Model Completeness and Π_2-rules: the case of Contact Algebras

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Abstract

We give a sufficient condition for deciding admissibility of non-standard inference rules inside a modal calculus $\mathcal{S}$ with the universal modality. The condition requires the existence of a model completion for the discriminator variety of algebras which are models of $\mathcal{S}$. We apply the condition to the case of symmetric strict implication calculus, i.e., to the modal calculus axiomatizing contact algebras. Such an application requires a characterization of duals of morphisms which are embeddings (in the model-theoretic sense). We supply also an explicit infinite set of axioms for the class of existentially closed contact algebras. The axioms are obtained via a classification of duals of finite minimal extensions of finite contact algebras.

Keywords: Contact Algebras, Non-Standard Inference Rules, Model Completeness, Existentially Closed Structures.

1 Introduction

The use of non-standard rules has a long tradition in modal logic starting from the pioneering work of Gabbay [18], who introduced a non-standard rule for irreflexivity. Non-standard rules have been employed in temporal logic in the context of branching time logic [7] and for axiomatization problems [19] concerning the logic of the real line in the language with the Since and Until modalities. General completeness results for modal languages that are sufficiently expressive to define the so-called difference modality have been obtained in [32]. For the use of the non-standard density rule in many-valued logics we refer to [27] and [29].

Recently, there has been a renewed interest in non-standard rules in the context of the region-based theories of space [30]. One of the key algebraic structures in these theories is that of *contact algebras*. These algebras form a discriminator variety, see e.g., [4]. Compingent algebras are contact algebras satisfying two $\forall\exists$-sentences (aka Π_2-sentences) [4,15]. De Vries [15] established a duality between complete compingent algebras and compact Hausdorff spaces. This duality led to new logical calculi for compact Hausdorff spaces in [2] for two-sorted modal language and in [4] for a uni-modal language with a strict implication. Key to these approaches is a development of logical calculi corresponding to contact algebras. In [4] such a calculus is called the *strict symmetric implication calculus* and is denoted by $\mathsf{S^2IC}$. The extra Π_2-axoms of compingent algebras then correspond to non-standard Π_2-rules, which turn out to be admissible in $\mathsf{S^2IC}$. This generates a natural question of investigating admissibility of Π_2-rules in $\mathsf{S^2IC}$ and in general in logical calculi corresponding to discriminator varieties of modal algebras. This is the question that we address in this paper. We connect admissibility of non-standard Π_2-rules with the model completion of the first-order theory of the corresponding algebraic structures. Motivated by this connection, we then provide (an infinite) axiomatization of the model completion of the theory of contact algebras. As far as we are aware this is a first systematic study of admissibility in the context of non-standard inference rules.

The definition of Π_2-rules we give below is taken from [4] and is close to that of Balbiani et al. [2].

Definition 1.1 [Π_2-rule] A Π_2-*rule* is a rule of the form

$$(\rho) \qquad \frac{F(\underline{\varphi}/\underline{x}, \underline{p}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

where F, G are formulas, $\underline{\varphi}$ is a tuple of formulas, χ is a formula, and $\underline{p}$ is a tuple of propositional letters which do not occur in $\underline{\varphi}$ and χ.

Little is known about the problem of recognizing *admissibility* for non-standard rules, although this problem was already raised in [32]. An immediate easy computation shows that whenever a system $\mathcal{S}$ admits *local uniform interpolants*, then the above rule (ρ) is admissble iff the formula $G(\underline{x}) \to E_{\underline{p}}F(\underline{x}, \underline{p})$ is provable in $\mathcal{S}$, where $E_{\underline{p}}F(\underline{x}, \underline{p})$ is the uniform pre-interpolant of $F(\underline{x}, \underline{p})$ wrt the variables $\underline{p}$. [1]

Local uniform interpolants rarely exist: among the systems where they are available we list K, GL, $\mathsf{S4.Grz}$, $\mathsf{S5}$ [6,20,22,28,33]. From the structural point of view, *global uniform interpolants* (i.e. uniform interpolants for the global consequence relation) are more informative, due to their relationship to

[1] We consider part of the definition of uniform pre- and post- interpolants, the fact that they are stable under substitution: in other words, substituing $\underline{\varphi}$ for the $\underline{p}$ in $E_{\underline{p}}F(\underline{x}, \underline{p})$ must give the same result as computing $E_{\underline{p}}F(\underline{\varphi}/\underline{x}, \underline{p})$ after the substitution (see [20] for a careful analysis).

compact congruences and model completions [22,24,31]. However, the above simple argument for recognizing admissibility of non-standard rules *seems not to go through* via global uniform interpolants. There is no direct implication (in both senses) between the existence of local and of global uniform interpolants: global uniform interpolants fail to exists for K [24] whereas local ones exists. Conversely, there are cases where global interpolants exist and local interpolants do not (this is easily seen from the results of [26] for locally tabular S4-logics, where existence of uniform local/global interpolants reduces to existence of ordinary local/global interpolants and hence to superamalagamation/amalgamation properties).

Non-standard rules are usually investigated in a system $\mathcal{S}$ of modal logic with a global modality. The global modality is known to supply a discriminator term for the class of $\mathcal{S}$-algebras [23]. In such contexts, by the results of [22], existence of uniform interpolants imply (actually it is equivalent to) existence of a model completion for the equational class of $\mathcal{S}$-algebras. An easy modification of the arguments in [22], shows also that the existence of global uniform interpolants for $\mathcal{S}$ implies the existence of a model completion $T_{\mathcal{S}}{}^{\star}$ for the theory $T_{\mathcal{S}}$ axiomatizing the universal class of *simple* $\mathcal{S}$-algebras. In this paper, we first show that the latter condition (namely existence of a model completion $T_{\mathcal{S}}{}^{\star}$ for $T_{\mathcal{S}}$) is sufficient to characterize non-standard $\mathcal{S}$-rules. This characterization yields effective recognizability of non-standard rules, if quantifier elimination in $T^{\star}_{\mathcal{S}}$ is effective. The latter is certainly the case when $\mathcal{S}$ is decidable and locally tabular. We apply this general result to the case of contact algebras, where we show that the model completion of the theory of simple algebras exists and provide also an axiomatization for it.

2 Π_2-rules and model completions

A *modal signature* Σ is a finite signature comprising Boolean operators $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$ as well as additional operators of any arity called the *modal* operators. Among modal operators, there is a distinguished unary operator $[\forall]$, called the *global* or *universal* modality. Out of Σ-symbols and out of a countable set of variables $x, y, z, \dots, p, q, r, \dots$ one can build the set of propositional Σ-*formulae*. Σ-formulae might be indicated both with the greek letters $\phi, \psi, \dots$ and the latin capital letters $F, G, \dots$. Notations such as $F(\underline{x})$ mean that the Σ-formula F contains at most the variables from the tuple $\underline{x}$. A *modal system* $\mathcal{S}$ (over the modal signature Σ) is a set of Σ-formulae comprising tautologies, the axioms:

$$[\forall]\phi \rightarrow \phi, \qquad [\forall]\phi \rightarrow [\forall][\forall]\phi,$$

$$\phi \rightarrow [\forall]\neg[\forall]\neg\phi, \qquad [\forall](\phi \rightarrow \psi) \rightarrow ([\forall]\phi \rightarrow [\forall]\psi),$$

$$\bigwedge_i [\forall](\phi_i \leftrightarrow \psi_i) \rightarrow (O(...\phi_i...) \leftrightarrow O(...\psi_i...)) \quad \text{(for all } O \in \Sigma\text{)}.$$

and closed under the rules of modus ponens (MP) (form ϕ and $\phi \rightarrow \psi$ infer ψ), uniform substitution (US) (from $F(\underline{x})$ infer $F(\underline{\psi}/\underline{x})$), and necessitation (N) (from ϕ infer $[\forall]\phi$). We often write $S \vdash \Phi$ or $\vdash_{\mathcal{S}} \phi$ for $\phi \in S$. *We let a* *modal signature* Σ *and a modal system* $\mathcal{S}$ *be fixed for the remaining part of this section.* Formulas in $\mathcal{S}$ will be called $\mathcal{S}$*-axioms*. We say that $\mathcal{S}$ is decidable iff the relation $\mathcal{S} \vdash \phi$ is decidable. We also say that $\mathcal{S}$ is *locally tabular* iff for every finite tuple of propositional variables $\underline{x}$ there are finitely many formulae $\psi(\underline{x}), \dots, \psi_n(\underline{x})$ such that for every further formula $\phi(\underline{x})$ there is some i in $1, \dots, n$ such that $\mathcal{S} \vdash \phi \leftrightarrow \psi_i$.

