Blackwell games

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Preface

In this paper I prove the determinacy of Blackwell games over a $G_{\delta\sigma}$ set. Blackwell games are infinite games of imperfect information, where both players simultaneously make their moves and then are informed of each other's move, and where payoff is determined by a Borel measurable function f on the set of possible resulting sequences of moves. Since the approach is very elementary, any reader with a basic knowledge of set theory should be able to grasp the definitions and proofs in this paper.

In Chapter 1, I informally introduce the concepts behind Blackwell Games, for those that are unfamiliar with Blackwell Games or Game Theory in general, and those that simply want to know what this paper is about.

In Chapters 2 and 3, I formally define Blackwell Games and other concepts, and prove several basic results that are used in the other chapters.

In Chapters 4 and 5, I give some new proofs of determinacy for Blackwell Games whose payoff function is the indicator function of an open or G_δ set. I show that open or G_δ games can be approximated with finite or open games. Finally, I prove that Blackwell Games over $G_{\delta\sigma}$ sets are determined as well.

The last chapter, Chapter 6, contains some odds and ends that I thought might be of interest to people, and that did not fit in elsewhere.

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Contents

Chapter 1

Introduction

This paper is about games.

Specifically, about Blackwell games. But Blackwell games are not normal games, like chess or poker. No, they are mathematical games. Abstract games. Games that no one could ever really play, because if someone were to play them, they would last for eternity.

This paper is about the 'determinacy of infinite two-person zero-sum games of incomplete information'.

In this chapter, I explain what that means. I start with simple, everyday games, which everyone can play, and which are *finite*(i.e. if you play them then you can predict, for some moment in time, that you will be finished then). Then I consider abstractions, and introduce the concept of determinacy for those games. Next, I examine 'infinite games', explain how they can be considered to be games if it is not possible to play them, and review some results that have been proved for those games. Then I jump back to finite games, but a different type of finite game this time, namely finite games of incomplete information (like Scissors-Paper-Stone, where one player does not know what the other player is doing *at that time*). And finally, I combine 'infinite' and 'incomplete information', and talk about Blackwell games, and about the specific results that I will prove in this paper.

But if you already know what 'determinacy of infinite two-person zero-sum games of incomplete information' means, skip this chapter. It's not for you.

1.1 Playing Games...

Consider a very simple type of game. Two players are playing against each other. One player (henceforth named player I) makes a move. Then the other player (henceforth named player II) makes his move. Then player I again makes a move. Then player II again makes a move. And they continue making moves, until they have played the number of rounds they agreed upon beforehand. Then they stop, and go to a giant Book, that contains every series of moves they could possibly have played during the game. And the players look up the series of moves they have actually played, and the Book tells them if its a draw, or if not, which one of them has won.

Sounds dull? Suppose that the moves the players can make, are writing crosses or circles in a 3×3 square, and that the Book lists as wins for player I or player II exactly those series of moves in which three crosses or three circles, respectively, are made on the same line. Then the game you're playing is actually a lot like Tic-Tac-Toe [1], isn't it? Or suppose that the moves the players can make, are coded chess moves on a chess board, and that the Book lists as wins for player I or II or draws, those series of 6350 moves in which black or white is mated or the game is drawn by the rules of chess, respectively. [2] Then you are really playing a kind of disguised, abstract chess game.

There are a number of games, for which you can write a Book of the type mentioned above, and then play a mathematical game that is really the same game in a different guise. Tic-Tac-Toe and Chess, for example. But also Connect-Four[3], or Checkers, or the Japanese game Go. And all these games have certain properties in common:

- Two players make moves, in turn
- There is no element of chance
- There is no element of physical skill; a player essentially selects a move, and then communicates it
- There is no hidden information; both players know all the moves made so far, and everything is on the table
- It is a zero-sum game: if one player loses, the other player wins
- There is a maximum number of rounds the game can last

There are also a lot of interesting games that do not fall in the class of games described above. A game like Monopoly contains an element of chance. Stratego contains hidden information. Poker contains both. In war, nobody wins, everybody loses. And eye-staring contests can go on forever. However, for now we will only deal with the class of games described above, the 'Finite Two-Person Zero-Sum Games of Perfect Information'. For each one of those games, we can write a Book, and then play the simple, abstract game I described above. So anything we prove about this 'Book-game' is true for all the other games in this class.

Anyone who's ever played Tic-Tac-Toe knows that it almost always ends in a draw. Figure 1.1 is a schematic of a good strategy for the first player in Tic-Tac-Toe. From every Tic-Tac-Toe-position in Figure 1.1, the first player can either

[1] also called Noughts-and-Crosses, or Boter-Kaas-en-Eieren in the Netherlands

[2] If you play using the rule that a game is drawn if 50 turns have passed without a piece having been captured or a pawn having moved, a game can last at most 6350 turns.

[3] called Vier-op-een-Rij in the Netherlands

draw or win. If from a position there is an arrow to another position, then one of the players can make a move to get to that other position. If it's the first player's turn to move, then there is one thick arrow, corresponding to a good move for the first player. If it's the second player's turn to move, then there are thin arrows for every move he can make, except those that are really stupid (i.e. lose immediately and unnecessarily), and those that are reflections or rotations of other moves. As a child, I made such a table, and if I played according to my table, I couldn't lose. There are similar tables for the second player, and if my opponent had such table, written down or in his mind, he couldn't lose either. So the game usually was a draw, and after a while we didn't play it anymore. The game was 'analyzed and found to be a draw'.

Something that is not as well-known, is the fact that the game Connect-Four was analyzed a few years ago [1], and was found to be a first-player win. That is to say, computer analysis found a strategy, a table like the one I made but much bigger and more complex, such that if a player plays according to that strategy, and that player has the first move, she can win. Fortunately, few people know this, and I read that the strategy was complex enough that you needed a computer to play the strategy correctly. But, mathematically speaking, the game is a win for the first player, and if the second player manages to draw or win, it is due to human fallibility.

A game like Chess or Go is a lot more complex than Connect-Four. If we tried to analyze it using the techniques and hardware now available, it would probably take more time than the universe has left to complete the calculations. However, in principle it is possible to analyze it. And then we would find either that White or Black has a winning strategy, or that both players have a drawing strategy [19].

