. If $\Phi^{-1}(\mathbf{K}) \neq \emptyset$, then there exists $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$ such that $\Phi(\mathbf{V}) = \mathbf{K}$. It follows from the equation (D) that $d(\mathbf{V}) = \omega$. Therefore, $\mathbf{V} \in [\mathbf{U}, \mathbf{CA}_2]$. But then by Theorem 7.11(3) $\Phi(\mathbf{V}) = \mathbf{Df}_2 \neq \mathbf{K}$, which is a contradiction. Thus, $\Phi^{-1}(\mathbf{K}) = \emptyset$ even though $d_1(\mathbf{K}) = d_2(\mathbf{K})$.

Suppose $\mathbf{V} \in \Lambda(\mathbf{CA}_2)$ and $\mathbf{K} \in \Lambda(\mathbf{Df}_2)$. For a property $\mathcal{P}$ of varieties of universal algebras, we say that Φ *preserves* $\mathcal{P}$ if $\Phi(\mathbf{V})$ has $\mathcal{P}$ whenever $\mathbf{V}$ does; and we say that Φ *reflects* $\mathcal{P}$ if every variety in $\Phi^{-1}(\mathbf{K})$ has $\mathcal{P}$ whenever $\mathbf{K}$ does.

Theorem 7.12. (1) *$\mathcal{P}$ is preserved by Φ if $\mathcal{P}$ is one of the following properties:* (a) *being finitely approximable;* (b) *being of finite depth;* (c) *being locally finite;* (d) *being pre locally finite;* (e) *being finitely generated.*

(2) *$\mathcal{P}$ is not preserved by Φ if $\mathcal{P}$ is the property of being pre finitely generated.*

(3) *$\mathcal{P}$ is reflected by Φ if $\mathcal{P}$ is one of the following properties:* (a) *being of finite depth;* (b) *being locally finite;* (c) *being finitely generated;*

(4) *$\mathcal{P}$ is not reflected by Φ if $\mathcal{P}$ is the property of* (a) *being pre locally finite;* (b) *being pre finitely generated.*

Proof. 1. (a) is obvious since every subvariety of $\mathbf{Df}_2$ is finitely approximable (see [2, Corollary 4.9.(2)]). (b) follows from the equation (D). (c) Suppose $\mathbf{V} \subseteq \mathbf{CA}_2$

is locally finite. Then $\mathbf{V}$ has finite depth. By (b) $\Phi(\mathbf{V})$ also has finite depth. Hence, $\Phi(\mathbf{V})$ is a proper subvariety of $\mathbf{Df}_2$. But every proper subvariety of $\mathbf{Df}_2$ is locally finite (see [2, Corollary 4.9.(1)]). Therefore, $\Phi(\mathbf{V})$ is locally finite. (d) The only pre locally finite subvarieties of $\mathbf{CA}_2$ and $\mathbf{Df}_2$ are $\mathbf{U}$ and $\mathbf{Df}_2$, respectively; and $\Phi(\mathbf{U}) = \mathbf{Df}_2$ by Theorem 7.11. (e) Suppose $\mathbf{V} \subseteq \mathbf{CA}_2$ is finitely generated. Then $\mathrm{Fin}\mathbf{V}_S$ is finite by Theorem 7.2. Hence, $\mathbb{F}(\mathrm{Fin}\mathbf{V}_S)$ is also finite. Since $\Phi(\mathbf{V})$ is generated by $\mathbb{F}(\mathrm{Fin}\mathbf{V}_S)$ and every finite simple $\mathbf{Df}_2$-algebra has finitely many simple subalgebras, $\Phi(\mathbf{V})$ contains finitely many finite simple $\mathbf{Df}_2$-algebras. Therefore, by the $\mathbf{Df}_2$-version of Theorem 7.2 (see [2, p. 33]), $\Phi(\mathbf{V})$ is finitely generated.

2. Observe that the Φ-images of pre finitely generated subvarieties of $\mathbf{CA}_2$ of depth 3 are varieties of $\mathbf{Df}_2$-algebras of both E_1 and E_2-depth 3. Also observe that subvarieties of $\mathbf{Df}_2$ of depth 3 are not pre finitely generated varieties (see [2, Theorem 5.4]). The result follows.

3. (a) directly follows from the equation (D). (b) The only non-locally finite subvariety of $\mathbf{Df}_2$ is $\mathbf{Df}_2$ itself. By Theorem 7.11 $\Phi^{-1}(\mathbf{Df}_2) = [\mathbf{U}, \mathbf{CA}_2]$. Hence, Theorem 5.5 implies that if $\mathbf{K} \in \Lambda(\mathbf{Df}_2)$ is locally finite, then $\Phi^{-1}(\mathbf{K})$ is either empty or contains varieties of cylindric algebras of finite depth. Since every subvariety of $\mathbf{CA}_2$ of finite depth is locally finite, Φ reflects the property of being locally finite. (c) follows from Theorem 7.2, its $\mathbf{Df}_2$-version and the fact that the reduct of a simple cylindric algebra is a simple $\mathbf{Df}_2$-algebra.

4. (a) Let $\mathbf{V} \subseteq \mathbf{CA}_2$ be such that $\mathbf{U} \subset \mathbf{V}$. Then by Corollary 5.11(2) $\mathbf{V}$ is not pre locally finite. However, $\Phi(\mathbf{V}) = \mathbf{Df}_2$ and $\mathbf{Df}_2$ is pre locally finite. Therefore, Φ does not reflect the property of being pre locally finite. (b) As follows from [2, Theorem 5.4], the variety $\Phi(\mathbf{V}_1)$ is pre finitely generated. Since $\Phi^{-1}(\Phi(\mathbf{V}_1)) = \{\mathbf{V}_1, \mathbf{V}^1\}$ and $\mathbf{V}^1$ is not pre finitely generated (see Section 6), we obtain that the property of being pre finitely generated is not reflected by Φ. $\square$

We conclude by mentioning that it is an open problem whether Φ reflects finite approximability. If every variety of cylindric algebras were finitely approximable, which we conjectured at the end of Section 5, then the answer to this problem would be positive.

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