We now consider the effect of the addition of Π_2-rules (see Definition 1.1) to a system $\mathcal{S}$.

Definition 2.1 [Proofs with Π_2-rules] Let Θ be a set of Π_2-rules. For a formula φ, we say that φ is *derivable* in $\mathcal{S}$ using the Π_2-rules in Θ, and write $\vdash_{\mathcal{S}+\Theta} \varphi$, provided there is a proof $\psi_1, \dots, \psi_n$ such that $\psi_n = \varphi$ and each ψ_i is an instance of an axiom of $\mathcal{S}$, or is obtained either by (MP) or (N) from some previous ψ_j's, or there is $j < i$ such that ψ_i is obtained from ψ_j by an application of one of the Π_2-rules $\rho \in \Theta$. The latter means the following, for ρ like in Definition 1.1: $\psi_j = F(\underline{\xi}/\underline{x}, \underline{p}) \to \chi$ and $\psi_i = G(\underline{\xi}/\underline{x}) \to \chi$, where F, G are formulas, $\underline{\xi}$ is a tuple of formulas, χ is a formula, and $\underline{p}$ is a tuple of propositional letters not occurring in $\underline{\xi}, \chi$.

We are interested in characterizing those Π_2-rules that can be freely used in a system without affecting its deductive power.

Definition 2.2 A rule ρ is *admissible* in the system $\mathcal{S}$ if for each formula φ, from $\vdash_{\mathcal{S}+\rho} \varphi$ it follows that $\vdash_{\mathcal{S}} \varphi$.

We may view our modal signature Σ as a first-order signature and Σ-formulae as terms in such a signature. For a modal system $\mathcal{S}$, an $\mathcal{S}$-algebra is a Boolean algebra with operations (one operation of suitable arity for each $O \in \Sigma$) satisfying $[\forall]\top = \top$ and $\phi = \top$ for every $\mathcal{S}$-axioms ϕ. We call an $\mathcal{S}$-algebra *simple* iff the universal first-order condition $\forall x\,([\forall]x = \top \vee [\forall]x = \bot)$ holds. This agrees with the standard definition from universal algebra, because it can be shown that congruences in a $\mathcal{S}$-algebra bijectively correspond to $[\forall]$*-filters*, i.e. to filters F satisfying the additional condition that $a \in F$ implies $[\forall]a \in F$. We call $T_{\mathcal{S}}$ the equational first-order theory of *simple non degenerate* $\mathcal{S}$*-algebras* (an $\mathcal{S}$-algebra is non degenerate iff $\bot \neq \top$). A standard Lindenbaum construction proves the *algebraic completeness theorem*, namely that for every ϕ we have $\mathcal{S} \vdash \phi$ iff the identity $\phi = \top$ holds in all $\mathcal{S}$-algebras (and hence iff $\phi = \top$ holds in all simple $\mathcal{S}$-algebras, because $\mathcal{S}$-algebras are a discriminator variety).

With each Π_2-rule ρ given in Definition 1.1, we can associate the following $\forall\exists$-statement in the *first-order* language of $\mathcal{S}$-algebras:

$$\Pi(\rho) := \forall \underline{x}, z \Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x}, \underline{y}) \not\leq z \Big).$$

Theorem 2.3 *Suppose that the universal theory* $T_{\mathcal{S}}$ *has a model completion* $T^\star_{\mathcal{S}}$*. Then a* Π_2*-rule* ρ *is admissible in* $\mathcal{S}$ *iff* $T^\star_{\mathcal{S}} \models \Pi(\rho)$.

Proof. In our general setting [4, Theorem 6.12] holds, replacing the system SIC mentioned there with our generic system $\mathcal{S}$ (the proof of this generalization is reported in the Appendix below as Theorem A.4 and follows the very same arguments as the analogous result of [4]). Using that theorem, we have to show that $T^\star_{\mathcal{S}} \models \Pi(\rho)$ holds iff every simple $\mathcal{S}$-algebra $\mathcal{B}$ can be embedded into some simple $\mathcal{S}$-algebra $\mathcal{C}$ which satisfies $\Pi(\rho)$. This is shown below using the fact that $\Pi(\rho)$ is a Π_2-sentence. Recall that models of $T^\star_{\mathcal{S}}$ are just the existentially closed simple $\mathcal{S}$-algebras (see [12, Proposition 3.5.15]).

Suppose for the left to right direction that $T^\star_{\mathcal{S}} \models \Pi(\rho)$ holds and let $\mathcal{B}$ be any simple $\mathcal{S}$-algebra. Then $\mathcal{B}$ embeds into an existentially closed simple $\mathcal{S}$-algebra $\mathcal{C}$ (this is a general model-theoretic fact [12]); as mentioned above, since $T_{\mathcal{S}}$ has a model completion $T^\star_{\mathcal{S}}$, the existentially closed simple $\mathcal{S}$-algebras are an elementary class and are precisely the models of $T^\star_{\mathcal{S}}$. Thus $\mathcal{B}$ embeds into $\mathcal{C}$ and $\mathcal{C}$ satisfies $\Pi(\rho)$, because $T^\star_{\mathcal{S}} \models \Pi(\rho)$.

Conversely, suppose that every simple $\mathcal{S}$-algebra $\mathcal{B}$ can be embedded into some simple $\mathcal{S}$-algebra $\mathcal{C}$ which satisfies $\Pi(\rho)$. Pick $\mathcal{B}$ such that $\mathcal{B} \models T^\star_{\mathcal{S}}$ and let $\Pi(\rho)$ be $\forall \underline{x} \exists \underline{y} H(\underline{x}, \underline{y})$, where H is quantifier free. Let $\underline{b}$ be a tuple from the support of $\mathcal{B}$. Then we have $\mathcal{C} \models \exists \underline{y} H(\underline{b}, \underline{y})$ for some extension $\mathcal{C}$ of $\mathcal{B}$. As $\mathcal{B}$ is existentially closed, this immediately entails that $\mathcal{B} \models \exists \underline{y} H(\underline{b}, \underline{y})$. Since the $\underline{b}$ was arbitrary, we conclude that $\mathcal{B} \models \Pi(\rho)$, as required. □

Checking whether a Π_2-rule is admissible or not now amounts to checking whether $T^\star_{\mathcal{S}} \models \Pi(\rho)$ holds or not. The latter can be done via quantifier elimination in $T^\star_{\mathcal{S}}$. We give sufficient conditions for this to be effective.