In fact it is mathematically provable, that for any game that is in our class of games, either one of the players has a winning strategy, or both players have a drawing strategy [4]. We call this: 'the game is *determined*'. The reasoning is as follows:
Suppose that it is player I's turn to move, and suppose we already know that for every position that results after one move, the game from that position is determined. Then there are three possibilities:

- Player I can make a move to get to a position from which she can win. Then player I can win from this position by making that move.
- Player I can't make a move to get to a position from which she can win, but she can make a move to get to a position from which she can draw. Then player I can at least draw from this position by making that move, and player II can draw or win from this position by waiting until player I has made her move, and then playing a drawing or winning strategy from the position he is in now (which exists, since player I cannot win from that position).
- Player I can't make a move to get to a position from which she can win or draw.

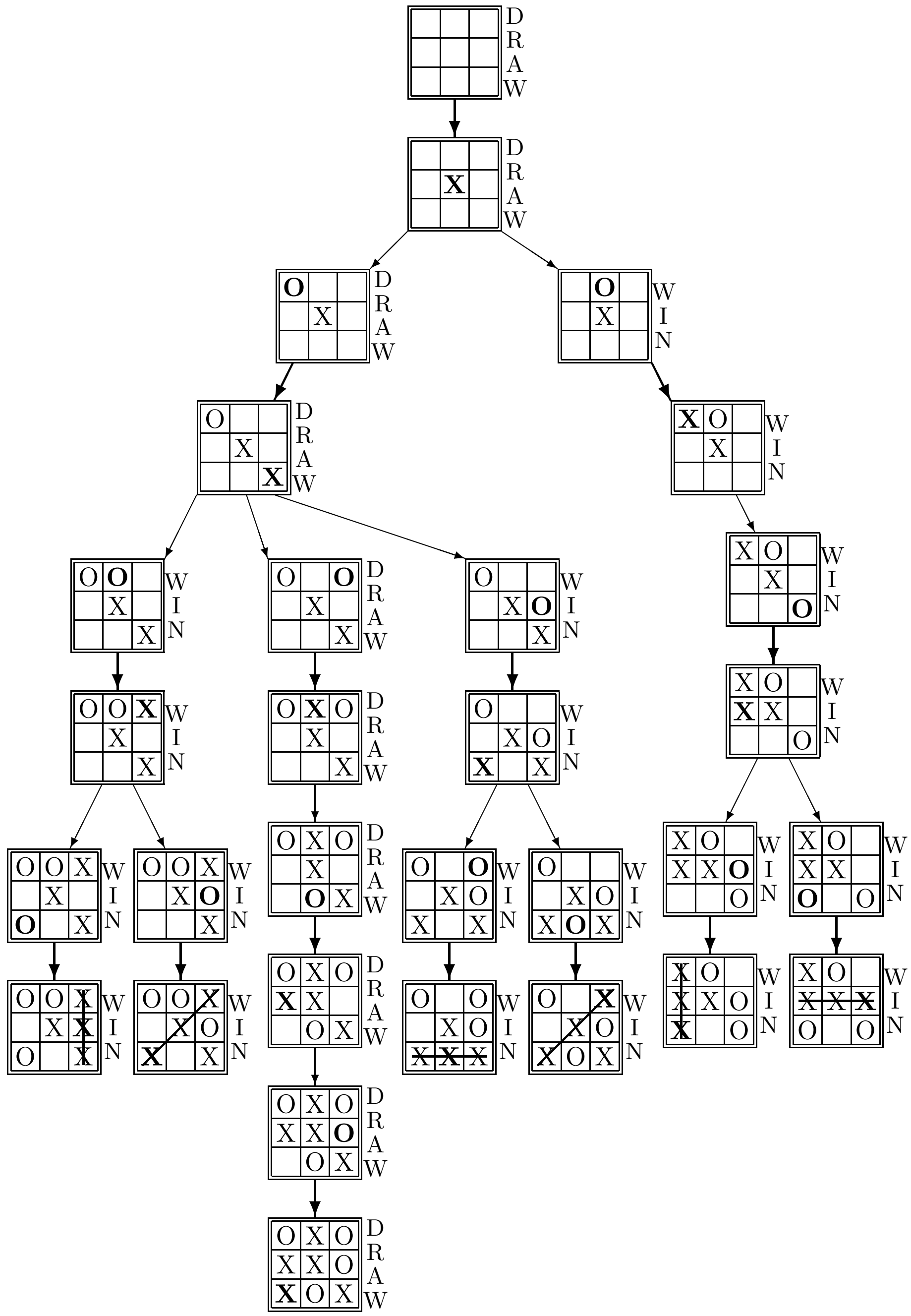

Figure 1.1: A reasonably good strategy for Tic-Tac-Toe

Then player II can win from this position, by waiting until player I has made her move, and then playing a winning strategy from the position he is now in (which exists, since player I cannot win or draw from that position).

Figure 1.1 is an example of how to use this method to find the values (win, draw or lose) of positions in Tic-Tac-Toe, and good moves to make in those positions.

There is, however, a snag. In order to prove that the game is determined if you start from the starting position, I use the supposition that the game is determined from any position where you have already made one move. This seems to be circular reasoning. But it is not circular... quite. Remember, the games we were talking about all lasted a maximum number of rounds. And the game we get after you make one move, can last one round less from that position. So, in order to prove determinacy of all games that last at most, say, 100 rounds, we need determinacy of all games that last at most 99 rounds. And in order to prove determinacy of all games that last at most 99 rounds, we need determinacy of all games that last at most 98 rounds. Etcetera, etcetera, all the way to: and in order to prove determinacy of all games that last at most 1 round, we need determinacy of all games that last at most 0 rounds. But what is a game that last at most 0 rounds? That is a game, in which one of the players has already won, or a game that is already tied. The positions at the bottom of Figure 1.1 are examples of such 'games'. Such a game is trivially determined. So all the games that last at most 1 round are also determined, and so are the games that last at most 2 rounds , 3 rounds, ..., 100 rounds. And as in Figure 1.1, we can calculate our values 'from the ground up'.

This is called a 'proof by induction'. And we can use it to prove that every game is determined if there is a maximum number of rounds the game can last, no matter how big that maximum number of rounds is. Game Theory usually deals only with such finite games. But Logicians like to play with infinities. So what happens if there is no such number? What if it were possible for the game to go on forever?

