The McKinsey-Tarski Theorem for Topological Evidence Logics

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Abstract. We prove an analogue of the McKinsey and Tarski theorem for the recently introduced dense-interior semantics of topological evidence logics. In particular, we show that in this semantics the modal logic $\mathsf{S4.2}$ is sound and complete for any dense-in-itself metrizable space. As a result $\mathsf{S4.2}$ is complete with respect to the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, the Baire space $\mathfrak{B}$, the Cantor space $\mathfrak{C}$, etc. We also show that an extension of this logic with the universal modality is sound and complete for any idempotent dense-in-itself metrizable space, obtaining as a result that this logic is sound and complete with respect to $\mathbb{Q}$, $\mathfrak{B}$, $\mathfrak{C}$, etc.

1 Introduction

Epistemic logics (i.e. the family of modal logics concerned with what an epistemic agent *believes* or *knows*) has by now a well-established semantics in the form of Kripke frames [11]. Hintikka [11] reasonably claims that the accessibility relation encoding knowledge must be minimally reflexive and transitive, which on the syntactic level translates to the corresponding logic of knowledge containing the axioms of $\mathsf{S4}$. This, paired with the fact (proven by McKinsey and Tarski [14]) that $\mathsf{S4}$ is the logic of topological spaces under the *interior semantics*, lays the ground for a topological treatment of knowledge. Moreover, treating the knowledge modality as the topological interior operator, and the open sets as "pieces of evidence" adds an evidential dimension to the notion of knowledge that one cannot obtain within the framework of Kripke frames.

Reading epistemic sentences using the interior semantics might be too simplistic: it equates "knowing" and "having evidence". In addition, the attempts to bring the notion of belief into this framework have not been very successful.

Following [18], a logic that allows us to talk about knowledge, belief and the relation thereof, about evidence (both basic and combined) and justification is introduced in [2]. This is the framework of *topological evidence models* (topo-e-models) and this paper builds on it.

McKinsey and Tarski also proved in [14] a stronger result—their celebrated theorem—namely, that there are single spaces (dense-in-themselves and metrizable) such as the real line, whose logic is $\mathsf{S4}$. The present paper aims to translate the spirit of this theorem to the framework of topo-e-models. To this respect, we introduce a notion of generic models over a language $\mathcal{L}$, which are topological spaces whose logic is precisely the sound and complete $\mathcal{L}$-logic of topo-e-models, and provide several examples of generic models for the different fragments of the language. More precisely, we show that in this new semantics the modal logic $\mathsf{S4.2}$ is sound and complete for any dense-in-itself metrizable space. As a result $\mathsf{S4.2}$ is complete with respect to the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, the Baire space $\mathfrak{B}$, the Cantor space $\mathfrak{C}$, etc. We also show that extensions of this logic (e.g., with the global modality) are sound and complete for any idempotent dense-in-itsef metrizable space such as $\mathbb{Q}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. Our proofs rely on a recent topological proof of the McKinsey and Tarski theorem [5]. Namely, an open and continuous onto map from any dense-in-itself metrizable space onto a finite rooted $\mathsf{S4}$-frame defined in [5] can be used to define an open and continuous onto map from such a space but now with the dense-interior topology onto a finite rooted $\mathsf{S4.2}$-frame.

This paper is structured as follows: in the present section we show how to use topological spaces to model epistemic sentences and introduce the framework of topological evidence models. In Section 2, we explain how McKinsey and Tarski's theorem encodes a notion of *generic model* which we then use to state and prove our main results. These results also include different fragments of the language within the framework of topo-e-models. Finally, we conclude in Section 3.

1.1 Logics of knowledge and belief

Below we list some logics of belief and knowledge which were mentioned in the introduction and will be used throughout this paper.

The modal logic $\mathsf{S4}$ is the least set of formulas in the language $\mathcal{L}_\Box$ which contains all the propositional tautologies, is closed under uniform substitution and the rules of modus ponens (from ϕ and $\phi \to \psi$ infer ψ) and necessitation (from ϕ infer $\Box\phi$) and contains the axioms:

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$;
(T) $\Box\phi \to \phi$ (*factivity of knowledge*);
(4) $\Box\phi \to \Box\Box\phi$ (*positive introspection*).

The modal logic $\mathsf{S5}$ contains the axioms and rules of $\mathsf{S4}$ plus the axiom:

(5) $\neg\Box\phi \to \Box\neg\Box\phi$ (*negative introspection*).

$\mathsf{S4.2}$ is $\mathsf{S4}$ plus the axiom:

(.2) $\neg\Box\neg\Box\phi \to \Box\neg\Box\neg\phi$.

$\mathsf{KD45}$ has the (K), (4) and (5) axioms plus:

(D) $\Box\phi \to \neg\Box\neg\phi$.

The logic Stal, with respect to a language with the K and B modalities, adds the axioms in Table 1 to the $\mathsf{S4}$ axioms for K.

(PI) $B\phi \to KB\phi$;
(NI) $\neg B\phi \to K\neg B\phi$;
(KB) $K\phi \to B\phi$;
(CB) $B\phi \to \neg B\neg\phi$;
(FB) $B\phi \to BK\phi$.

Table 1. Extra axioms for Stal

1.2 The Interior Semantics: the McKinsey-Tarski Theorem

Let Prop be a countable set of propositional variables and consider a modal language $\mathcal{L}_\Box$ defined as follows: $\phi := p \,|\, \phi \wedge \phi \,|\, \neg\phi \,|\, \Box\phi$, with $p \in \mathsf{Prop}$.

A *topological model* is a topological space (X, τ) together with a valuation $V : \mathsf{Prop} \to 2^X$. The semantics of a formula ϕ is defined recursively as follows: $\|p\| = V(p)$; $\|\phi \wedge \psi\| = \|\phi\| \cap \|\psi\|$, $\|\neg\phi\| = X \setminus \|\phi\|$, $\|\Box\phi\| = \mathrm{Int}\, \|\phi\|$, where Int is the interior operator of the topology.

We now give some examples of topological spaces (which will be used throughout the remainder of this paper) in which we model epistemic sentences.

Example 1. (The real line) Let $\mathbb{R}$ be the set of real numbers. We define the *natural topology* $\tau_\mathbb{R}$ on $\mathbb{R}$, as the topology generated by the basis of open intervals

$$\mathcal{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}.$$

Equivalently, $U \subseteq \mathbb{R}$ is an open set if, for each $x \in U$, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$.

Example 2. (The rational numbers) The *natural topology* $\tau_\mathbb{Q}$ on the set of rational numbers $\mathbb{Q}$ is simply the subspace topology[3] $\tau_\mathbb{R}|_\mathbb{Q}$ or, equivalently, the topology generated on $\mathbb{Q}$ by the basis of open intervals $\{(a, b) : a, b \in \mathbb{R}, a < b\}$, where $(a, b) = \{x \in \mathbb{Q} : a < x < b\}$.