Corollary 2.4 *Let $\mathcal{S}$ be decidable and locally tabular. Assume also that simple $\mathcal{S}$-algebras enjoy the amalgamation property. Then admissibility of Π_2-rules in $\mathcal{S}$ is effective.*

Proof. Local tabularity of $\mathcal{S}$ implies local finiteness [2] of $T_{\mathcal{S}}$. For universal locally finite theories in a finite language, amalgamability is a necessary and sufficient condition for existence of a model completion [25,34]. Quantifier elimination in $T^\star_{\mathcal{S}}$ is effective because there are only finitely many non-equivalent formulae in a fixed finite number of variables, because of Lemma A.3 from the Appendix and because of the following folklore lemma. □

Lemma 2.5 *The quantifier-free formula $R(\underline{x})$ provably equivalent in $T^\star_{\mathcal{S}}$ to an existential formula $\exists \underline{y} H(\underline{x}, \underline{y})$ is the strongest quantifier free formula $G(\underline{x})$ implied (modulo $T_{\mathcal{S}}$) by $H(\underline{x}, \underline{y})$.*

Proof. Recall that $T_{\mathcal{S}}$ and $T^\star_{\mathcal{S}}$ are co-theories [12], i.e. they prove the same universal formulae. Thus we have the following chain of equivalences:

$$\dfrac{\dfrac{\dfrac{\dfrac{T_{\mathcal{S}} \vdash H(\underline{x}, \underline{y}) \to G(\underline{x})}{T^\star_{\mathcal{S}} \vdash H(\underline{x}, \underline{y}) \to G(\underline{x})}}{T^\star_{\mathcal{S}} \vdash \exists \underline{y} H(\underline{x}, \underline{y}) \to G(\underline{x})}}{T^\star_{\mathcal{S}} \vdash R(\underline{x}) \to G(\underline{x})}}{T_{\mathcal{S}} \vdash R(\underline{x}) \to G(\underline{x})}$$

[2] Recall that a class of algebras is *locally finite* if every finitely generated algebra in this class if finite, see [11, Section 14.2] for the connection between local finiteness and local tabularity.

yielding the claim. □

We point out that there might be different ways (other than Corollary 2.4) to exploit Theorem 2.3 in order to decide admissibility of Π_2-rules (for instance, as mentioned in the introduction, computability of global interpolants offers a powerful opportunity, given the relationship between model completions and uniform global interpolants [22]). However Corollary 2.4 gives a simple criterion, independent of more sophisticated machinery, which is useful for the application of this paper. In Section 4, we give an example of the application of Corollary 2.4 for recognizing an admissible rule.

The usefulness of Corollary 2.4 lies in the fact that its only real requirement is the amalgamation property, besides local tabularity. Whenever local tabularity holds, finitely presented algebras are finite, thus it is sufficient to establish amalgamability for *finite* algebras (this is easily seen by compactness of first-order logic, because, in the end, amalgamation property can be established by showing the consistency of some joined Robinson diagrams). Whenever a “good” duality is established, amalgamation of finite algebras turns out to be equivalent to dual amalgamation for finite frames, which is usually much easier to check. We will now give a couple of simple (non-)examples.

Example 2.6 If the modal signature contains only the global modality $[\forall]$, we have the locally tabular logic $\mathsf{S5}$. Finite simple non degenerate $\mathsf{S5}$-algebras are dual to finite nonempty sets and onto maps, for which dual amalgamation trivially holds (by standard pullback construction), see, e.g., [11, Thm. 14.23].

Example 2.7 The logic of difference [14,32] has in addition to the global modality a unary operator D subject to the axioms

$$[\forall]\phi \leftrightarrow (\phi \wedge \neg D\neg\phi), \qquad \phi \to D\neg D\neg\phi, \qquad DD\phi \to \phi \vee D\phi.$$

This logic axiomatizes Kripke frames where the accessibility relation is inequality. Local finiteness can be established for instance by the method of irreducible models [21]. Amalgamation however fails. To see this, notice that the simple frames for this logic are sets endowed with a relation E such that $w_1 \neq w_2 \to w_1 E w_2$. Now let $X = \{x_1, \dots, x_5\}$, $Y = \{y_1, \dots, y_5\}$ and $Z = \{z_1, z_2\}$. Let $x_i E_X x_j$ iff $i \neq j$ for $1 \leq i, j \leq 5$, $y_i E_Y y_j$ iff $i \neq j$ for $1 \leq i, j \leq 5$ and $z_i E_Z z_j$ for $i, j = 1, 2$. Let also $f : X \to Z$ and $g : Y \to Z$ be such that $f(x_1) = f(x_2) = f(x_3) = g(y_1) = g(y_2) = z_1$ and $f(x_4) = f(x_5) = g(y_3) = g(y_4) = g(y_4) = z_5$. Then it is easy to see that f and g are p-morphisms. If a dual amalgam exists, then there must exist a frame (U, E_U) and onto p-morphims $h : U \to X$ and $j : U \to Y$ such that $f \circ h = g \circ j$. However, an easy argument shows that U should contain more than 5 points. Moreover, for $u, v \in U$ with $u \neq v$ we should have $u E_U v$. But then there will be distinct points in U mapped by f to some x_i, which would entail that x_i is reflexive, which is a contradiction.

3 Symmetric Strict Implication and Contact Algebras

In this section we first review some material from [4]. Let us consider the modal signature comprising, besides the global modality $[\forall]$, a binary operator $\rightsquigarrow$, which we call *strict implication*, subject to the following axioms (we keep the same numeration as in [4] and add axiom (A0) which is seen as a definition of $[\forall]$ in [4]).

(A0) $[\forall]\varphi \leftrightarrow (\top \rightsquigarrow \varphi)$,
(A1) $(\bot \rightsquigarrow \varphi) \wedge (\varphi \rightsquigarrow \top)$,
(A2) $[(\varphi \vee \psi) \rightsquigarrow \chi] \leftrightarrow [(\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)]$,
(A3) $[\varphi \rightsquigarrow (\psi \wedge \chi)] \leftrightarrow [(\varphi \rightsquigarrow \psi) \wedge (\varphi \rightsquigarrow \chi)]$,
(A4) $(\varphi \rightsquigarrow \psi) \rightarrow (\varphi \rightarrow \psi)$,
(A5) $(\varphi \rightsquigarrow \psi) \leftrightarrow (\neg\psi \rightsquigarrow \neg\varphi)$,
(A8) $[\forall]\varphi \rightarrow [\forall][\forall]\varphi$,
(A9) $\neg[\forall]\varphi \rightarrow [\forall]\neg[\forall]\varphi$,
(A10) $(\varphi \rightsquigarrow \psi) \leftrightarrow [\forall](\varphi \rightsquigarrow \psi)$,
(A11) $[\forall]\varphi \rightarrow (\neg[\forall]\varphi \rightsquigarrow \bot)$,

Inference rules are modus ponens and necessitation. It can be shown (see [4]) that this system (called *symmetric strict implication calculus* $\mathsf{S^2IC}$) matches our requirements from Section 2. Moreover $\mathsf{S^2IC}$ is locally tabular and simple $\mathsf{S^2IC}$-algebras are those $\mathsf{S^2IC}$-algebras $\mathcal{B}$ where we have that $a \rightsquigarrow b$ is either $\bot$ or $\top$. Thus in a simple non-degenerate $\mathsf{S^2IC}$-algebra, the operation $\rightsquigarrow$ is in fact the characteristic function of a binary relation $\prec$. It can be proved that the characteristic function of a binary relation $\prec$ on a Boolean algebra gives rise to an $\mathsf{S^2IC}$-algebra structure iff it satisfies the following conditions:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$;
(S5) $a \prec b$ implies $a \leq b$;
(S6) $a \prec b$ implies $\neg b \prec \neg a$.

Non-degenerate Boolean algebras endowed with a relation $\prec$ satisfying the above conditions (S1)-(S6) are called *contact algebras.* [3] Since the theory of non degenerate simple $\mathsf{S^2IC}$-algebras is essentially the same (in fact, it is a syntactic variant) as the universal theory Con of contact algebras, we shall investigate the latter in order to apply Corollary 2.4. What we have to show in order to check the hypotheses of such a corollary is just that Con is amalgamable.

To prove amalgamability, we need a duality theorem. In [5,10,16] a duality theorem is established for the category of contact algebras and $\prec$-*maps* (a map $\mu : (\mathcal{B}, \prec) \rightarrow (\mathcal{C}, \prec)$ among contact algebras is said to be a $\prec$-map iff it is a Boolean homomorphism such that $a \prec b$ implies $\mu(a) \prec \mu(b)$). We shall make use of that theorem but we shall modify it, because for amalgamation we need

[3] It is more common to use in contact algebras the *contact relation* δ [30] which is given by $a\delta b$ iff $a \not\prec \neg b$. However, we stick with our notation to stay close to our main reference [4].

a duality for contact algebras and *embeddings* in the model theoretic sense (this means that an embedding is an injective map that not only *preserves* but also *reflects* the relation $\prec$). We first recall the duality theorem of [5], giving just the minimum information that is indispensable for our purposes.