1.2 ... unto Infinity

What do we mean by an infinite game? Well, take for example Chess with the 50-turn rule taken out, as well as the rule that it's a draw if you make the same move in the same position three times. Then a game can potentially go on forever. We can say that, if the game lasts an infinite number of rounds, then the result is a draw. Let's call this version 'Infinite Chess'. In the abstract version, the Book of results now becomes an infinitely big Book that lists which player wins, loses or draws, for every possible sequences of moves that can be played, including sequences that are infinitely long. But how can you play such an infinite game?

Suppose that player I plays according to a strategy. Remember, we can visualize a strategy as a giant table, that has an entry for every position that can occur, and in which player I can look up the move she should make in that position (if she wants to keep to her strategy). Now visualize a strategy for player I, and another strategy for player II. Then we can construct the sequence of moves that they are going to play, look it up in the Book of Game Results, and see whether it is a win for player I, a win for player II, or a draw. This can be done even in the case of infinite games. So we can still play the game, but now by having the players pick the strategies they are going to use beforehand, combining them, and looking up the result.

Back to Chess. Let us suppose, for the sake of argument, that White can win in a normal game of Chess. Then, by using the same strategy, White can win in a game of Infinite Chess. That is, no matter what strategy Black uses, the resulting sequence is a win for White. So we can use terms like winning strategy, losing strategy, and determinacy, just as with finite games.

However, our proof using induction is only valid for finite games. If the game can last infinitely many rounds, and we make a single move, then the game can still last infinitely many rounds more, and the reasoning we used before truly is circular now. It is imaginable, that for every strategy that one player can come up with, the other player has a strategy that it, and vice versa. Then both players would only have losing strategies, and the game would not be determined. So 'are infinite games determined?' is a valid question. Unfortunately, the answer is not a simple 'yes' or 'no'.

1.3 Borel Wins!

Consider, again, the game of Chess, in the abstract Book version. Now consider the Book of Infinite Chess II, that has as entries sequences of infinitely many chess moves, and for each infinite sequence lists whatever the old Book of Chess lists for the subsequence consisting of the first 6350 moves, i.e. the Book of Infinite Chess II lists that a given sequence of infinitely many moves is a win for player I if and only if the old, finite Book of Chess lists that the sequence of the first 6350 moves is a win for player I, which is only if, interpreting the moves as ordinary chess moves, white wins somewhere in the first 6350 turns. Technically this Book is the Book of an infinite game. But all the moves played after round 6350 don't matter. And since Chess is determined, so is this game.

Of course, this game is infinite only in the technical sense of the word: the 'idea' behind this game is finite. But consider, for example, an infinite game whose Book was created using the following method:

1. We gather together 'countably infinitely' many different Books of infinite games, I.e.it is possible to number the Books, in such a way that each Book has a different (whole) number. There exist collections of Books that are so huge that it is impossible to number them like that, because there are 'more' Books than there are whole numbers. Such collections are called 'uncountably infinite', and we do *not* use them here.

2. We write a new Book. To simplify things, assume that all the old Books are without draws, i.e. all the sequences of moves are either wins for player

> I or wins for player II. Then, for each possible sequence of moves, we write in the new Book that the sequence is a win for player I, if and only if it is a win for player I in *at least one* of the old Books. Otherwise we write that it is a win for player II.

If the Books we start with are all of games that are essentially finite, i.e. infinite only in a technical sense, then the resulting Book is of a type that is called 'open'. Generally, open games are really infinite, in more than the techical sense of the word. However, it is *still* possible to prove that this game is determined [6], although it is slightly more difficult than in the case of essentially finite Books.

The method for creating new Books, described above, is called 'taking the countable union'. There is also another method, called 'taking the countable intersection', where a sequence of moves is a win for player I in the new Book if and only if it is a win for player I in *all* of the original Books.

We can start with other Books than those of essentially finite games. If we start with Books that are all of open games, and take the countable intersection, we get a Book of a type that is called 'G_δ'. If we start with Books of G_δ-games, and take the countable union, we get a Book of a type that is called '$G_{\delta\sigma}$'. If we start with Books of $G_{\delta\sigma}$-games, and take the countable intersection again, we get a Book of a type that is called '$G_{\delta\sigma\delta}$'. Et cetera, et cetera, ad infinitum. All the Books that can be obtained using countable union and countable intersection, including the finite Books we began with, are collectively known as 'Borel' Books. And for all 'Borel' Books, the games played with those Books are determined. This was first proven for G_δ games [18], then for $G_{\delta\sigma}$ games [5] and $G_{\delta\sigma\delta}$ games [16], and finally for all Borel games [7] [10] [11].

So all infinite games that are Borel are determined. Non-Borel games *may* be determined, but we will never be able to prove it. If we assume the *Axiom of Choice* (a mathematical assumption that cannot be proven or disproved, and that most mathematicians treat as a given), then we can construct a game that is not determined [13]. On the other hand, if we do not assume the Axiom of Choice, then we can assume instead that all games of this type are determined (the *Axiom of Determinacy*), without fear that we will run into an inconsistency[4] [12] [14]. In short, without assuming something else we cannot prove anything one way or the other.

1.4 Scissors, Paper, Stone

We will now consider another game, namely the game of 'Scissors, Paper, Stone'. This game is played by two persons. The players count to three together, and on three, they both put their hand forward, balled in a fist ('Stone'), flat with the palm down ('Paper'), or with middle- and forefinger pointing forwards, spread, and the thumb and other fingers curled inwards ('Scissors'). If both players throw the same, it is a draw. If one player throws Paper and the other player throws Scissors, then Scissors wins ('Scissors cut Paper'). If one player throws

[4]assuming the existence of a certain very large cardinal

Scissors and the other player throws Stone, then Stone wins ('Stone blunts Scissors'). And finally, if one player throws Paper and the other player throws Stone, then Paper wins ('Paper wraps Stone').

This game is not of the type we were talking about before. The players do not make a move in turn, but simultaneously. Or to put it another way, neither of the players know what move the other is playing while they are making theirs. This is an example of a game with 'Imperfect Information'. But what, exactly, are the consequences of this difference?

Well, for one thing, strategies are no longer simple instructions of the type 'in this position, make that move'. For suppose that one player uses the strategy 'throw Stone'. Then the other player simply uses the strategy 'throw Paper', and wins. But that strategy, even though it wins here, is in general not a particularly good strategy, since it loses from the strategy 'throw Scissors'. All strategies of the type 'throw *this* ' are losing strategies.