Example 3. (The Baire space and the Cantor space) Let ω^ω be the set of infinite sequences of natural numbers, and ω^* be the set of finite such sequences. For $s \in \omega^*$ and $\alpha \in \omega^\omega$ we say $s \lhd \alpha$ whenever s is an *initial segment* of α, i.e., whenever $s = \langle s_1, ..., s_n\rangle$ with $s_i = \alpha(i)$ for $1 \le i \le n$. For $s \in \omega^*$, let $O(s)$ denote the set of sequences of natural numbers that have s as an initial segment, i.e. $O(s) = \{\alpha \in \omega^\omega : s \lhd \alpha\}$. The *Baire space* $\mathfrak{B} = (\omega^\omega, \tau_\mathfrak{B})$ is the topological space that has ω^ω as its underlying set together with the topology $\tau_\mathfrak{B}$ generated by the basis

$$\mathcal{B}_\mathfrak{B} = \{O(s) : s \in \omega^*\}.$$

We can analogously define the *Cantor space* $\mathfrak{C}$ on the set $\mathbf{2}^\omega$ of countable sequences of zeros and ones. The Cantor space has a visual representation in

[3] Given a topological space (X, τ) and a set $Y \subseteq X$, we can define the *subspace topology* $\tau|_Y$ on Y as the set

$$\tau|_Y := \{U \cap Y : U \in \tau\}.$$

Note that $(Y, \tau|_Y)$ is trivially a topological space.

the form of the infinite binary tree. This is a tree whose nodes are the finite sequences of zeros and ones. It has the empty sequence as the root and each node $\langle i_1, ..., i_n \rangle \in \mathbf{2}^*$ has exactly two successors, namely $\langle i_1, ..., i_n, 0 \rangle$ as its left successor and $\langle i_1, ..., i_n, 1 \rangle$ as its right successor. The elements of the Cantor space can be identified with *branches* of this tree, where a branch is a countable collection of nodes $\{s_0, s_1, s_2, ...\}$ such that s_0 is the empty sequence (i.e. the root of the tree) and each s_{k+1} is an immediate successor of s_k. The basic open sets $O(s)$ are identified with "fans", each fan being the subtree that spurs from one node. An open set is any union of some of these fans. $\alpha \in \mathbf{2}^\omega$ is in a basic open set $O(s)$ whenever the corresponding branch "enters" the fan.

Example 4. (The binary tree $\mathcal{T}_2$) If we consider the nodes of the infinite binary tree instead of its branches to be the points of our space, we can equip it with a topology by setting the basic open sets to be those of the form $O(s)$, where $s = \langle a_0, ..., a_n \rangle$ and $t \in O(s)$ if and only if t is a finite sequence of length greater than or equal to $n+1$ with its $n+1$ first elements being $a_0, ..., a_n$.

The interior semantics on topological spaces generalises the Kripke semantics on preordered frames[4]. If we are reading $\Box$ as an epistemic operator, we can translate the semantics of [11] into this topological framework, with the addition that having a topological space allows us to have an *evidential* view of knowledge. Indeed, if we read $\Box$ as a knowledge modality, we interpret the open sets in the topology to be pieces of evidence the agent has, and we say that P entails Q whenever $P \subseteq Q$, then the interior semantics defined above gives us that the agent *knows* ϕ whenever she has a piece of evidence which entails ϕ.

Let us revisit some of the examples above in this light.

Example 5. An underfunded ornithologist measures the weight of a bird. Her devices of measurement produce results with a margin of error of ±10g. Let us code the set of possible worlds with the positive real numbers $(0, \infty)$, where at world x the weight of the bird is precisely x grams. Now, suppose the actual world is $x_0 = 509$ and the ornithologist obtains a measurement of 500g ± 10g. Then the open interval $(490, 510)$ is her piece of evidence. With this, there are things she knows and things she does not know. She does not know, for instance, the proposition "the bird is heavier than 500g" to be true. She knows, however, that the bird is heavier than 400g. This proposition can be interpreted as the set of worlds $P = (400, \infty)$ and she has a piece of evidence which includes the actual world and entails this proposition: $x_0 \in (490, 510) \subseteq P$.

Example 6. Let us equate a world with an infinite stream of data, represented by a sequence of natural numbers. We are thus in our Baire space. Our epistemic agent this time is a scientist, and her evidence comes in the form of *observations*,

[4] Given a preordered set $(X, \leq)$, the collection of upwards-closed sets defines an Alexandroff topology on X, i.e., a topology closed under infinite intersections. Conversely, given an Alexandroff topological space (X, τ) the relation $x \leq y$ iff $x \in U$ implies $y \in U$, for all $U \in \tau$, defines a preorder. This correspondence is 1-1 and moreover $x \in \operatorname{Int} P$ iff $y \in P$ for all $y \geq x$. For details, see, e.g., [3].

which are finite streams of data that the scientist is able to grasp. A world is compatible with her observation whenever the stream of data is an initial segment of said world. If she observes $s = \langle a_1, ..., a_n \rangle$, then the set of worlds compatible with it (the corresponding *piece of evidence* in our sense) is precisely the basic open set $O(s)$.

In this setting, open sets correspond to *verifiable* propositions: if P is an open set and the actual world x_0 is in P, then there exist a basic open set $O(s)$ such that $x_0 \in O(s) \subseteq P$. Thus this scientist can potentially make an observation, s, which will allow her to know P. Similarly, closed sets correspond to *refutable* propositions and clopen sets to *decidable* propositions. For more details on this interpretation, see [12].

1.3 McKinsey and Tarski: S4 as a topological logic of knowledge

Modelling knowledge as topological interior gives us an intuitive, evidence-based idea of what knowledge amounts to. Moreover, the interior semantics generalises the Kripke semantics for preorders and:

Theorem 1 (McKinsey and Tarski, [14]). S4 *is sound and complete with respect to topological spaces under the interior semantics.*

McKinsey and Tarski also proved a stronger result. We do not need to consider the class of all topological spaces to obtain the logic S4. They showed that, instead, we can take some particular, "natural" topological space used to model knowledge, whose logic is S4.

Definition 1. *A topological space* (X, τ) *is called* dense-in-itself *if no singleton is an open set, i.e., if* $\{x\} \notin \tau$ *for all* $x \in X$. *We say* (X, τ) *is* metrizable *if there exists a metric*[5] d *on* X *which generates* τ.