We say that a binary relation R on a topological space X is *closed* if R is a closed subset of $X \times X$ in the product topology. Let StR be the category having (i) as objects the pairs (X, R), where X is a (non empty) Stone space and R is a closed, reflexive and symmetric relation on X, and (ii) as arrows the continuous maps $f : (X, R) \to (X', R')$ which are *stable* (i.e. such that xRy implies $f(x)R'f(y)$ for all points x, y in the domain of f). We define a contravariant functor

$$(-)^\star : \mathsf{StR}^{op} \to \mathsf{Con}_s$$

into the category Con_s of contact algebras and $\prec$-maps as follows:

- for an object (X, R), the contact algebra $(X, R)^\star$ has $\mathsf{Clop}(X)$ the clopens of X as carrier set (with union, intersection and complement as Boolean operations) and its relation $\prec$ is given by $C \prec D$ iff $R[C] \subseteq D$ (here we used the abbreviation $R[C] = \{x \in X \mid sRx \text{ for some } s \in C\}$);
- for a stable continuous map $f : (X, R) \to (X', R')$, the map $f^\star$ is the inverse image along f.

Theorem 3.1 ([5,16]) *The functor $(-)^\star$ establishes an equivalence of categories.*

We now intend to restrict this equivalence to the category Con_e of contact algebras and embeddings. To this aim we need to identify a suitable subcategory StR_e of StR. Now StR_e has the same objects as StR, however a stable continuous map $f : (X_1, R_1) \to (X_2, R_2)$ is in StR_e iff it satisfies the following additional condition:

$$\forall x, y \in X_2\ [xR_2y \ \Leftrightarrow\ \exists \tilde{x}, \tilde{y} \in X_1 \text{ s.t. } f(\tilde{x}) = x,\ f(\tilde{y}) = y \ \&\ \tilde{x}R_1\tilde{y}] \qquad (1)$$

Notice that, since R_2 is reflexive, it turns out that a map satisfying (1) must be surjective. We call the stable maps satisfying (1) *regular stable maps*, because it can be shown that these maps are just the regular epimorphisms in the category StR.

Theorem 3.2 *The functor $(-)^\star$, suitably restricted in its domain and codomain, establishes an equivalence of categories between StR_e and Con_e.*

Proof. We need to show that f satisfies condition (1) above iff $f^\star$ is an embedding between contact algebras, i.e. iff it satisfies the condition

$$(R_1[f^{-1}(U)] \subseteq f^{-1}(V) \ \Leftrightarrow\ R_2[U] \subseteq V) \quad \forall\, U, V \in \mathsf{Clop}(X_2) \qquad (2)$$

where $\mathsf{Clop}(X_2)$ is the set of clopens of the Stone space X_2. We tranform condition (2) up to equivalence. First notice that, by the adjunction between direct and inverse image, (2) is equivalent to

$$(f(R_1[f^{-1}(U)]) \subseteq V \ \Leftrightarrow\ R_2[U] \subseteq V) \quad \forall\, U, V \in \mathsf{Clop}(X_2) \qquad (3)$$

Now, in compact Hausdorff spaces, closed relations and continuous functions map closed sets to closed sets, hence $f(R_1[f^{-1}(U)])$ is closed and so, since clopens are a base for closed sets, (3) turns out to be equivalent to

$$(f(R_1[f^{-1}(U)]) = R_2[U]) \quad \forall\, U \in \mathsf{Clop}(X_2) \tag{4}$$

We now claim that (4) is equivalent to

$$f(R_1[f^{-1}(\{x\})]) = R_2[\{x\}] \quad \forall x \in X_2 \tag{5}$$

In fact, (5) implies (4) because all operations $f(-), R[-], f^{-1}(-)$ preserve set-theoretic unions. The converse implication holds because of Esakia's lemma below applied to the down-directed system $\{U \in \mathsf{Clop}(X_2) \mid x \in U\}$. Notice that Esakia's lemma applies because $f \circ R_1 \circ f^{op}$ and R_2 are symmetric relations, since R_1 and R_2 are symmetric (here we view f and $f^{-1} = f^{op}$ as relations via their graphs).

Now it is sufficient to observe that (5) is equivalent to the conjunction of (1) and stability. □

We will now prove a version of Esakia's lemma for our spaces. Esakia's lemma normally speaks about the inverse of a relation R, but here we need a version which holds for R-images because our relation is symmetric.

Lemma 3.3 (Esakia, Lemma 3.3.12 in [17]) *Let X be a compact Hausdorff space, and R a point-closed*[4] *symmetric binary relation on X. Then for each downward directed family $\mathcal{C} = \{C_i\}_{i\in I}$ of nonempty closed subsets of X, we have $R[\bigcap_{i\in I} C_i] = \bigcap_{i\in I} R[C_i]$.*

Proof. The inclusion $R[\bigcap_{i\in I} C_i] \subseteq \bigcap_{i\in I} R[C_i]$ is trivial. Now suppose $x \in \bigcap_{i\in I} R[C_i]$. Then $x \in R[C_i]$ for each C_i and, by symmetry, $R[x] \cap C_i$ is nonempty for each $i \in I$. But as C_i-s are downward directed, all the finite intersections $R[x] \cap C_{i_1} \cap ... \cap C_{i_n}$ (with $i_j \in I$ for $j \in \{1, ..., n\}$) are nonempty. By compactness, the infinite intersection (which equals $R[x] \cap \bigcap_{i\in I} C_i$) is nonempty and so, by symmetry, $x \in R[\bigcap_{i\in I} C_i]$. □

Now we are ready to show that Corollary 2.4 applies.

Theorem 3.4 *The universal theory* Con *of contact algebras has the amalgamation property. Therefore, as it is also locally finite,* Con *has a model completion.*

Proof. As we observed in Section 2, it is sufficient to prove amalgamation for finite algebras (by local finiteness and by a compactness argument based on Robinson diagrams). Finite algebras are dual to discrete Stone spaces, hence it is sufficient to show the following.

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[4] A binary relation R on a topological space X is said to be *point-closed* if $\forall x \in X$ $R[x]$ is closed in X. A closed relation in a compact Hausdorff space maps closed sets to closed sets via $R[-]$, hence it is point-closed.

(+) Given finite nonempty sets X_A, X_B, X_C endwed with reflexive and symmetric relations R_A, R_B, R_C and given regular stable maps $f : (X_B, R_B) \to (X_A, R_A)$, $g : (X_B, R_B) \to (X_A, R_A)$, there exist (X_D, R_D) (with reflexive and symmetric R_D) and regular stable maps $\pi_1 : (X_D, R_D) \to (X_B, R_B)$, $\pi_2 : (X_D, R_D) \to (X_C, R_C)$, such that $f \circ \pi_1 = g \circ \pi_2$.

Statement (+) is easily proved by taking as $(X_D, R_D), \pi_1, \pi_2$ the obvious pullback with the two projections. □

4 A Set of Axioms for $\mathsf{Con}^\star$

Theorem 3.4 gives the possibility of applying Corollary 2.4 to recognize admissible rules. We give here another algorithm, slightly different from that of Corollary 2.4. We recall that $\mathsf{Con}^\star$ is the theory of existentially closed contact algebras [12]. The following result (given that Con is locally finite) is folklore (a detailed proof of the analogous statement for Brouwverian semilattices is in the ArXiv version of [9] as [8, Proposition 2.16]).