But consider the strategy 'throw Scissors 1/3 of the time, throw Paper 1/3 of the time, and throw Stone the remaining 1/3 of the time'. Against any other strategy, this strategy loses, draws and wins 1/3 of the time each, for an average result of 0. Clearly, this strategy is better than a losing strategy. What is more, there exists no better strategy (in terms of worst-case behavior), since against this same strategy played by the opponent any strategy will result in (on average) a draw, and therefore no strategy can have value greater than 0.

This kind of strategy is called a 'mixed strategy'. We can visualize it as a giant table which lists, for every position in the game, and every move that can be made in that position, what the chance should be that the player makes that move. If we play this strategy against an opposing strategy, and we assign values to winning and losing ('the loser pays the winner one dollar'), then we can calculate the average profit/loss one player can expect to make from the other, playing those strategies. This means that strategies are no longer winning, losing or drawing: they now have a *value*, the profit/loss they can expect to make (on average) against the best counterstrategy of the opponent. And a game is called determined if, for some value v, one of the players has a strategy with which she can always expect to make (on average) at least v\$, no matter what the other plays, while the other player has a strategy with which he can always expect to lose (on average) at most v\$, no matter what the other plays.

One more example to show how a mixed strategy works. Suppose that player I has, in her hand, not visible to player II, either a nickel or a quarter. Player II tries to guess which coin player I has in her hand. If he guesses right, then he gets the coin. If he guesses wrong, then he owes player I the average, 15 cents. This is a game of the same type as Scissors-Paper-Stone, since for all practical purposes it does not matter whether player I picks her coin before, or at the same time that player II makes his guess. The payoff of this game, the amount player II can win, is:

		Player II guesses	
		a nickel	a quarter
Player I has	A nickel	5 c	-15 c
	A quarter	-15 c	25 c

A good strategy for player I is to put the nickel in her hand two-thirds of the time, and the quarter one-third of the time. Then, no matter what guess her opponent makes, she will win, on average, $1\frac{2}{3}$ cent. Coincidentally, a good strategy for player II is to guess the nickel two-thirds of the time, and the quarter one-third of the time, because then, no matter what player I has in her hand, at least he will lose no more than, on average, $1\frac{2}{3}$ cent. Of course, in this case the best strategy for player II is not to play at all, but that's another matter entirely.

1.5 Anyone for Blackwell Games?

Consider the abstract version of this game. Both players, simultaneously, make a move. They each take note of the other's move. Then again they make a move. Then again they make a move. And they continue making moves, until they have played a number of rounds they agreed upon beforehand. Then they stop, go to a giant Book, look up the sequence of moves they have played, and one player pays the other the indicated amount.

It is known that each 1-round game of this type is determined, as long as the number of different *possible moves* each player can select in that round is finite [15]. If the number of possible moves is not finite, then we can construct a game that is not determined, so we will only bother with the case of finitely many possible moves. And using a 'proof by induction' (again), we can prove that any game of this type is determined, as long as there is a maximum number of rounds the game can last.

Note that the first restriction, that the number of different possible moves each player can select is finite, is necessary. Take, for instance, the game where two persons simultaneously yell a number, and the one with the higher number wins. Then, no matter what one player's strategy, whether it is picking a number, or some kind of 'probability distribution' on the numbers, his opponent can devise a counterstrategy that wins from this strategy 99% of the time, or more often if necessary. This is true for both players. So any strategy in this game, for any player, is a losing strategy, and this game is not determined.

But what about the second restriction? Again we ask ourselves, what happens if we allow games to last forever? Then we get Blackwell Games, named after David Blackwell, the first one to describe and study these games [2]. 'Infinite Games of Perfect Information' usually fall under Mathematical Logic. 'Finite Games of Imperfect Information' usually fall under Game Theory. Blackwell Games are 'Infinite Games of Imperfect Information', and as such lie at the crossing of Mathematical Logic and Game Theory.

As before, playing such a game is a matter of picking strategies (independently) and calculating the (average expected) result of the combination. Also as before, the Books involved can be essentially finite sets, open sets, G_δ sets, $G_{\delta\sigma}$ sets, Borel sets, or something completely different. And in the case of an open or G_δ set, determinacy has been proven [2] [3]. But in the case of general

Borel sets, the problem is still open.

In this paper, I try to give an overview of what is known so far about Blackwell Games. But more importantly, I move to the next step of the hierarchy of Borel sets, i.e. I prove:

Theorem *Let S be a $G_{\delta\sigma}$ set. Then the Blackwell game $\Gamma(S)$ is determined.*

And *that*, finally, is what this paper is about.

Chapter 2

Definitions and Terminology

In this chapter we define the concepts we use. Most of the definitions in the first and second section are fairly standard, and similar to the usual game theoretic definitions for games of perfect information (except that strategies give probability distributions on moves instead of choices of moves). The definitions in the third section formalize the concept of 'stopping and paying the current value of the game'.

2.1 Blackwell Games

Let X and Y be two finite, nonempty sets, and put $Z = X \times Y$.

Definition 2.1 A *position* or *finite play* (*of length* k) is a finite sequence p (of length k) of pairs $(x, y) \in Z$. An *(infinite) play* is a countably infinite sequence w of pairs $(x, y) \in Z$. A *move* made in a play w or p is an element of the sequence w, p respectively.

Notation e denotes the position of length 0, i.e. the empty sequence.
w usually denotes an infinite play, p denotes a finite play or position.
$p_{|n}$, $w_{|n}$ denote the sequences consisting of the first n moves made in p, w respectively.
$p^\frown p'$, $p^\frown w$ denote the sequences consisting of the finite sequence p followed by the finite sequence p' or the infinite sequence w, respectively.
We will sometimes write a sequence $((x_1, y_1), (x_2, y_2), \ldots)$ as $(x_1, y_1, x_2, y_2, \ldots)$.

Notation W denotes the set of all plays, i.e. $W = Z^{\mathbb{N}}$
P denotes the set of all positions, i.e. $P = Z^{<\omega}$.
W_n denotes the set of all finite plays of length n, i.e. $W_n = Z^n$, for $n \in \mathbb{N}$.

We give W the usual topology by letting the basic open sets be the sets of the form $\{w \in W \mid w_{|n} \in H\}$ for some $n \in \mathbb{N}$ and some set $H \subseteq W_n$ of finite plays of length n.