Remark 1. All the spaces presented as examples in subsection 1.2 are both dense-in-themselves and metrizable. The corresponding metric for the spaces $\mathbb{R}$, $\mathbb{I}$ and $\mathbb{Q}$ is $d(x, y) = |x - y|$, and clearly no singleton contains an open interval in these spaces. The binary tree $\mathcal{T}_2$ clearly has no open singletons and it is a regular space with a countable basis and thus metrizable. $\mathfrak{B}$ and $\mathfrak{C}$ are homeomorphic to dense-in-themselves metrizable subspaces of $\mathbb{R}$ (for details on these claims, see [7,15]).

Theorem 2 (McKinsey and Tarski, [14]). S4 *is the logic of any dense-in-itself metrizable space.*[6]

[5] I.e. a map $d : X \times X \to [0, \infty)$ satisfying for all $x, y, z \in X$: (i.) $d(x, y) = 0$ iff $x = y$; (ii.) $d(x, y) = d(y, x)$; (iii.) $d(x, z) \leq d(x, y) + d(y, z)$. A metric d on X induces a topology τ_d: we say that a set $U \subseteq X$ is open if, for every $x \in U$, there exists some $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $y \in U$.

[6] The original formulation of this theorem talked about dense-in-itself, metrizable, *separable* spaces. It was shown in [16] that the separability condition can be dropped.

We thus have a semantics based on evidence that allows us to talk about knowledge and whose logic is a philosophically suitable epistemic logic. Moreover, we have some specific spaces which provide "nice" ways to conceptualise knowledge and whose logic is still S4.

This semantics, however, is not the topic of this paper. Instead, we will be working with the *dense interior* semantics. Understanding the conceptual reasons to move away from the interior and introducing this semantics is the aim of the next subsection.

1.4 Dense interior

The relation between belief and knowledge has historically been a main focus of epistemology. One would want to have a formal system that accounts for knowledge and belief together, which requires careful consideration regarding the way in which they interact. Canonically, knowledge has been thought of as "true, justified belief". However, Gettier's counterexamples of cases of true, justified belief which do not amount to knowledge shattered this paradigm [8].

Stalnaker [18] argues that a relational semantics is insufficient to capture Gettier's considerations in [8] and, trying to stay close to most of the intuitions of Hintikka in [11], provides an axiomatisation for a system of knowledge and belief. This system, Stal, has two modal operators, B and K, and on top of the S4 axioms and rules for K it adds the axioms of Table 1.

In this logic, knowledge is an S4.2 modality, belief is a KD45 modality and the following formulas can be proven: $B\phi \leftrightarrow \neg K \neg K \phi$ and $B\phi \leftrightarrow BK\phi$. "Believing p" is the same as "not knowing you don't know p" and belief becomes "subjective certainty", in the sense that the agent cannot distinguish whether she believes or knows p, and believing amounts to believing that one knows.

Now, modelling epistemic sentences via the interior semantics defined above forces us to equate "knowing" with "having evidence". Moreover, attempts to introduce belief in this framework have had some flagrant issues. To give some examples, the framework considered in [19], in which knowledge is interior and belief is read as the dual of the *derived set operator*[7], makes knowledge amount to true belief, which clearly falls short. [1] takes a Stalnakerian stand but it confines us to work with *hereditarily extremally disconnected spaces* (h.e.d)[8], which seems to be a rather restricted class of spaces. None of the "natural" spaces provided above as examples are h.e.d.

In [2] a new semantics is introduced, building on the idea of *evidence models* of [4] which exploits the notion of evidence-based knowledge allowing to account for notions as diverse as *basic evidence* versus *combined evidence*, *factual, misleading* and *nonmisleading evidence*, etc. It is a semantics whose logic maintains a Stalnakerian spirit with regards to the relation between knowledge and belief,

[7] $BP = \neg d(\neg P)$, where $d(P) = \{x : \forall U \in \tau(x \in U \text{ implies } \exists y \in P \cap U, y \neq x)\}$.

[8] A space is *extremally disconnected* (e.d.) if the closure of an open set is open, and *hereditarily* so if all its subspaces are e.d.

which behaves well dynamically and which does not confine us to work with "strange" classes of spaces.

This is the *dense interior semantics*, defined on *topological evidence models*.

1.5 The logic of topological evidence models

We briefly present here the framework introduced in [2]. Our language is now $\mathcal{L}_{\forall KB\Box\Box_0}$, which includes the modalities K (knowledge), B (belief), $[\forall]$ (infallible knowledge), $\Box_0$ (basic evidence), $\Box$ (combined evidence).

Definition 2 (The dense interior semantics). *We interpret sentences on* topological evidence models *(i.e. tuples (X, τ, E_0, V) where (X, τ, V) is a topological model and E_0 is a subbasis of τ) as follows: $x \in [\![K\phi]\!]$ iff $x \in \mathrm{Int}[\![\phi]\!]$ and $\mathrm{Int}[\![\phi]\!]$ is* dense[9]*; $x \in [\![B\phi]\!]$ iff $\mathrm{Int}[\![\phi]\!]$ is dense; $x \in [\![[\forall]\phi]\!]$ iff $[\![\phi]\!] = X$; $x \in [\![\Box_0\phi]\!]$ iff there is $e \in E_0$ with $x \in e \subseteq [\![\phi]\!]$; $x \in [\![\Box\phi]\!]$ iff $x \in \mathrm{Int}[\![\phi]\!]$. Validity is defined in the standard way.*

We see that "knowing" does not equate "having evidence" in this framework, but it is rather something stronger: in order for the agent to know P, she needs to have a piece of evidence for P which is *dense*, i.e., which has nonempty intersection with (and thus cannot be contradicted by) any other potential piece of evidence she could gather.

Fragments of the logic. The following logics are obtained by considering certain fragments of the language (i.e. certain subsets of the modalities above).

"K-only", $\mathcal{L}_K$	S4.2.
"Knowledge", $\mathcal{L}_{\forall K}$	S5 axioms and rules for $[\forall]$, plus S4.2 for K, plus $[\forall]\phi \to K\phi$ and $\neg[\forall]\neg K\phi \to [\forall]\neg K\neg\phi$.
"Combined evidence", $\mathcal{L}_{\forall\Box}$	S5 for $[\forall]$, S4 for $\Box$, plus $[\forall]\phi \to \Box\phi$.
"Evidence", $\mathcal{L}_{\forall\Box\Box_0}$	S5 for $[\forall]$, S4 for $\Box$, plus the axioms $\Box_0\phi \to \Box_0\Box_0\phi$, $[\forall]\phi \to \Box_0\phi$, $\Box_0\phi \to \Box\phi$, $(\Box_0\phi \wedge [\forall]\psi) \to \Box_0(\phi \wedge [\forall]\psi)$.

We will refer to these logics respectively as $\mathsf{S4.2}_K$, $\mathsf{Logic}_{\forall K}$, $\mathsf{Logic}_{\forall\Box}$ and $\mathsf{Logic}_{\forall\Box\Box_0}$. K and B are definable in the evidence fragments[10], thus we can think of the logic of $\mathcal{L}_{\forall\Box\Box_0}$ as the "full logic".