Theorem 4.1 *Let $(\mathcal{B}, \prec)$ be a contact algebra. We have that $(\mathcal{B}, \prec)$ is existentially closed iff for any finite subalgebra $(\mathcal{B}_0, \prec) \subseteq (\mathcal{B}, \prec)$ and for any finite extension $(\mathcal{C}, \prec) \supseteq (\mathcal{B}_0, \prec)$ there exists an embedding $(\mathcal{C}, \prec) \hookrightarrow (\mathcal{B}, \prec)$ such that the following diagram commutes*

$$\begin{array}{ccc} (\mathcal{B}_0, \prec) & \hookrightarrow & (\mathcal{B}, \prec) \\ \downarrow & \nearrow & \\ (\mathcal{C}, \prec) & & \end{array}$$

Example 4.2 Consider the Π_2-rule:

$$(\rho 9) \qquad \frac{(p \rightsquigarrow p) \wedge (\varphi \rightsquigarrow p) \wedge (p \rightsquigarrow \psi) \to \chi}{(\varphi \rightsquigarrow \psi) \to \chi}$$

This rule is admissible in $\mathsf{S^2IC}$ [4, Theorem 6.15]. We will now give an alternative and more automated proof of this result. Translating $\Pi(\rho 9)$ into the equivalent language of contact algebras, we obtain (see statement ($S9$) from Section 6.3 of [4])

$$x \prec y \Rightarrow \exists z \ (z \prec z \wedge x \prec z \prec y) \qquad (6)$$

According to Theorem 2.3, we have to show that (6) is provable in $\mathsf{Con}^\star$. Note that (6) expresses interesting (order-)topological properties. It is valid on (X, R) iff R is a Priestley quasi-order [5, Lemma 5.2]. Also it is valid on a compact Hausdorff space X iff X is a Stone space [3, Lemma 4.11].

If we follow the procedure of Corollary 2.4 (which is based on Lemma 2.5), we first compute the quantifier-free formula equivalent in $\mathsf{Con}^\star$ to $\exists z \ (z \prec z \wedge x \prec z \prec y)$ by taking the conjunction of the (finitely many) quantifier-free first-order formulae $\phi(x, y)$ which are implied (modulo Con) by $z \prec z \wedge x \prec z \prec y$: this is, up to equivalence, $x \prec y$. Now, in order to show the admissibility of $(\rho 9)$ is sufficient to observe that $\mathsf{Con} \models x \prec y \Rightarrow x \prec y$.

As an alternative, we can rely on Theorem 4.1 and show that (6) is true in every existentially closed contact algebra. To this aim, it is sufficient to enumerate all contact algebras $\mathcal{B}_0$ generated by two elements a, b such that $\mathcal{B}_0 \models a \prec b$ and to show that all such algebras embed in a contact algebra $\mathcal{C}$ generated by three elements a, b, c such that $\mathcal{C} \models c \prec c \wedge a \prec c \prec b$ (this can be done automatically for instance using a model finder tool). Both of the above procedures are heavy and not elegant, but they are nevertheless mechanical and do not require ingenious *ad hoc* constructions (such as e.g., the construction of Lemma 5.4 in [4]).

Theorem 4.1 implicitly supplies an infinite set of axioms for the model completion of the theory of contact algebras. Such an axiomatization is not however very informative, as it comes from generic model-theoretic facts. In this section, we supply a better axiomatization, following the same strategy used in [13] for the case of amalgamable locally finite varieties of Heyting algebras and in [9] for the case of Brouwverian semilattices. The strategy consists of classifying minimal extensions via the so-called 'signatures'.

It is evident that the Theorem 4.1 still holds if we limit its statement to finite *minimal* extensions $(\mathcal{C}, \prec)$ of $(\mathcal{B}_0, \prec)$ (such an extension $(\mathcal{C}, \prec)$ is said to be minimal iff it is proper and every proper extension contains it, up to isomorphism). Using our Duality Theorem 3.2 restricted to the finite discrete case, we can characterize the dual spaces $(X_\mathcal{C}, R_\mathcal{C})$ and $(X_{\mathcal{B}_0}, R_{\mathcal{B}_0})$ and the dual stable map $f : (X_\mathcal{C}, R_\mathcal{C}) \to (X_{\mathcal{B}_0}, R_{\mathcal{B}_0})$ corresponding to such minimal extensions.

Proposition 4.3 *Let $(\mathcal{B}_0, \prec) \hookrightarrow (\mathcal{C}, \prec)$ be an embedding between finite contact algebras, with dual regular stable map $f : (X_\mathcal{C}, R_\mathcal{C}) \to (X_{\mathcal{B}_0}, R_{\mathcal{B}_0})$. The embedding is minimal iff (up to isomorphism) there are a finite set Y, finite subsets $S_1, S_2 \subseteq Y$ and elements $x \in X_{\mathcal{B}_0}, x_1 \in X_\mathcal{C}, x_2 \in X_\mathcal{C}$ such that:*

(i) *$X_{\mathcal{B}_0}$ is the disjoint union $Y \oplus \{x\}$;*

(ii) *$X_\mathcal{C}$ is the disjoint union $Y \oplus \{x_1, x_2\}$;*

(iii) *f restricted to Y is the identity map and $f(x_1) = f(x_2) = x$;*

(iv) *the restrictions of $R_\mathcal{C}$ and of $R_{\mathcal{B}_0}$ to Y coincide;*

(v) *$R_\mathcal{C}[x_1] \setminus \{x_1\} = S_1$ and $R_\mathcal{C}[x_2] \setminus \{x_2\} = S_2$;*

(vi) *$R_{\mathcal{B}_0}[x] \setminus \{x\} = S_1 \cup S_2$.*

Proof. First notice that, as a consequence of (1), if the cardinality of $X_{\mathcal{B}_0}$ and of $X_\mathcal{C}$ are the same, then f is an isomorphism. This is seen as follows: we already observed that condition (1) implies surjectivity and in case of the same finite cardinality surjectivity implies injectivity. Preservation and reflection of the relation follow by stability and (1) again.

In addition, if the cardinality of $X_\mathcal{C}$ is equal to the cardinality of $X_{\mathcal{B}_0}$ plus one (this is precisely the case mentioned in the statement of the proposition), then f cannot be properly factored, hence it is minimal. We show that all minimal maps arise in this way.

In general, if the cardinality of $X_{\mathcal{C}}$ is bigger than the cardinality of $X_{\mathcal{B}_0}$, we can define the following factorization of f. Pick some $x \in X_{\mathcal{B}_0}$ having more than one preimage and split $f^{-1}(\{x\})$ as $T_1 \cup T_2$, where T_1, T_2 are disjoint and non-empty. We have that $X_{\mathcal{C}}$ is the disjoint union $X \oplus T_1 \oplus T_2$ for some set X and $X_{\mathcal{B}_0}$ is the disjoint union $Y \oplus \{x\}$ for some set Y. Define a discrete dual space (Z, R_Z) as follows. Z is the disjoint union $Y \oplus \{x_1, x_2\}$ for new x_1, x_2 and R_Z is the reflexive and symmetric closure of the following sets of pairs: (i) the pairs (z_1, z_2) for $z_1 R_{\mathcal{B}_0} z_2$ and $z_1, z_2 \in Y$; (ii) the pairs (x_i, u) for $u \in f(R_{\mathcal{C}}[T_i])$ $(i = 1, 2)$; (iii) the pair (x_1, x_2), but only in case $T_1 \cap R_{\mathcal{C}}[T_2] \neq \emptyset$. Then it is easily seen that f factorizes as $h \circ \tilde{f}$ in StR_e, where: (I) $\tilde{f}$ maps T_1 to x_1, T_2 to x_2 and acts as f on X; (II) h is the identity on Y and maps both x_1, x_2 to x.