Definition 2.2 A *basic open* subset of W is a set $B \subseteq W$ such that for some $n \in \mathbb{N}$, for all $w \in B$, for all $w' \in W$, if $w_{|n} = w'_{|n}$, then $w' \in B$.
An *open* subset of W is a countable union of basic open subsets of W.
A *closed* subset of W is a countable intersection of basic open subsets of W.
A F_σ subset of W is a countable union of closed subsets of W.
A G_δ subset of W is a countable intersection of open subsets of W.
A $G_{\delta\sigma}$ subset of W is a countable union of G_δ subsets of W.
A $F_{\sigma\delta}$ subset of W is a countable intersection of F_σ subsets of W.
A *Borel* subset of W is a subset that belongs to every collection of subsets of W that contains every basic open subset of W and is closed under countable union and countable intersection.

Remark 2.3 This topology on W is equivalent to the product topology on $Z^{\mathbb{N}}$ (taking the discrete topology on Z). Since Z is finite, it is a compact space, and it follows by Tychonoff's Theorem that W is a compact space as well.

Definition 2.4 A play w *hits* or *passes through* a position p if $w_{|n} = p$, where n is the length of p. A position p' *follows* p, and p *precedes* p', if $p'_{|n} = p$ and $p' \neq p$. Consequently, p precedes or is equal to p' if $p'_{|n} = p$. We denote this by $p \subset w$, $p \subset p'$, and $p \subseteq p'$, respectively.

Remark 2.5 The basic open subsets of W are exactly those of the form $\{w \in W \mid w \text{ hits a position in } H\}$ for some finite set H of positions. The open subsets of W are exactly those of the form $\{w \in W \mid w \text{ hits a position in } H\}$ for some countable set H of positions. The G_δ subsets of W are exactly those of the form $\{w \in W \mid \#\{p \in H \mid w \text{ hits } p\} = \infty\}$ for some countable set H of positions.

Definition 2.6 Let $f : W \to \mathbb{R}$ be a bounded Borel (measurable) function (i.e. a bounded function such that $f^{-1}[O]$ is a Borel set for every open set $O \subseteq \mathbb{R}$). The *Blackwell game* $\Gamma(f)$ with *payoff function* f is the two-person zero-sum infinite game of imperfect information played as follows: Player I chooses an element $x_1 \in X$ (*makes the move* x_1) and, simultaneously, player II chooses an element $y_1 \in Y$. Then both players are told $z_1 = (x_1, y_1)$, and the game *is at* or *has reached* position (z_1). Then player I chooses $x_2 \in X$ and, simultaneously, player II chooses $y_2 \in Y$. Then both players are told $z_2 = (x_2, y_2)$, and the game is at position (z_1, z_2). Then both players simultaneously choose $x_3 \in X$ and $y_3 \in Y$, etc. Thus they produce a play $w = (z_1, z_2, \ldots)$. Then player II pays player I the amount $f(w)$, ending the game.

Definition 2.7 Let $f : W \to \mathbb{R}$ be a bounded Borel function, and $p = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n))$ a position. The *subgame* $\Gamma(f, p)$ *starting from position* p is played like $\Gamma(f)$, except that the players start at round $n + 1$, and the first n moves are supposed to have been $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$.

Remark 2.8 The game $\Gamma(f, p)$ is played exactly the same as the game $\Gamma(g)$, where g is the payoff function defined by $g(w) = f(p^\frown w)$.

Notation If S is a Borel subset of W, then $\Gamma(S)$ stands for the game $\Gamma(I_S)$, where I_S is the indicator function of S, defined by $I_S(w) := 1$ if $w \in H$, $I_S(w) := 0$ if $w \notin H$.

2.2 Strategies and Values

Definition 2.9 A *strategy* for player I in a Blackwell game $\Gamma(f)$ is a function σ assigning to each position p a probability distribution on X. More formally, σ is a function $P \to [0,1]^X$ satisfying $\forall p \in P : \sum_{x\in X} \sigma(p)(x) = 1$.
Analogously, a *strategy* for player II is a function τ assigning to each position p a probability distribution on Y.
Player I *plays according to* σ if, in any position p, the chance that player I will play $x \in X$ is equal to $\sigma(p)(x)$. Similarly, player II *plays according to* τ if, in any position p, the chance that player II will play $y \in Y$ is equal to $\tau(p)(x)$.

Remark 2.10 If player I plays according to a strategy σ, and player II plays according to a strategy τ, then we can calculate the payoff player I can expect to win on average. For a finite game, we could simply calculate, for each possible sequence of moves, the probability that that sequence is actually played, multiply it by the associated payoff, and then add together the results for all possible sequences. For Blackwell games, which are usually infinite, we use probability measures and integrals of the payoff function instead. The requirement in Definition 2.6 that payoff functions are bounded and Borel measurable, is to insure that these integrals exist and are finite. In Chapter 6 we will look at a way of extending the definitions to games with non-measurable (but still bounded) payoff functions.

Definition 2.11 Let σ and τ be strategies for players I, II in a Blackwell game $\Gamma(f)$. σ and τ determine a probability measure $\mu_{\sigma,\tau}$ on W, induced by

$$\mu_{\sigma,\tau}\{w \mid w \supset p\} = \prod_{i=1}^{n} \big(\sigma(p_{|(i-1)})(x_i) \bullet \tau(p_{|(i-1)})(y_i)\big) \tag{2.1}$$

for any position $p = (x_1, y_1, \ldots, x_n, y_n) \in P$.
The *expected income* of player I, if she plays according to σ and player II plays according to τ, is the expectation of $f(w)$ under this probability distribution:

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \int f(w) d\mu_{\sigma,\tau}(w) \tag{2.2}$$

Remark 2.12 Similarly, strategies σ and τ in a subgame $\Gamma(f, p)$ starting at position p define a conditional probability measure on the set of all sequences that hit p, i.e. such that $\mu\{w \mid w \supset p\} = 1$. The expected income of player I in

$\Gamma(f, p)$ is the expectation of $f(w)$ under this conditional probability distribution. By the strict definition of strategies, a strategy in the subgame has to be defined on all positions, including those positions that are outside the subgame (those positions that are neither equal to p nor following p). Since the probability distributions given by the strategies at positions outside the subgame do not affect the expectation of the outcome, we will only require strategies to be defined on positions in the subgame. In fact we occasionally assume that a strategy is *not* defined on positions outside the subgame, for instance when we combine strategies for subgames starting from different positions.