2 Generic spaces for the logic of topo-e-models

McKinsey and Tarski's theorem [14] stating that S4 is the logic of any dense-in-itself metrizable space (such as the real line $\mathbb{R}$) under the interior semantics tells us that we have a space which gives a somewhat "natural" way of capturing

[9] A set $U \subseteq X$ is dense whenever $\mathrm{Cl}\,U = X$ or equivalently whenever $U \cap V \neq \varnothing$ for all nonempty open set V.

[10] $K\phi \equiv \Box\phi \wedge [\forall]\Box\Diamond\phi$ and $B\phi \equiv \neg K\neg K\phi$.

knowledge yet it is "generic" enough so that its logic is precisely the logic of all topological spaces. Whatever is not provable in the logic of knowledge $\mathsf{S4}$ will find a refutation in $\mathbb{R}$ and whatever is true in $\mathsf{S4}$ will hold in every model based on the topology of the real line.

Translating this idea to the framework of topo-e-models is the aim of this paper. We wish to find topological evidence models which capture the logics presented in the preceding chapter, that is, special spaces whose logic under the dense interior semantics is exactly the logic of topo-e-models. We start by formalising the idea of "generic".

Definition 3 (Generic models). *Let $\mathcal{L}$ be a language and (X, τ) a topological space. We will say that (X, τ) is a* generic model for $\mathcal{L}$ *if the sound and complete $\mathcal{L}$-logic over the class of all topological evidence models is sound and complete with respect to the family*

$$\{(X, \tau, E_0) : E_0 \text{ is a subbasis of } \tau\}.$$

If $\Box_0$ is not in the language, then a generic model is simply a topological space which is sound and complete with respect to the corresponding $\mathcal{L}$-logic.

Since McKinsey and Tarski's original paper (which appeared in 1944), a number of simplified proofs of this result have been obtained. For an overview, we refer to [3]. Many of these proofs are built on the following idea. It is a well-known fact that $\mathsf{S4}$ is sound and complete with respect to finite rooted preorders (see e.g. [6]). One then constructs an *interior map* (a surjective map which is continuous and open[11]) from a dense-in-itself metrizable space (X, τ) onto any such preorder $(W, \leq)$. It can be proven that given such a map $f : X \to W$ and a valuation V on $(W, \leq)$, if we define $V^f(p) := \{x \in X : fx \in V(p)\}$ it is the case that, for any formula ϕ in the language of $\mathsf{S4}$, $x \models \phi$ in (X, τ, V^f) if and only if $fx \models \phi$ in $(W, \leq, V)$. Completeness is then a straightforward consequence, for if $\mathsf{S4} \not\vdash \phi$, then there is a model based on a finite rooted preorder $(W, \leq, V)$ refuting ϕ and thus we can refute ϕ on (X, τ, V^f). The next subsection builds on a recent proof of the McKinsey-Tarski theorem, contained in [5], which is purely topological.

2.1 S4.2 as the logic of $\mathbb{R}$

This section is devoted to the proof of our analogue of McKinsey and Tarski's theorem:

Theorem 3. *$\mathsf{S4.2}_K$ is the logic of any dense-in-itself metrizable space if we read K as dense interior. That is, for any formula ϕ in the language $\mathcal{L}_K$, and any dense-in-itself metrizable space (X, τ), we have that $\mathsf{S4.2}_K \vdash \phi$ if and only if $(X, \tau) \models \phi$ with the dense interior semantics.*

[11] A map $f : (X, \tau) \to (Y, \sigma)$ is *continous* is $U \in \sigma$ implies $f^{-1}[U] \in \tau$ and *open* if $U \in \tau$ implies $f[U] \in \sigma$.

Before tackling this proof, we will need to introduce some auxiliary notions.

Given a topological space (X, τ) define τ^+ to be the collection of dense open sets in (X, τ) plus the empty set:

$$\tau^+ = \{U \in \tau : \mathrm{Cl}\, U = X\} \cup \{\varnothing\}.$$

Recall that a topological space is *extremally disconnected* if the closure of any open set is an open set. The following is straightforward to check.

Lemma 1. *(X, τ^+) is an extremally disconnected topological space and, for any valuation V and any formula ϕ in the modal language $\mathcal{L}_K$ we have that $[\![\phi]\!]^{(X,\tau,V)}$ under the dense interior semantics coincides with $\|\phi\|^{(X,\tau^+,V)}$ under the interior semantics.*

Lemma 2. *For any topological space (X, τ), we have that $(X, \tau^+) \models \mathsf{S4.2}$ under the interior semantics.*

Proof. Follows from the above together with the soundness and completeness of $\mathsf{S4.2}$ with respect to extremally disconnected spaces (see, e.g., [3,1]).

Now, we will be using the known result that $\mathsf{S4.2}$ is sound and complete with respect to the class of finite rooted frames $(W, \leq)$ in which $\leq$ is a reflexive, transitive and weakly directed[12] relation [6]. Note that if a frame is rooted and weakly directed, for every pair of points $x, y \in W$, and given that $r \leq x, y$ where r is the root of W, weak directedness grants us the existence of some z such that $z \geq x, y$. But this means that, for every pair of points x and y, the set $\uparrow x \cap \uparrow y$ is nonempty, and thus for every pair of nonempty upsets U and V we have that $U \cap V \neq \varnothing$. This means that every nonempty upset is dense in such a frame, and therefore that the topology of upsets $\tau = \mathrm{Up}(W)$ coincides with τ^+. This fact, paired with the previous lemma, immediately gives us the folowing result.

Lemma 3. *Let $\mathfrak{F} = (W, \leq)$ be a reflexive, transitive and weakly directed rooted frame. Then the dense interior semantics on $(W, \mathrm{Up}(W))$ coincides with the interior semantics on it, which in turn coincides with the standard Kripke semantics on $(W, \leq)$. In other words, in any model based on such a frame*

$$x \models K\phi \text{ if and only if } y \models \phi \text{ for all } y \geq x.$$

Moreover, we have the following:

Lemma 4. *Let (X, τ) be some topological space and $(W, \leq, V)$ be a finite, rooted, reflexive, transitive and weakly directed Kripke model. Let*

$$f : (X, \tau^+) \twoheadrightarrow (W, \mathrm{Up}(W))$$

be an onto interior map and define

$$V^f(p) := \{x \in X : fx \in V(p)\}.$$

Then for every $x \in X$ we have that $(X, \tau, V^f), x \models \phi$ under the dense interior semantics if and only if $(W, \leq, V), fx \models \phi$ under the Kripke semantics.