Now h produces the data required by the proposition and $\tilde{f}$ must be an isomorphism if f is minimal. □

Notice that the above conditions (i)-(vi) determine uniquely the finite minimal extension over the contact algebras dual to $(X_{\mathcal{B}_0}, R_{\mathcal{B}_0})$ except for a detail: they do not specify whether we have $x_1 R_{\mathcal{C}} x_2$ or not. So the data x, S_1, S_2 and $Y = X_{\mathcal{B}_0} \setminus \{x\}$ (lying *inside* $X_{\mathcal{B}_0}$) determine in fact *two* minimal expansions of the contact algebra dual to $(X_{\mathcal{B}_0}, R_{\mathcal{B}_0})$.

The next step is to *re-dualize* the data of Proposition 4.3 inside a given finite contact algebra. We first need some notation.

Definition 4.4 Let $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$ be a finite subalgebra of the contact algebra $(\mathcal{B}, \prec_{\mathcal{B}})$. Then, for $b \in \mathcal{B}$, we define $[b]^{\prec_{\mathcal{B}_0}} := \bigcap\{x \in \mathcal{B}_0 \mid b \prec_{\mathcal{B}} x\}$.

The notion of a signature given below dualizes and internalizes the data of Proposition 4.3 (the further bit $\star$ is used to distinguish the two possible minimal extensions).

Definition 4.5 Let $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$ be a finite contact algebra. We call a *signature* in $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$ a tuple $(b, \tilde{c_1}, \tilde{c_2})$, where $b \in \mathcal{B}_0$ is an atom, and $\tilde{c_1}, \tilde{c_2} \in \mathcal{B}_0$ are such that $[b]^{\prec_{\mathcal{B}_0}} \wedge \neg b = \tilde{c_1} \vee \tilde{c_2}$. A *marked signature* in $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$ is a tuple $(b, \tilde{c_1}, \tilde{c_2}, \star)$, where $(b, \tilde{c_1}, \tilde{c_2})$ is a signature and $\star \in \{0, 1\}$.

We are now ready to produce our first-order axiomatization of existentially closed contact algebras.

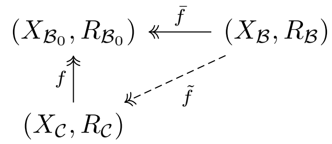
Theorem 4.6 *A contact algebra $(\mathcal{B}, \prec_{\mathcal{B}})$ is existentially closed if and only if, for any finite subalgebra $(\mathcal{B}_0, \prec_{\mathcal{B}_0}) \subseteq (\mathcal{B}, \prec_{\mathcal{B}})$, the following conditions hold:*

(i) *for every marked signature $(b, \tilde{c_1}, \tilde{c_2}, 1)$ in $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$, there exist $b_1, b_2 \in \mathcal{B} \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = \bot$, $[b_i]^{\prec_{\mathcal{B}_0}} = \tilde{c_i} \vee b$ for $i \in \{1, 2\}$ and $b_1 \not\prec_{\mathcal{B}} \neg b_2$;*

(ii) *for every marked signature $(b, \tilde{c_1}, \tilde{c_2}, 0)$ in $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$, there exist $b_1, b_2 \in \mathcal{B} \setminus \{0\}$ such that $b = b_1 \vee b_2$, $b_1 \wedge b_2 = \bot$, $[b_i]^{\prec_{\mathcal{B}_0}} = \tilde{c_i} \vee b$ for $i \in \{1, 2\}$ and $b_1 \prec_{\mathcal{B}} \neg b_2$.*

Proof. $(\Rightarrow)$ Let $(\mathcal{B}_0, \prec_{\mathcal{B}_0}) \hookrightarrow (\mathcal{B}, \prec_{\mathcal{B}})$ be a finite subalgebra, and let $(b, \tilde{c_1}, \tilde{c_2}, \star)$ be a signature in $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$. Let $(\mathcal{B}_0, \prec_{\mathcal{B}_0}) \hookrightarrow (C, \prec_{\mathcal{C}})$ be the finite minimal extension whose dual satisfies the conditions (i)-(vi) of Proposition 4.3 (and also $x_1 R_{\mathcal{C}} x_2$ iff $\star = 1$). Then it is clear that there exist b_1, b_2 satisfying (i) (for the case $\star = 1$) or (ii) (for the case $\star = 0$) inside $(C, \prec_{\mathcal{C}})$. Thanks to Theorem 4.1, we know that there exists an embedding $(C, \prec_{\mathcal{C}}) \hookrightarrow (B, \prec_{\mathcal{B}})$ that fixes $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$. Via this embedding, the required b_1, b_2 are moved to $(B, \prec_{\mathcal{B}})$: they still satisfy the conditions required by (i) and (ii) because such conditions can be expressed as first-order ground conditions with parameters in $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$ [5] and hence they are preserved through embeddings.

($\Leftarrow$) Here we use our Duality Theorem 3.2. We are given a finite contact subalgebra $(\mathcal{B}_0, \prec_{\mathcal{B}_0})$ of $(\mathcal{B}, \prec_{\mathcal{B}})$ and a finite minimal extension $(\mathcal{C}, \prec_{\mathcal{C}})$ of its. The situation, in the dual category is the following:



where $\bar{f}$ is dual to the inclusion $(\mathcal{B}_0, \prec_{\mathcal{B}_0}) \hookrightarrow (\mathcal{B}, \prec_{\mathcal{B}})$ and f satisfies the conditions (i)-(vi) of Proposition 4.3. We need to define $\tilde{f}$ so that the above triangle commutes in the category StR_e. Recall that the two spaces $X_{\mathcal{B}_0}, X_C$ are discrete, but X_B is not.

By hypothesis, we know that there exist non empty disjoint clopens $U_{b_1}, U_{b_2} \in \mathsf{Clop}(X_B)$ such that $\bar{f}^{-1}(\{x\}) = U_{b_1} \cup U_{b_2}$. According to Definition 4.4, we have that the clopen defined by $[b_i]^{\prec_{\mathcal{B}_0}}$ is the intersection of the family $\bar{f}^{-1}(T)$ varying T among the subsets of $X_{\mathcal{B}_0}$ such that $R_{\mathcal{B}}[U_{b_i}] \subseteq \bar{f}^{-1}(T)$, i.e. varying T among the subsets of $X_{\mathcal{B}_0}$ such that $\bar{f}(R_{\mathcal{B}}[U_{b_i}]) \subseteq T$. Since this intersection is precisely $\bar{f}^{-1}(\bar{f}(R_{\mathcal{B}}[U_{b_i}]))$, according to our hypothesis, we have $\bar{f}^{-1}(\bar{f}(R_{\mathcal{B}}[U_{b_i}])) = \bar{f}^{-1}(S_i) \cup \bar{f}^{-1}(\{x\})$. Since $\bar{f}^{-1}$ is injective, we conclude

$$\bar{f}(R_{\mathcal{B}}[U_{b_1}])) = S_1 \cup \{x\} \quad \text{and} \quad \bar{f}(R_{\mathcal{B}}[U_{b_2}])) = S_2 \cup \{x\}. \tag{7}$$

In case $x_1 R_{\mathcal{C}} x_2$ holds, we use hypothesis (i) and in case it does not hold, we use hypothesis (ii). To sum up, recalling that $b_1 \not\prec \neg b_2$ dualizes to $R_{\mathcal{B}}[U_{b_1}] \cap U_{b_2} \neq \emptyset$, we get

$$x_1 R_{\mathcal{C}} x_2 \iff R_{\mathcal{B}}[U_{b_1}] \cap U_{b_2} \neq \emptyset. \tag{8}$$

We define $\tilde{f}$ as follows: $\tilde{f}(z) = \bar{f}(z)$ for $z \notin U_{b_1} \cup U_{b_2}$, $\tilde{f}(z) = x_1$ for $z \in U_{b_1}$, $\tilde{f}(z) = x_2$ for $z \in U_{b_2}$. By Proposition 4.3(iii), it is clear that $\tilde{f} \circ f = \bar{f}$. The continuity of $\tilde{f}$ is also immediate.

