NOTES ON ALGEBRAS OF TOPOLOGY

1. Three algebras coming from topology

1.1. Interior algebras.

Definition 1.1. *An* S4*-algebra (alternatively, a* closure algebra *or an* interior algebra*) is a pair* $(B, \Box)$ *where* B *is a Boolean algebra and* $\Box : B \to B$ *a modal operator such that for each* $a, b \in B$ *we have*

(1) $\Box 1 = 1$,
(2) $\Box(a \wedge b) = \Box a \wedge \Box b$,
(3) $\Box a \leq \Box\Box a$,
(4) $\Box a \leq a$.

If we let $\Diamond a = \neg\Box\neg a$, then the S4-axioms can be rewritten as:

(1) $\Diamond 0 = 0$,
(2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,
(3) $\Diamond\Diamond a \leq \Diamond a$,
(4) $a \leq \Diamond a$.

Given a topological space (X, τ) the algebra $(\mathcal{P}(X), \mathrm{Int})$, where $\mathcal{P}(X)$ is the powerset of X and Int the interior operator, is an S4-algebra.

Exercise 1.2. *Prove the above claim.*

1.2. Algebras of open and regular open sets.

We also know that $(\tau, \cap, \cup, \to, \emptyset)$ is a Heyting algebra, where for $U, V \in \tau$ we have

$$U \to V = \mathrm{Int}((X \setminus U) \cup V).$$

Recall that an open set $U \in \tau$ is called *regular open* if $\mathrm{Int}(\mathrm{Cl}(U)) = U$, where Cl is the closure operator. Let $\mathrm{RO}(X)$ denote the set of all regular open subsets of X. Then $(\mathrm{RO}(X), \cap, \dot\cup, \dot\neg, \emptyset, X)$ is a Boolean algebra, where for $U, V \in \mathrm{RO}(X)$ we have

$$U \dot\cup V = \mathrm{Int}(\mathrm{Cl}((U \cup V))$$

and

$$\dot\neg U = \mathrm{Int}(X \setminus U).$$

Exercise 1.3. *Verify the above claim.*

The above motivates the following definition.

Definition 1.4. *Let* A *be a Heyting algebra. An element* $a \in A$ *is called* regular *if* $a = \neg\neg a$*. Let* $Rg(A)$ *be the set of all regular elements of* A*.*

Exercise 1.5. *Show that* $(Rg(A), \wedge, \dot\vee, \neg, 0, 1)$ *forms a Boolean algebra, where for each* $a, b \in Rg(A)$ *we have*

$$a \dot\vee b = \neg\neg(a \vee b).$$

Exercise 1.6. *Show that the map* $\neg\neg : A \to Rg(A)$ *is a Heyting algebra homomorphism.*

2. Pre-orders and Alexandroff topologies

A *pre-ordered set* or a *pre order* is a set with a reflexive and transitive binary relation on it. Let $(X, \leq)$ be a pre order. A subset $U \subseteq X$ is called an *up-set* if $x \in U$ and $x \leq y$ imply $y \in U$. Given a pre order $(X, \leq)$ we can define a topology

$$\tau_{\leq} = \{U \subseteq X : U \text{ is an up-set}\}.$$

A topological space (X, τ) is called an *Alexandroff space* if τ is closed under arbitrary intersections.

Exercise 2.1. *Show that* $(X, \tau_{\leq})$ *is an Alexandorff space.*

Given a topological space (X, τ) define a relation $\leq_\tau$ on X by setting

$$x \leq_\tau y \text{ iff every open set containing } x \text{ also contains } y.$$

Exercise 2.2. *Show that*

(1) $x \leq_\tau y$ *iff* $x \in \mathrm{Cl}(y)$,
(2) $\leq_\tau$ *is reflexive and transitive,*
(3) $(X, \leq)$ *is isomorphic to* $(X, \leq_{\tau_{\leq}})$,
(4) *if* (X, τ) *is an Alexandroff space, then* (X, τ) *is homeomorphic to* $(X, \tau_{\leq_\tau})$.