[12] A relation $\leq$ is *weakly directed* whenever $x \leq y, z$ implies that there exists $t \geq y, z$.

Proof. Straightforward induction on the structure of ϕ.

Definition 4. *Given topological spaces* (X, τ) *and* (Y, σ)*, we will refer to an open (resp. continuous, interior) map* $f : (X, \tau^+) \to (Y, \sigma)$ *as a* dense-open *(resp.* dense-continuous, dense-interior*) map* $f : (X, \tau) \to (Y, \sigma)$.

Given all the above, in order to prove completeness it suffices to show that there exists a dense-interior map from any dense-in-itself metrizable space (X, τ) onto any finite $\mathsf{S4.2}$ frame. This way, if a formula ϕ is not a theorem of $\mathsf{S4.2}$, then it will be refuted on some such frame and therefore, by using this map plus Lemma 4, we can construct a valuation on (X, τ) which refutes ϕ. And indeed:

Theorem 4. *Given a dense-in-itself metrizable space* (X, τ) *and a finite rooted* $\mathsf{S4.2}$ *frame* $(W, \leq)$ *there exists an onto dense-interior map* $\bar{f} : (X, \tau) \twoheadrightarrow (W, \leq)$.

Proof. See Appendix A.1.

This finishes the proof of Theorem 3.

2.2 Adding belief

The logic Stal introduced in Section 1.1 is the logic of topo-e-models for the belief and knowledge fragment. The formula $B\phi \leftrightarrow \neg K \neg K \phi$ is provable in Stal (see [18]). In particular, for any formula ϕ in the language $\mathcal{L}_{KB}$, there exists a formula ψ in the language $\mathcal{L}_K$ such that $\models_{\mathsf{Stal}} \phi \leftrightarrow \psi$ (indeed, we get ψ by substituting every instance of B in ϕ with $\neg K \neg K$).

And thus we have the following:

Theorem 5. Stal *is sound and complete with respect to any dense-in-itself metrizable space with the dense interior semantics.*

Proof. Soundness follows from the fact that Stal is sound with respect to topo-e-models. For completeness, suppose $\mathsf{Stal} \not\vdash \phi$ and take ψ in the language $\mathcal{L}_K$ such that $\models_{\mathsf{Stal}} \phi \leftrightarrow \psi$. Then $\mathsf{S4.2} \not\vdash \psi$, hence by Theorem 3, for any dense-in-itself metrizable space (X, τ), there is a point $x \in X$ and valuation V such that $(X, \tau^+, V), x \not\models \psi$. By soundness and the fact that $\models_{\mathsf{Stal}} \phi \leftrightarrow \psi$, we conclude that ϕ is false at x as well.

2.3 The global modality $[\forall]$ and the logic of $\mathbb{Q}$

Three fragments including the global modality $[\forall]$ will be considered in the present subsection: the *knowledge fragment* (the one which includes the K and $[\forall]$ modalities), the *factive evidence fragment* (including $\Box$ and $[\forall]$) and the *evidence fragment* (including $[\forall]$, $\Box$ and $\Box_0$).

First let us concentrate on the factive evidence fragment. Recall that the logic of this fragment, $\mathsf{Logic}_{\forall\Box}$, consists of $\mathsf{S5}_{\forall}$ plus $\mathsf{S4}_{\Box}$ plus the axiom $[\forall]\phi \to \Box\phi$.

This logic is not complete with respect to $\mathbb{R}$. Consider the following formula:

$$[\forall](\Box p \vee \Box \neg p) \to ([\forall]p \vee [\forall]\neg p) \tag{Con}$$

It is the case that (Con) is not derivable in the logic yet it is always true in $\mathbb{R}$. More generally:

Theorem 6 (Shehtman, [17]). *A topological space* (X, τ) *satisfies* (Con) *if and only if it is connected*[13]*.*

Instead of considering connected spaces and adding (Con) as an axiom to our logic (an axiom which would be hard to justify epistemically), we will show completeness of this fragment (plus the other two mentioned above which include the global modality) with respect to a dense-in-itself, metrizable yet disconnected space, namely $\mathbb{Q}$. This parallels a similar result of [17] stating that $\mathbb{Q}$ is sound and complete with respect to $\mathsf{S4}$ with the global modality.

The knowledge fragment $\mathcal{L}_{\forall K}$. Similarly to Subsection 2.1, we will use completeness of the logic with respect to a class of finite frames, namely:

Lemma 5 ([9]). $\mathsf{Logic}_{\forall K}$ *is sound and complete with respect to finite models of the form* (W, R, V) *where* W *is a finite set,* R *is a preorder with a final cluster*[14] *and* K *and* $[\forall]$ *are respectively interpreted as the Kripke modality for* R *and the universal modality.*

Once again, we can easily check the following statement.

Lemma 6. *Let* $\mathfrak{M} = (W, R, V)$ *be a finite preordered model with a final cluster,* (X, τ) *a topological space and* $f : X \twoheadrightarrow W$ *an onto dense-interior map. Then for any formula* ϕ *we have* $(X, \tau, V_f), x \models \phi$ *iff* $\mathfrak{M}, fx \models \phi$*, where* $V_f(p) = f^{-1}[V(p)]$*.*

Then, to prove completeness, it suffices to find such a map from $\mathbb{Q}$. And indeed:

Theorem 7. *Given a finite preorder with a final cluster* (W, R)*, there exists an onto dense-interior map* $f : (\mathbb{Q}, \tau_{\mathbb{Q}}) \twoheadrightarrow (W, R)$*.*

Proof. See Appendix A.2.

The factive evidence fragment $\mathcal{L}_{\forall\Box}$ It is proved in [9] that $\mathsf{Logic}_{\forall\Box}$ is sound and complete with respect to finite relational models of the form $(X, \leq, V)$ where $\leq$ is a preorder.

Thus, to prove completeness of this logic with respect to $\mathbb{Q}$ it suffices to find a suitable open and continuous map from $\mathbb{Q}$ onto any such finite frame. And indeed (by a proof similar to the one of Theorem 7) we obtain:

Theorem 8. *Let* $(W, \leq)$ *be any finite preordered frame. Then there exists an open, continuous and surjective map* $f : (\mathbb{Q}, \tau_{\mathbb{Q}}) \twoheadrightarrow (W, \mathrm{Up}_{\leq}(W))$*.*

Again, noting that if we define $V^f(p) = \{x \in \mathbb{Q} : fx \in V(p)\}$ we obtain $x \models \phi$ in $(\mathbb{Q}, \tau_{\mathbb{Q}}, V^f)$ if and only if $fx \models \phi$ in $(W, \leq, V)$, completeness follows.

[13] A space X is *connected* if there is no proper subset $A \subseteq X$ such that both A and $X \backslash A$ are open. $\mathbb{R}$ is a connected space.