Let us check *stability*, namely that for $y_1, y_2 \in X_{\mathcal{B}}$ such that $y_1 R_{\mathcal{B}} y_2$, we have $\tilde{f}(y_1) R_{\mathcal{C}} \tilde{f}(y_2)$. We distinguish three cases:

1. $y_1, y_2 \notin U_{b_1} \cup U_{b_2}$
2. $y_1 \in U_{b_1}$, $y_2 \notin U_{b_1} \cup U_{b_2}$
3. $y_1, y_2 \in U_{b_2} \cup U_{b_2}$

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[5] If $a_1, \ldots, a_n$ are the elements of $\mathcal{B}_0$, then $[b_i]^{\prec_{\mathcal{B}_0}} = \tilde{c_i} \vee b$ can be written as $\bigwedge_{j=1}^{n} (b_i \prec a_j \leftrightarrow (\tilde{c_i} \vee b \leq a_j))$.

(by the symmetry of $R_{\mathcal{B}}$, this enumeration is exhaustive, up to exchanging the role of U_{b_1} and of U_{b_2}). Case 1 is covered by the stability of $\bar{f}$ and Proposition 4.3(iv). Case 3 is covered by the reflexivity of $R_{\mathcal{C}}$ and (8). In Case 2, we have $y_2 \in R_{\mathcal{B}}[U_{b_1}]$, thus $\tilde{f}(y_2) = \bar{f}(y_2) \in S_1$ by (7) (and by the fact that $\bar{f}(y_2) \neq x$). Thus we conclude $x_1 = \tilde{f}(y_1) R_{\mathcal{C}} \tilde{f}(y_2)$ by Proposition 4.3(v).

It remains to prove that for all $z_1, z_2 \in X_{\mathcal{C}}$ such that $z_1 R_{\mathcal{C}} z_2$ we have

$$\exists y_1, y_2 \in X_{\mathcal{B}} \text{ s.t. } \tilde{f}(y_1) = z_1 \ \& \ \tilde{f}(y_2) = z_2 \ \& \ y_1 R_{\mathcal{B}} y_2 \tag{9}$$

(this is condition (1)). Again we distinguish three cases:

a. $z_1, z_2 \notin \{x_1, x_2\}$
b. $z_1 = x_1$, $z_2 \notin \{x_1, x_2\}$
c. $z_1 = x_1$, $z_2 = x_2$.

Case a is covered by Proposition 4.3(iii) and by the fact that $\bar{f}$ satisfies condition (1). Case c is covered by (8). In Case b, $z_2 \in S_1$ by Proposition 4.3(v) and by (7) there are $y_1 \in U_{b_1}$ and y_2 such that $y_1 R_{\mathcal{B}} y_2$ and $\bar{f}(y_2) = z_2$. Then, $\tilde{f}(y_1) = x_1 = z_1$ and $\tilde{f}(y_2) = \bar{f}(y_2) = z_2$ (we have $\tilde{f}(y_2) = \bar{f}(y_2)$ because $\bar{f}(y_2) = z_2 \neq x$, so that $y_2 \notin U_{b_1} \cup U_{b_2}$). □

Theorem 4.6 gives a first order axiomatization because the reference to finite subalgebras can be replaced by a suitable string of universal quantifiers. However, the above axiomatization is infinite, thus determining whether there exists a finite axiomatization (and supplying one, in case of a positive answer) remains at the moment an open question.

In principle, the axiomatization supplied by Theorem 4.6 should be naturally convertible (using Lemma A.3 below) into a basis for admissible Π_2-rules for the symmetric strict implication calculus, once the notion of a basis for admissible Π_2-rules is suitably defined. We leave this task for future research. Connections with the literature on admissibility of standard inference rules in contact algebras [1] should also be developed: our non-standard rules have the particular shape (ρ) outlined in Definition 1.1 and they trivialize if they are standard (i.e., if p does not occur in the formula F from the premise); however it could be interesting to analyze more general formats for non-standard rules encompassing standard inference rules.

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Appendix

A An Admissibility Criterion

We report here the statement and the proof of Theorem 6.12 of [4], generalized to a system $\mathcal{S}$ satisfying the conditions of Section 2. We need the same series of results as in [4], starting from Theorem 6.6 of [4] (we need only a slightly simplified version of the last theorem because we do not consider proofs with assumptions):

Theorem A.1 *For every set of* Π_2*-rules* Θ *and for every formula* ψ*, we have that* $T_{\mathcal{S}} \cup \{\Pi(\rho) \mid \rho \in \Theta\} \models \psi = \top \iff \vdash_{\mathcal{S}+\Theta} \psi$.

Proof. The right-to-left direction is a trivial induction on the length of a proof witnessing $\vdash_{\mathcal{S}+\Theta} \psi$. For the other side, we need a modified Lindembaum construction. Suppose that $\nvdash_{\mathcal{S}+\Theta} \psi$. For each rule $\rho_i \in \Theta$, we add a countably infinite set of fresh propositional letters to the set of existing propositional letters. Then we build the Lindenbaum algebra $\mathcal{B}$ over the expanded set of propositional letters, where the elements are the equivalence classes $[\varphi]$ under provable equivalence in $\mathcal{S} + \Theta$. Next we construct a maximal $[\forall]$-filter M of $\mathcal{B}$ such that $\neg[\forall]\psi \in M$ and for every rule $\rho_i \in \Theta$

$$(\rho_i) \qquad \frac{F_i(\underline{\varphi}/\underline{x}, \underline{p}) \to \chi}{G_i(\underline{\varphi}/\underline{x}) \to \chi}$$

and formulas $\underline{\varphi}, \chi$:

(†) if $[G_i(\underline{\varphi}) \to \chi] \notin M$, then there is a tuple $\underline{p}$ such that $[F_i(\underline{\varphi}, \underline{p}) \to \chi] \notin M$.

To construct M, let $\Delta_0 := \{\neg[\forall]\varphi\}$, a consistent set. We enumerate all formulas φ as $(\varphi_k : k \in \mathbb{N})$ and all tuples $(i, \underline{\varphi}, \chi)$ where $i \in \mathbb{N}$ and $\underline{\varphi}, \chi$ are as in the particular rule ρ_i, and we build the sets $\Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_n \subseteq \dots$ as follows (notice that, according to the construction below, for all n and $\theta \in \Delta_n$, we have $\vdash_{\mathcal{S}+\Theta} \theta \leftrightarrow [\forall]\theta$).

- For $n = 2k$, if $\nvdash_{\mathcal{S}+\Theta} \bigwedge \Delta_n \to [\forall]\varphi_k$, let $\Delta_{n+1} = \Delta_n \cup \{\neg[\forall]\varphi_k\}$; otherwise let $\Delta_{n+1} = \Delta_n$.
- For $n = 2k + 1$, let $(l, \underline{\varphi}, \chi)$ be the k-th tuple. If $\nvdash_{\mathcal{S}+\Theta} \bigwedge \Delta_n \to (G_l(\underline{\varphi}) \to \chi)$, let $\Delta_{n+1} = \Delta_n \cup \{\neg[\forall](F_l(\underline{\varphi}, \underline{p}) \to \chi)\}$, where $\underline{p}$ is a tuple of proposi-

tional letters for ρ_l not occurring in $\underline{\varphi}, \chi$, and any of θ with $\theta \in \Delta_n$ (we can take $\underline{p}$ from the countably infinite additional propositional letters which we have reserved for the rule ρ_l). Otherwise, let $\Delta_{n+1} = \Delta_n$.

Let M be

$$\{\ [\theta] \mid \text{ there are } \theta_1, \ldots, \theta_n \in \bigcup_{n\in\mathbb{N}} \Delta_i \text{ such that } \vdash_{\mathcal{S}+\Theta} \theta_1 \wedge \cdots \wedge \theta_n \to \theta\}\ .$$

It is clear that M is a proper $[\forall]$-filter not containing $[\psi]$.[6] Also, by the even steps of the construction of the sets Δ_n, it contains either $[[\forall]\theta]$ or $[\neg[\forall]\theta]$ for every θ, thus M is a maximal $[\forall]$-filter. Finally, the odd steps of the construction of the sets Δ_n ensure that M satisfies (†): in fact, if $[G_i(\underline{\varphi}) \to \chi] \notin M$, then by step $n = 2k+1$, we have $[\neg[\forall](F_l(\underline{\varphi}, \underline{p}) \to \chi)] \in M$ and if also $[F_i(\underline{\varphi}, \underline{p}) \to \chi] \in M$, then $[[\forall](F_i(\underline{\varphi}, \underline{p}) \to \chi)] \in M$ (because M is a $[\forall]$-filter) and so M would not be proper, a contradiction. Therefore, we can conclude that M satisfies all the desired properties.