Definition 2.13 Let $\Gamma(f)$ be a Blackwell game, let p be a position in $\Gamma(f)$, and let σ be a strategy for player I in $\Gamma(f)$. The *restriction* of σ to the positions of $\Gamma(f, p)$ is the strategy σ', defined only on positions q at or after p, and satisfying $\sigma'(q) = \sigma(q)$ on those positions. In this case, σ is called an *extension* of σ'.

Definition 2.14 Let $\Gamma(f)$ be a Blackwell game. The *value* of a strategy σ for player I in $\Gamma(f)$ is the expected income player I can guarantee if she plays according to σ, i.e.

$$\operatorname{val}(\sigma \text{ in } \Gamma(f)) = \inf_{\tau} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{2.3}$$

Similarly, the *value* of a strategy τ for player I in $\Gamma(f)$ is the amount to which player II can restrict player I's income if he plays according to τ, i.e.

$$\operatorname{val}(\tau \text{ in } \Gamma(f)) = \sup_{\sigma} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{2.4}$$

Definition 2.15 Let $\Gamma(f)$ be a Blackwell game. The *lower value* of $\Gamma(f)$ is the smallest upper bound on the income that player I can guarantee, i.e.

$$\operatorname{val}^{\downarrow}(\Gamma(f)) = \sup_{\sigma} \inf_{\tau} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{2.5}$$

Similarly, the *upper value* of $\Gamma(f)$ is the largest lower bound on the restrictions player II can put on player I's income, i.e.

$$\operatorname{val}^{\uparrow}(\Gamma(f)) = \inf_{\tau} \sup_{\sigma} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{2.6}$$

If $\operatorname{val}^{\uparrow}(\Gamma(f)) = \operatorname{val}^{\downarrow}(\Gamma(f))$, then $\Gamma(f)$ is called *determined*, and we will denote $\operatorname{val}(\Gamma(f)) = \operatorname{val}^{\uparrow}(\Gamma(f)) = \operatorname{val}^{\downarrow}(\Gamma(f))$.

Remark 2.16 For any $\epsilon > 0$, player I has a strategy σ such that for any strategy τ for player II, $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \geq \operatorname{val}^{\downarrow}(\Gamma(f)) - \epsilon$. On the other hand, for any $\epsilon > 0$, for any strategy σ for player I, player II has a strategy τ such that $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \leq \operatorname{val}^{\downarrow}(\Gamma(f)) + \epsilon$. The lower value of $\Gamma(f)$ is the unique number with both these properties, and can be thought of as the value of the game $\Gamma(f)$ for player I.
Analogously for the upper value. Clearly, for all games $\Gamma(f)$, $\operatorname{val}^{\downarrow}(\Gamma(f)) \leq \operatorname{val}^{\uparrow}(\Gamma(f))$.

Definition 2.17 Let $\Gamma(f)$ be a Blackwell game, and let $\epsilon > 0$.
A strategy σ for player I in $\Gamma(f)$ is *optimal* if $\mathrm{val}(\sigma \text{ in } \Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma(f))$. A strategy σ for player I in $\Gamma(f)$ is ϵ-*optimal* if $\mathrm{val}(\sigma \text{ in } \Gamma(f)) > \mathrm{val}^{\downarrow}(\Gamma(f)) - \epsilon$. Similarly, a strategy τ for player II in $\Gamma(f)$ is *optimal* if $\mathrm{val}(\tau \text{ in } \Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma(f))$, and ϵ-*optimal* if $\mathrm{val}(\tau \text{ in } \Gamma(f)) < \mathrm{val}^{\uparrow}(\Gamma(f)) + \epsilon$.

Remark 2.18 Note that a strategy σ for player I is ϵ-optimal if there exists a value $u > v - \epsilon$ such that for all τ, $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \geq u$. This is strictly stronger than the condition that for all τ, $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) > v - \epsilon$.

2.3 Stopping and Truncating

Definition 2.19 A *stopping position* in a Blackwell game $\Gamma(f)$ is a position p, such that for all plays w, w' that hit p, $f(w) = f(w')$. We will denote this value by $f(p)$. A *stopset* in a Blackwell game $\Gamma(f)$ is a set H of stopping positions, such that no stopping position $p \in H$ precedes another stopping position $p' \in H$.

Remark 2.20 Any position after a stopping position is, of course, itself a stopping position. Usually we don't bother with these positions. This is reflected in our definition of stopsets.

Remark 2.21 If p is a stopping position, any moves made at or after p will not affect the outcome of the game. It is often convenient to assume that both players will stop playing if a stopping position is reached. If $\Gamma(f)$ is a Blackwell game, and H is a stopset, we will write $\Gamma_H(f)$ to explicitly denote that players stop playing at the positions in H. The probability distributions given by strategies at stopping positions do not affect the outcome of the game, and similarly to Remark 2.12, we will only require a strategy to be defined on nonstopping positions. In fact we will occasionally assume that a strategy is *not* defined on stopping positions, for instance when we combine strategies that are consistent on all nonstopping positions.

Definition 2.22 Let $\Gamma(f)$ be a Blackwell game. If, for some n, all positions in W_n are stopping positions, then $\Gamma(f)$ is called *finite (of length n)*. If $\Gamma(f)$ is finite, we can stop after playing n rounds, and we will denote this by writing $\Gamma_n(f)$.

Remark 2.23 We will often define a payoff function f using the following format:

$$\begin{aligned} f(p) &= \text{formula1 for } p \in H \\ f(w) &= \text{formula2 if } w \text{ does not hit any position in } H \end{aligned}$$

where H is a stopset-to-be, i.e. a set of positions such that no position $p \in H$ precedes another position $p' \in H$.

Definition 2.24 Let f, g be two payoff functions, and H a stopset in $\Gamma(g)$. $\Gamma_H(g)$ is an *equivalent truncated subgame* of $\Gamma(f)$ (*truncated at* H), if for any play $w \in W$ that does not hit a stopping position in H, $f(w) = g(w)$, and for any $p \in H$, $g(p) = \text{val}(\Gamma(f, p))$.
$\Gamma_H(g)$ is a *truncated subgame, equivalent for player I [player II]*, if for any play $w \in W$ that does not hit a stopping position in H, $f(w) = g(w)$, and for any $p \in H$, $g(p) = \text{val}^{\downarrow}(\Gamma(f, p))$ $[g(p) = \text{val}^{\uparrow}(\Gamma(f, p))]$. In all three cases, $\Gamma(f)$ is called an *extension* of $\Gamma_H(g)$.