[14] I.e. a set $A \subseteq W$ such that wRa for all $a \in A$ and all $w \in W$.

Adding basic evidence: the evidence fragment $\mathcal{L}_{\forall\Box\Box_0}$. Let us now account for basic evidence. We take the fragment consisting of the modal operators $\Box$, $[\forall]$ and $\Box_0$. Recall that we interpret formulas of this fragment on topo-e-models (X, τ, E_0, V), where E_0 is a subbasis for (X, τ), in the following way: $x \in [\![\Box_0\phi]\!]$ if and only if there exists $e \in E_0$ with $x \in e \subseteq [\![\phi]\!]$.

The logic of this fragment is $\mathsf{Logic}_{\forall\Box\Box_0}$, as discussed in Section 1.5. It is proven in [2] that this logic is sound and complete with respect to finite *pseudo-models*, i.e., structures of the form $(X, \leq, E_0^X, V)$, where $\leq$ is a preorder and E_0^X is a subbasis for $\mathrm{Up}(X)$ with $X \in E_0$.

Completeness is an immediate corollary of the following result:

Theorem 9. *Let $\mathfrak{M} = (X, \leq, E_0^X, V)$ be a pseudo-model as defined above and $f : \mathbb{Q} \twoheadrightarrow X$ be an onto interior map. Then if we define $V^{\mathbb{Q}}(p) = f^{-1}[V(p)]$ and $E_0^{\mathbb{Q}} := \{e \subseteq \mathbb{Q} : f[e] \in E_0^X\}$, we have that $\mathfrak{N} = (\mathbb{Q}, \tau_{\mathbb{Q}}, E_0^{\mathbb{Q}}, V^{\mathbb{Q}})$ is a topo-e-model and, for every ϕ in the language, $\mathfrak{N}, x \models \phi$ iff $\mathfrak{M}, fx \models \phi$.*

Proof. See Appendix A.3.

To summarise the results in this subsection we obtain:

Theorem 10. *$(\mathbb{Q}, \tau_{\mathbb{Q}})$ is a generic model for the fragments $\mathcal{L}_{\forall\Box}$, $\mathcal{L}_{\forall K}$ and $\mathcal{L}_{\forall\Box\Box_0}$.*

Proof. The result follows from Theorems 7, 8 and 9, respectively.

A condition for generic models. We will now generalize the results in the present subsection to a class of spaces. One can easily see that the only part in the proof of Theorem 7 which uses a special property of $\mathbb{Q}$ which $\mathbb{R}$ does not have is that we partition $\mathbb{Q}$ in n subspaces which are homeomorphic to $\mathbb{Q}$ itself. Given a dense-in-itself metrizable space which admits such partition, all the proofs in the present subsection will work *mutatis mutandis*. We will now give a necessary and sufficient condition for such a space to have this property.

Definition 5 (Idempotent spaces). *A topological space (X, τ) is* idempotent *whenever (X, τ) is homeomorphic to the sum $(X, \tau) \oplus (X, \tau)$.*[15]

Then the following holds:

Lemma 7. *A topological space (X, τ) is idempotent if and only if it can be partitioned in n subspaces homeomorphic to itself for each $n \geq 1$.*

Proof. If (X, τ) admits a partition in two subspaces homeomorphic to itself, since these are disjoint their union (which is X) is homeomorphic to their sum, which is homeomorphic to $X \oplus X$.

Conversely, if (X, τ) is idempotent we can reason recursively to find that X is homeomorphic to the sum $X_1 \oplus ... \oplus X_n$ where each X_i is a copy of X. Let $f : X_1 \oplus ... \oplus X_n \to X$ be a homeomorphism. Then $\{f[X_1], ..., f[X_n]\}$ constitutes a partition of X in n subspaces, each of them homeomorphic to X.

[15] $(X, \tau) \oplus (Y, \sigma)$ is the space which has the disjoint union $(X \times \{1\}) \cup (Y \times \{2\})$ as its underlying set and $\tau \oplus \sigma = \{(U \times \{1\}) \cup (V \times \{2\}) : U \in \tau, V \in \sigma\}$ as its topology.

And thus, we have the general result:

Corollary 1. *Any dense-in-itself idempotent metrizable space is sound and complete with respect to* $\mathsf{Logic}_{\forall K}$, $\mathsf{Logic}_{\forall\Box}$ *and* $\mathsf{Logic}_{\forall\Box\Box_0}$.

All the spaces introduced in Section 1, except for $\mathbb{R}$ and $\mathcal{T}_2$, are dense-in-themselves, metrizable and idempotent spaces. And thus:

Theorem 11. *The rational line* $\mathbb{Q}$*, the Cantor space* $\mathfrak{C}$ *and the Baire space* $\mathfrak{B}$ *are generic spaces for the fragments* $\mathcal{L}_K$, $\mathcal{L}_{KB}$, $\mathcal{L}_{\forall\Box}$, $\mathcal{L}_{\forall K}$ *and* $\mathcal{L}_{\forall\Box\Box_0}$.

Completeness of $\mathsf{Logic}_{\forall\Box\Box_0}$ with respect to $\mathbb{Q}$ with a particular subbasis. While so far in the present section we have shown several of the logics in [2] to be sound and complete with respect to single generic models, we failed to provide a single topo-e-model for the fragment involving the basic evidence modality. Instead, we showed that the corresponding logic is sound and complete with respect to the class of topological evidence models based on $(\mathbb{Q}, \tau_{\mathbb{Q}})$ with arbitrary subbases. But can we find one subbasis $\mathcal{S}$ such that the logic of the single space $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S})$ is precisely $\mathsf{Logic}_{\forall\Box\Box_0}$?

This would need to be a subbasis which is not a basis (for otherwise $\Box\phi \leftrightarrow \Box_0\phi$ would be a theorem of the logic). One obvious candidate is perhaps the most paradigmatic case of subbasis-which-is-not-a-basis, namely

$$\mathcal{S} = \{(a, \infty), (-\infty, b) : a, b \in \mathbb{Q}\}.$$

We will show that this subbasis does not lead to a complete logic. To show why, consider the following formula, with three propositional variables p_1, p_2, p_3:

$$\gamma = \bigwedge_{i=1,2,3} (\Box_0 p_i \wedge [\exists]\Box_0\neg p_i) \bigwedge_{i\neq j\in\{1,2,3\}} [\exists](\Box_0 p_i \wedge \neg\Box_0 p_j),$$

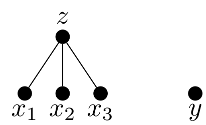
where $[\exists]$ is the dual of $[\forall]$ (i.e. $[\exists]\phi = \neg[\forall]\neg\phi$). Then γ is consistent in the logic yet it cannot be satisfied by any model based on $\mathbb{Q}$ with the aforementioned subbasis.