By (†), the quotient of $\mathcal{B}$ by M satisfies each $\Pi(\rho_i)$; such a quotient is a simple algebra, because M is maximal as a $[\forall]$-filter. Moreover, since $[\neg[\forall]\psi] \in M$, we have that $[\neg[\forall]\psi]$ maps to $\top$, so $[[\forall]\psi]$ maps to $\bot$ in the quotient. Thus, $[\varphi]$ does not map to $\top$ in the quotient, and hence $T_{\mathcal{S}} \cup \{\Pi(\rho) \mid \rho \in \Theta\} \not\models \psi = \top$. □

Definition A.2 Given a quantifier-free first-order formula $\Phi(\underline{x})$, we associate with it the term (aka the propositional modal formula) $\Phi^*(\underline{x})$ as follows:

$$(t(\underline{x}) = u(\underline{x}))^* = [\forall](t(\underline{x}) \leftrightarrow u(\underline{x}))$$
$$(\neg\Psi)^*(\underline{x}) = \neg\Psi^*(\underline{x})$$
$$(\Psi_1(\underline{x}) \wedge \Psi_2(\underline{x})^* = \Psi_1^*(\underline{x}) \wedge \Psi_2^*(\underline{x}).$$

The following lemma is immediate:

Lemma A.3 *Let $\mathcal{B}$ be a simple $\mathcal{S}$-algebra and let $\Phi(\underline{x})$ be a quantifier-free formula. Then we have $\mathcal{B} \models \Phi(\underline{a}/\underline{x})$ iff $\mathcal{B} \models (\Phi(\underline{a}/\underline{x}))^* = \top$, for every tuple $\underline{a}$ from $\mathcal{B}$.*

Theorem A.4 (Admissibility Criterion) *A Π_2-rule ρ is admissible in $\mathcal{S}$ iff for each simple $\mathcal{S}$-algebra $\mathcal{B}$ there is a simple $\mathcal{S}$-algebra $\mathcal{C}$ such that $\mathcal{B}$ is a substructure of $\mathcal{C}$ and $\mathcal{C} \models \Pi(\rho)$.*

Proof. (⇒) Suppose that the rule ρ

$$(\rho) \quad \frac{F(\underline{\varphi}/\underline{x}, \underline{p}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

[6] The fact that M is proper comes from the fact that $\nvdash_{\mathcal{S}+\Theta} \bigwedge \Delta_n \to \bot$. This is clear for even n and for $n = 0$. For odd $n = 2k+1$, suppose that $\vdash_{\mathcal{S}+\Theta} \bigwedge \Delta_k \to [\forall](F_l(\underline{\varphi}, \underline{p}) \to \chi)$ and that $\nvdash_{\mathcal{S}+\Theta} \bigwedge \Delta_k \to (G_l(\underline{\varphi}) \to \chi)$. Then, by the axiom $[\forall]\phi \to \phi$ from Section 2, we have $\vdash_{\mathcal{S}+\Theta} F_l(\underline{\varphi}, \underline{p}) \to (\bigwedge \Delta_k \to \chi)$ and also (applying the rule ρ_l of the k-th tuple) $\vdash_{\mathcal{S}+\Theta} G_l(\underline{\varphi}) \to (\bigwedge \Delta_k \to \chi)$, yielding a contradiction.

is admissible in $\mathcal{S}$. It is sufficient to show that there exists a model $\mathcal{C}$ of the theory

$$T = T_{\mathcal{S}} \cup \{\Pi(\rho)\} \cup \Delta(\mathcal{B})$$

where $\Delta(\mathcal{B})$ is the diagram of $\mathcal{B}$ [12, p. 68]. Suppose for a contradiction that T has no models, hence is inconsistent. Then, by compactness, there exists a quantifier-free first-order formula $\Psi(\underline{x})$ and a tuple $\underline{x}$ of variables corresponding to some $\underline{a} \in \mathcal{B}$ such that

$$T_{\mathcal{S}} \cup \{\Pi(\rho)\} \models \neg\Psi(\underline{a}/\underline{x}) \text{ and } \mathcal{B} \models \Psi(\underline{a}/\underline{x}).$$

By Theorem A.1, $\mathcal{S}+\rho$ is complete with respect to the simple $\mathcal{S}$-algebras satisfying $\Pi(\rho)$. Therefore, by Lemma A.3, we have $T_{\mathcal{S}} \cup \{\Pi(\rho)\} \models (\neg\Psi(\underline{x}))^* = \top$ and also $\vdash_{\mathcal{S}+\rho} (\neg\Psi(\underline{x}))^*$, where $(-)^*$ is the translation given in Definition A.2. By admissibility, $\vdash_{\mathcal{S}} (\neg\Psi(\underline{x}))^*$. Thus, for the valuation v into $\mathcal{B}$ that maps $\underline{x}$ to $\underline{a}$, we have $v((\neg\Psi(\underline{x}))^*) = 1$, so $v((\Psi(\underline{x}))^*) = 0$. This contradicts the fact that $\mathcal{B} \models \Psi(\underline{a}/\underline{x})$. Consequently, T must be consistent, and hence it has a model.

($\Leftarrow$) Suppose $\vdash_{\mathcal{S}} F(\underline{\varphi}, \underline{p}) \to \chi$ with $\underline{p}$ not occurring in $\underline{\varphi}, \chi$. Let $\mathcal{B}$ be a simple $\mathcal{S}$-algebra and let v be a valuation on $\mathcal{B}$. By assumption, there is a simple $\mathcal{S}$-algebra $\mathcal{C}$ such that $\mathcal{B}$ is a substructure of $\mathcal{C}$ and $\mathcal{C} \models \Pi(\rho)$. Let $i : \mathcal{B} \hookrightarrow \mathcal{C}$ be the inclusion. Then $v' := i \circ v$ is a valuation on $\mathcal{C}$. For any $\underline{c} \in C$, let v'' be the valuation that agrees with v' except for the fact that it maps the $\underline{p}$ into the $\underline{c}$. Since $\vdash_{\mathcal{S}} F(\underline{\varphi}/\underline{x}, \underline{p}) \to \chi$, by the algebraic completeness theorem[7] we have $v''(F(\underline{\varphi}/\underline{x}, \underline{p}) \to \chi) = \top$. This means that for all $\underline{c} \in \mathcal{C}$, we have $F(v'(\underline{\varphi}), \underline{c}) \leq v'(\chi)$. Therefore, $\mathcal{C} \models \forall \underline{y}\Big(F(v'(\underline{\varphi}), \underline{y}) \leq v'(\chi)\Big)$. Since $\mathcal{C} \models \Pi(\rho)$, we have $\mathcal{C} \models G(v'(\underline{\varphi})) \leq v'(\chi)$. Thus, as $G(v'(\underline{\varphi})) \leq v'(\chi)$ holds in $\mathcal{C}$, we have that $G(v(\underline{\varphi})) \leq v(\chi)$ holds in $\mathcal{B}$. Consequently, $v(G(\underline{\varphi}) \to \chi) = \top$. Applying the algebraic completeness theorem again yields that $\vdash_{\mathcal{S}} G(\underline{\varphi}) \to \chi$ because $\mathcal{B}$ is arbitrary, and hence ρ is admissible in $\mathcal{S}$. □

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[7] This is Theorem A.1 for $\Theta = \emptyset$.