Remark 2.25 $\Gamma_H(g)$ is an equivalent truncated subgame of $\Gamma(f)$ iff it is a truncated subgame equivalent for both player I and player II.

Remark 2.26 If $\Gamma_H(g)$ is a truncated subgame of $\Gamma(f)$ equivalent for player I and/or II, then any stopping position p in $\Gamma(f)$, is a stopping position in $\Gamma_H(g)$. For if p is preceded by a position in H, then p is preceded by a stopping position and hence is a stopping position itself, and if p is not preceded by a position in H, then we observe that any play $w \supset p$ that does not hit any position in H has payoff $f(p)$, and any game starting at any position $q \in H, q \supseteq p$ has value $f(p)$ as well.

Definition 2.27 Let, for $n \in I\!N$, f_n be a payoff function, and H_n a set of stopping positions in $\Gamma(f_n)$. If for all $n \in I\!N$, $\Gamma_{H_n}(f_n)$ is a truncated subgame of $\Gamma_{H_{n+1}}(f_{n+1})$, and equivalent to $\Gamma_{H_{n+1}}(f_{n+1})$ [for player I, II], , then the series of games $(\Gamma_{H_n}(f_n))_{n \in I\!N}$ is called a *nested series of equivalent truncated subgames* [equivalent for player I, II].

Definition 2.28 Let $\Gamma(f)$ be a Blackwell game, let $\Gamma_H(g)$ be a truncated subgame of $\Gamma(f)$ (truncated at H, and equivalent for player I or II), and let σ be a strategy for player I in $\Gamma(f)$. The *restriction* of σ to the positions of $\Gamma_H(g)$ is the strategy σ', defined only on those positions q that are not at or after any position $p \in H$, and satisfying $\sigma'(q) = \sigma(q)$ on those positions. In this case σ is called an *extension* of σ'.

2.4 Notational Conventions

- I, II: the two players
- X, Y: the two sets out of which I and II choose their moves
- Z: $X \times Y$
- w, w': plays
- W: the set of all plays
- W_n: the set of all finite plays of length n
- S: a subset of W

- O: an open subset of W
- D: a G_δ subset of W
- p, p': finite plays or positions
- e: the starting position
- P: the set of all positions
- H, H': sets of positions, usually stopsets or stopsets-to-be
- $w \supset p$: the play w hits position p
- $p' \supset p$: the position p' follows p
- $p' \supseteq p$: the position p' follows or is equal to p
- $p^\frown p', p^\frown w$: the plays obtained by concatenating sequences p and p' or w
- $p_{|n}, w_{|n}$, the plays consisting of the first n moves in p or w

- f, g, h: payoff functions
- $f(w)$: the payoff player I gets from player II for the play w
- $f(p)$: if p is a stopping position, the payoff player I gets from player II for any play w that passes through p
- $\Gamma(f)$: the game with payoff function f
- $\Gamma_H(f)$: the game with payoff function f and stopset H
- $\Gamma_n(f)$: the finite game with payoff function f and stopset W_n
- $\Gamma(S)$: the game with payoff function I_S
- $\Gamma(f, p)$: the game with payoff function f, starting from position p
- σ: a strategy for player I
- τ: a strategy for player II
- $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))$: the expected payoff when playing σ against τ in the game $\Gamma(f)$
- $\mathrm{val}(\sigma \text{ in } \Gamma(f))$: the value of σ in the game $\Gamma(f)$
- $\mathrm{val}(\Gamma(f))$: the value of the game $\Gamma(f)$
- $\mathrm{val}^{\downarrow}(\Gamma(f))$: the lower value of the game $\Gamma(f)$
- $\mathrm{val}^{\uparrow}(\Gamma(f))$: the upper value of the game $\Gamma(f)$
- u, v: values

Chapter 3

The Tools We Use

In this chapter we lay the groundwork for the results of the next chapters. Most of the lemma's and theorems in this chapter are intuitively fairly obvious. The general idea is be to approximate complex games with simpler games. In the first section, we prove lemma's that deal with expectations, and approximating complex payoff functions with simpler payoff functions. In the second section, we prove determinacy of the simplest games, namely finite games. In the third section, we show how to construct strategies and values of complex games from strategies and values of truncated subgames.

3.1 Payoff Time

We start with some basic notions, such as comparing a game with another game whose payoff function is everywhere higher, or adding a constant amount to the payoff.

Lemma 3.1 *Let f, g be two payoff functions such that for all $w \in W$, $f(w) \leq g(w)$. Then* $\mathrm{val}^{\downarrow}(\Gamma(f)) \leq \mathrm{val}^{\downarrow}(\Gamma(g))$ *and* $\mathrm{val}^{\uparrow}(\Gamma(f)) \leq \mathrm{val}^{\uparrow}(\Gamma(g))$.

Proof:

$$
\begin{array}{lll}
 & \forall w \in W: & f(w) \leq g(w) \qquad (3.1)\\
\Rightarrow & \forall \mu: & \int_{w\in W} f(w)d\mu(w) \leq \int_{w \in W} g(w)d\mu(w) \qquad (3.2)\\
\Rightarrow & \forall \sigma, \tau: & E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \leq E(\sigma \text{ vs } \tau \text{ in } \Gamma(g)) \qquad (3.3)\\
\Rightarrow & \forall \sigma: & \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \leq \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(g)) \qquad (3.4)\\
\Rightarrow & & \sup_\sigma \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \leq \sup_\sigma \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(g)) \qquad (3.5)\\
\Rightarrow & & \mathrm{val}^{\downarrow}(\Gamma(f)) \leq \mathrm{val}^{\downarrow}(\Gamma(g)) \qquad (3.6)
\end{array}
$$

Similarly for the upper value.

□

Lemma 3.2 *Let f be a payoff function, and let $c \in I\!R$. Then* $\mathrm{val}^{\downarrow}(\Gamma(f+c)) = \mathrm{val}^{\downarrow}(\Gamma(f)) + c$ *and* $\mathrm{val}^{\uparrow}(\Gamma(f+c)) = \mathrm{val}^{\uparrow}(\Gamma(f)) + c$.