Indeed, note that, in any topo-e-model, $[\![\Box_0\phi]\!]$ is a union of elements in the subbasis. In particular, with the subbasis $\mathcal{S}$ as defined above, we have that $[\![\Box_0\phi]\!]$ is always of the form $[\![\Box_0\phi]\!] = (-\infty, a) \cup (b, \infty)$ for some $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ (here, we call $(-\infty, -\infty) = (\infty, \infty) = \varnothing$ and $(-\infty, \infty) = \mathbb{Q}$).

Moreover, if the set $[\![\Box_0\phi \wedge [\exists]\Box_0\neg\phi]\!]$ is nonempty, then it is straightforward to see that $[\![\Box_0\phi]\!]$ has to be either of the form (a, ∞) or of the form $(-\infty, a)$ for some $a \in \mathbb{R}$.

By this observation, the first conjunct of γ gives that $[\![\Box_0 p_i]\!]$ is of the form (a, ∞) or $(-\infty, a)$ for some $a \in \mathbb{R}$. By the second conjunct, the sets $[\![\Box_0 p_i]\!]$ and $[\![\Box_0 p_j]\!]$ need to be incomparable for $i \neq j$. But of course, at least two of the sets $[\![\Box_0 p_i]\!]$ have to be of the same form (either $(-\infty, a_i)$ and $(-\infty, a_j)$ or (a_i, ∞) and (a_j, ∞)), and thus it obviously cannot be the case that three such sets are incomparable. Therefore $(\mathbb{Q}, \tau_{\mathbb{Q}}, \mathcal{S}) \models \neg\gamma$.

However, γ is consistent. To show this, we use the fact (see [2]) that the logic is complete with respect to *quasi-models*, i.e. structures of the form $(X, \leq, E_0, V)$, where $\leq$ is a preorder and E_0 is a collection of $\leq$-upsets. $[\forall]$ is interpreted globally, $\Box$ is interpreted as the Kripke modality for $\leq$ and $x \in [\![\Box_0 \phi]\!]$ if and only if there is some $e \in E_0$ with $x \in e \subseteq [\![\phi]\!]$. Let $(X, \leq)$ be the following poset:



and call $e_i = \{x_i, z\}$ for $i = 1, 2, 3$. Let $E_0 = \{e_1, e_2, e_3, \{y\}, X\}$ and $V(p_i) = e_i$ for $i = 1, 2, 3$. It is clear that $(X, \leq, E_0, V)$ is a quasi-model and that $z \models \Box_0 p_i$, $x_i \models \Box_0 p_i \wedge \neg \Box_0 p_j$ for $j \neq i$, and $y \models \Box_0 \neg p_i$. Thus $z \models \gamma$ and γ is therefore consistent in the logic. Since every model based on $\mathbb{Q}$ with E_0 as a subbasis makes $\neg\gamma$ true yet $\neg\gamma \notin \mathsf{Logic}_{\forall\Box\Box_0}$, incompleteness follows.

We conjecture that no particular subbasis will give us completeness. Proving this result, or otherwise finding such a subbasis, constitutes an interesting line of future work.

3 Conclusions and future work

We have shown that there are topological spaces which are *generic enough* to capture the logic of topological evidence models, mirroring the McKinsey-Tarski theorem within the framework of topological evidence logics.

A number of questions still remain open. One potential direction for future work is to see whether the completeness results in this paper extend to strong completeness (it is shown in [13] that, under the interior semantics, $\mathsf{S4}$ is strongly complete with respect to any dense-in-itself metrizable space).

It will also be interesting to add a dynamic dimension to this work: one of the advantages of topo-e-models over other topological treatments of evidence logics is how well these models behave dynamically. In [2], dynamic extensions for these logics which include modalities for public announcement or evidence addition are given, along with sound an complete axiomatisations. Thus, one may wonder whether our models are also generic for these logics.

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A Appendices

A.1 Proof of Theorem 4

Let us take a finite rooted preorder $\mathfrak{F} = (W, \leq)$ and a dense-in-itself metrizable space (X, τ) and construct a dense-interior onto map $\bar{f} : (X, \tau) \twoheadrightarrow (W, \leq)$.[16] For this construction, we will use the following two lemmas. Their proofs[17] can be found respectively in [5, Lemmas 4.13 and 4.22] and [10, Thm. 41].

[16] We wish to thank Guram Bezhanishvili for the idea of this construction.
[17] Lemma 8 is a cornerstone of the proof of McKinsey and Tarski's theorem.

Lemma 8. *(i) If $\mathfrak{F} = (W, \leq)$ is a finite rooted preorder, and (G, τ) is a dense-in-itself metrizable space, there exists a continuous, open and surjective map $f : (G, \tau) \to (W, \mathrm{Up}_{\leq}(W))$.*

(ii) (Partition Lemma) *Let X be a dense-in-itself metrizable space and $n \geq 1$. Then there is a partition $\{G, U_1, ..., U_n\}$ of X such that G is a dense-in-itself closed subspace of X with dense complement and each U_i is an open set.*

Lemma 9. *Given a dense-in-itself metrizable space X and $n \geq 1$, X can be partitioned in n dense sets.*

Note that $\mathfrak{F}$ has a final cluster, i.e., a set $A \subseteq W$ with the property that $w \leq a$ for all $w \in W$ and all $a \in A$. Indeed, let $r \in W$ be the root and let $x, y \in W$ be any two maximal elements (which exist, on account that $\mathfrak{F}$ is finite). Since $r \leq x$ and $r \leq y$, by weak directedness, there is a z such that $x, y \leq z$. But by maximality of x and y, we have that $z \leq x$ and $z \leq y$, hence, by transitivity, $x \leq y$ and $y \leq x$: the maximal elements of $\mathfrak{F}$ form a final cluster. Let this cluster be $A = \{a_1, ..., a_n\}$.

If $W = A$, then we simply partition X in n dense sets $\{A_1, ..., A_n\}$ as per Lemma 9 and we take $\bar{f}$ to map each $x \in A_i$ to a_i. It is a straightforward check that $\bar{f}$ is dense-open (the image of a dense open set is W) and dense-continuous (the preimage of a nonempty upset is X). Otherwise, let us call $B := W \setminus A$, which is a finite rooted preorder. Let $\{G, U_1, ..., U_n\}$ be a partition of X as given by the Partition Lemma. Since G is a dense-in-itself metrizable space and B is a finite rooted preorder, by Lemma 8(i), there exists an onto interior map (with respect to the subspace topology of G) $f : G \twoheadrightarrow B$. We extend this map to $\bar{f} : X \twoheadrightarrow W$ by mapping each $x \in U_i$ to a_i.

