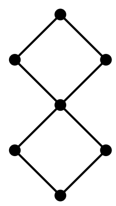
Mathematical structures in logic
Exercise class 5

Priestley and Esakia duality

March 6, 2018

1. Draw the poset dual to the Heyting algebra drawn below.



2. Let $(X, \tau, \leq)$ be a Priestley space. Show that
   (a) the set $\uparrow x$ is closed for each $x \in X$;
   (b) the sets $\uparrow F$ and $\downarrow F$ are closed for each closed subset F of (X, τ).

3. Let $\mathbf{D}$ be a bounded distributive lattice and let $X_D = (X, \tau, \leq)$ be its dual Priestley space. Show that for every clopen upset U of $X_{\mathbf{D}}$ there exists $a \in D$ such that $U = \varphi(a)$, where $\varphi(a)$ is the set of prime filters on $\mathbf{D}$ containing a (*Hint:* You will most likely have to use compactness twice, first for a cover of U^{c} and then for a cover of U.)

4. (a) Let $h : A \to B$ be a Boolean homomorphism between Boolean algebras A and B. Show that if h is surjective, then its dual $h_* = h^{-1} : X_B \to X_A$ is injective.
   (b) Let $f : X \to Y$ be a continuous map between Stone spaces X and Y. Show that if f is surjective, then its dual $f^* = f^{-1} : \mathsf{Clop}(Y) \to \mathsf{Clop}(X)$ is injective.

5. Let $(X, \tau, \leq)$ be a Priestley space. Let Y be a closed subset of X.
   (a) Show that $(Y, \tau_Y, \leq_Y)$ where τ_Y is the subspace topology[1] and $\leq_Y$ is the induced order is also a Priestley space.
   (b) If $(X, \tau, \leq)$ is an Esakia space and Y is a closed upset, then $(Y, \tau_Y, \leq_Y)$ is also an Esakia space.
   (c) (**Additional**.) Show that there exists an Esakia space $(X, \tau, \leq)$ and a closed non-upset $Y \subseteq X$ such that $(Y, \tau_Y, \leq_Y)$ is not an Esakia space. (*Hint:* consider the Alexandroff compactification of the natural numbers, find a suitable ordering on it so it becomes an Esakia space (there are several such orderings and not all of them will work), and find such a Y.)

[1] If (X, τ) is a topological space and Y is a subset of X, the *subspace topology* τ_Y is defined as $\tau_Y := \{U \cap Y : U \in \tau\}$.