We now show that $\bar{f}$ is the desired map. It is surjective by construction. It is dense-open, for given a nonempty dense open set $U \subseteq X$, we have that $U \cap G$ is an open set in the subspace topology of G and therefore $\bar{f}[U \cap G] = f[U \cap G]$ is an upset in B. On the other hand, $U \setminus G = U \cap (X \setminus G)$ is the intersection of two dense open sets and therefore is dense open, which means it has nonempty intersection with each of the U_i and hence $\bar{f}[U \setminus G] = A$. Therefore, $\bar{f}[U]$ is the union of an upset in B with A, and thus is an upset in W.

To see that $\bar{f}$ is dense-continuous, take a nonempty upset $U \subseteq W$, which will be a disjoint union $U = B' \cup A$, with B' being an upset in B. Then $\bar{f}^{-1}[B'] = f^{-1}[B']$ is an open set in X and $\bar{f}^{-1}[A] = U_1 \cup ... \cup U_n = X \setminus G$. Therefore, $\bar{f}^{-1}[U]$ is the union of an open set and a dense open set and thus a dense open set. This concludes the proof.

A.2 Proof of Theorem 7

Let $(W, \leq)$ be a finite preorder with a final cluster. We have the following:

Lemma 10. $(W, \leq)$ *is a p-morphic image of a finite disjoint union of finite rooted* $\mathsf{S4.2}$ *frames, via a dense-open and dense-continuous p-morphism.*

Proof. Let $x_1, ..., x_n$ be the minimal elements of W. Now, for $1 \leq i \leq n$ take $W'_i = \uparrow x_i \times \{i\}$. Define an order on $W' = W'_1 \cup ... \cup W'_n$ by: $(x, i) \leq (y, j)$ iff $i = j$ and $x \leq y$. Then $W'_1, ..., W'_n$ are pairwise disjoint finite rooted $\mathsf{S4.2}$ frames (with $A \times \{i\}$ as a final cluster) and $(x, i) \mapsto x$ is a p-morphism from W' onto W. It is easy to see that this mapping is dense-open (for every nonempty open set is dense in W) and dense-continuous (for the preimage of a nonempty W-upset is a W'-upset which contains all the final clusters, and thus is dense).

We can use Lemma 10 to construct the map: let $W'_1, ..., W'_n$ be the family of pairwise disjoint finite rooted $\mathsf{S4.2}$ frames whose union W' has $(W, \leq)$ as a p-morphic image.

Take $z_1, ..., z_{n-1} \in \mathbb{R} \backslash \mathbb{Q}$ and consider the intervals $A_1 = (-\infty, z_1)$, $A_n = (z_{n-1}, \infty)$ and $A_i = (z_{i-1}, z_i)$ for $1 < i < n$. Now, each A_i, as a subspace, is homeomorphic to $\mathbb{Q}$ (and thus a dense-in-itself metrizable space). From each $(A_i, \tau|_{A_i})$ we can find a dense-open, dense-continuous and surjective map f_i onto W'_i. Then $f = f_1 \cup ... \cup f_n$ is a dense-interior map onto W' which, when composed with the p-morphism in Lemma 10, gives us the desired map.

A.3 Proof of Theorem 9

We show that $E_0^{\mathbb{Q}}$ is a subbasis for $\mathbb{Q}$. First, given that $X \in E_0^X$ and $f[\mathbb{Q}] = X$, we have that $\mathbb{Q} \in E_0^{\mathbb{Q}}$, thus $\bigcup E_0^{\mathbb{Q}} = \mathbb{Q}$.

Now, suppose $p \in U \in \tau_{\mathbb{Q}}$. We show that there exist $e_1^q, ..., e_n^q \in E_0^{\mathbb{Q}}$ such that $p \in e_1^q \cap ... \cap e_n^q \subseteq U$. Note that $fp \in f[U]$ which is an open set. Since E_0^X is a subbasis for $(X, \leq)$ this means that there exist $e_1^x, ..., e_n^x \in E_0^X$ with $fp \in e_1^x \cap ... \cap e_n^x \subseteq f[U]$. Now set

$$e_i^q := f^{-1}[e_i^x] \backslash \{y \notin U : fy \in f[U]\}.$$

The fact that $e_i^q \in E_0^{\mathbb{Q}}$ follows from the fact that $f[e_i^q] = e_i^x$. Indeed, if $y \in f[e_i^q]$ then $y \in ff^{-1}[e_i^x] = e_i^x$ and conversely if $y \in e_i^x$, then either $y \in f[U]$ (in which case $y = fz$ for some $z \in U$ and thus $z \in f^{-1}[e_i^x]$ and therefore $z \notin \{z' \notin U : fz' \in f[U]\}$, which implies $z \in e_i^q$) or $y \notin f[U]$ (in which case $y = fz$ for some z by surjectivity and $z \notin \{z' \notin U : fz' \in f[U]\}$, thus $z \in e_i^q$). In either case, $y \in f[e_i^q]$.

Finally, note that $e_1^q \cap ... \cap e_n^q \subseteq U$. Indeed, for any $x \in e_1^q \cap ... \cap e_n^q$ we have that $fx \in e_1^x \cap ... \cap e_n^x \subseteq f[U]$, and thus by the definition of the e_i^q's it cannot be the case that $x \notin U$.

So for $p \in U \in \tau_{\mathbb{Q}}$ we have found elements $e_1^q, ... e_n^q \in E_0^{\mathbb{Q}}$ such that $p \in e_1^q \cap ... \cap e_n^q \subseteq U$, and therefore $E_0^{\mathbb{Q}}$ is a subbasis.

Now set a valuation $V^{\mathbb{Q}}(p) = \{x \in \mathbb{Q} : fx \in V(p)\}$ and let us show that, for any formula ϕ in the language and any $x \in \mathbb{Q}$, we have that $(\mathbb{Q}, \tau_{\mathbb{Q}}, E_0^{\mathbb{Q}}, V^{\mathbb{Q}}), x \models \phi$ if and only if $(X, \leq, E_0^X, V), fx \models \phi$. This is done by an induction on formulas; the only induction step that requires some attention is the one referring to $\Box_0$.

Let $x \models \Box_0 \psi$. This means that there exists some $e \in E_0^{\mathbb{Q}}$ with $x \in e$ and $y \models \psi$ for all $y \in e$. But then $fx \in f[e] \in E_0^X$ and by the induction hypothesis we have $fy \models \psi$ for all $fy \in f[e]$ and thus $fx \models \Box_0\psi$. Conversely, if $fx \in e' \subseteq [\![\psi]\!]^X$ for some $e' \in E_0^X$, we have that $x \in f^{-1}[e'] \in E_0^{\mathbb{Q}}$ and $fy \models \psi$ for each $y \in f^{-1}[e']$ and thus, by induction hypothesis, $y \models \psi$. Therefore $x \models \Box_0\psi$.