Proof:

$$
\begin{array}{lrcll}
 & \forall w \in W : & (f+c)(w) = f(w) + c & & (3.7)\\
\Rightarrow & \forall \mu : & \int_{w\in W} (f+c)(w) d\mu(w) = \int_{w\in W} f(w) d\mu(w) + c & & (3.8)\\
\Rightarrow & \forall \sigma, \tau : & E(\sigma \text{ vs } \tau \text{ in } \Gamma(f+c)) = E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) + c & & (3.9)\\
\Rightarrow & \forall \sigma : & \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(f+c)) = \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) + c & & (3.10)\\
\Rightarrow & & \sup_\sigma \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(f+c)) = \sup_\sigma \inf_\tau E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) + c & & (3.11)\\
\Rightarrow & & \mathrm{val}^\downarrow(\Gamma(f+c)) = \mathrm{val}^\downarrow(\Gamma(f)) + c & & (3.12)
\end{array}
$$

Similarly for the upper value.

□

Remark 3.3 Similarly, it can be proven that if $a \geq 0$, then $\mathrm{val}^\downarrow(\Gamma(af)) = a\mathrm{val}^\downarrow(\Gamma(f))$. It is not always true, however, that $\mathrm{val}^\downarrow(\Gamma(-f)) = -\mathrm{val}^\downarrow(\Gamma(f))$, or that $\mathrm{val}^\downarrow(\Gamma(-f)) = -\mathrm{val}^\uparrow(\Gamma(f))$ (see also Example 3.18). However, if we reverse the position of the players *and* inverse the payoff function, then the equation holds. I.e. if f is a payoff function, then if we define f_{rev} to be the payoff function defined on sequences of pairs $(y, x) \in Y \times X$ by setting $f_{\mathrm{rev}}(w_{\mathrm{rev}}) = f(w)$ iff w_{rev} is obtained from w by reversing all pairs (x, y), and if we define $\Gamma_{\mathrm{rev}}(-f_{\mathrm{rev}})$ to be the Blackwell game with payoff function $-f_{\mathrm{rev}}$ in which player I chooses moves from Y and player II chooses moves from X, then $\mathrm{val}^\downarrow(\Gamma_{\mathrm{rev}}(-f_{\mathrm{rev}})) = -\mathrm{val}^\uparrow(\Gamma(f))$. Among other things, this implies together with Lemma 3.2 that if we have proven determinacy for all Blackwell games on open, G_δ, $G_{\delta\sigma}$ sets, then we also have determinacy of Blackwell games on closed, F_σ, $F_{\sigma\delta}$ sets.

A payoff function can often be written as the limit of a series of 'simpler' payoff functions. If this convergence is uniform, then we can draw conclusions about the values.

Lemma 3.4 *Let $(f_i)_i$ be a sequence of functions $f_i : W \to [a, b]$ such that $(f_i)_i$ converges uniformly to a function $f : W \to [a, b]$. Then*

$$\lim_{i\to\infty} \mathrm{val}^\downarrow(\Gamma(f_i)) = \mathrm{val}^\downarrow(\Gamma(f)) \tag{3.13}$$

and

$$\lim_{i\to\infty} \mathrm{val}^\uparrow(\Gamma(f_i)) = \mathrm{val}^\uparrow(\Gamma(f)) \tag{3.14}$$

Proof:
Let $\epsilon > 0$. Uniform convergence means that there exists an $i_0 \in \mathbb{N}$, such that for all $i > i_0$: $f_i - \epsilon \leq f \leq f_i + \epsilon$. Then by applying Lemmas 3.1 and 3.2, we have that for all $i > i_0$: $\mathrm{val}^\downarrow(\Gamma(f_i)) - \epsilon \leq \mathrm{val}^\downarrow(\Gamma(f)) \leq \mathrm{val}^\downarrow(\Gamma(f_i)) + \epsilon$. Hence, $\lim_{i\to\infty} \mathrm{val}^\downarrow(\Gamma(f_i)) = \mathrm{val}^\downarrow(\Gamma(f))$.
Similarly for the upper value.

□

Corollary 3.5 *Let $n \in \mathbb{N}$. Let $(f_i)_i$ be a sequence of functions $f_i : W_n \to [a,b]$ such that $(f_i)_i$ converges pointwise to a function $f : W_n \to [a,b]$. Then* $\mathrm{val}(\Gamma_n(f)) = \lim_{i\to\infty} \mathrm{val}(\Gamma_n(f_i))$.

Proof:
This follows from the last Lemma, since W_n is finite, and hence pointwise convergence implies uniform convergence.

□

If we have pointwise convergence without uniform convergence, we cannot say anything about the values of games. However, we can say something about expectations, once the strategies involved are known.

Lemma 3.6 *Let $(f_i)_i$ be a sequence of functions $f_i : W \to [a,b]$ such that $(f_i)_i$ converges pointwise to a function $f : W \to [a,b]$. Then for any two strategies σ, τ,* $\lim_{i\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma(f_i)) = E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))$

Proof:
This follows from the Majorized Convergence Theorem of Lebesgue.

□

3.2 Finite Games

To say something about complex games, we have to know about simple games. So we will start with the simplest variant of our game: the game that lasts only one round. But first, we need a lemma.

Lemma 3.7 *Let C be a closed convex set in $\mathbb{R}^n$, and let $b \in \mathbb{R}^n - C$. Then there exists a hyperplane separating b and C, i.e. a vector $y \in \mathbb{R}^n - \{0\}$ and $d \in \mathbb{R}$ such that*

$$y^T b > d \tag{3.15}$$
$$y^T z < d \text{ for each } z \in C \tag{3.16}$$

Proof:
Since the theorem is trivial if $C = \emptyset$, we assume $C \neq \emptyset$. As C is closed, there exists a vector c in C that is nearest to b, i.e. that minimizes $\|z - b\|$. And because $b \notin C$, $\|c - b\| > 0$. We define

$$y = b - c \tag{3.17}$$
$$d = \frac{1}{2}(y^T b + y^T c) \tag{3.18}$$

Then $y^T b - d = y^T(b - \frac{1}{2}(b + c)) = \frac{1}{2}(b - c)^T(b - c) > 0$, proving that (3.15) holds.

Now suppose that (3.16) does not hold. Then for some $z \in C$, $y^T z \geq d$. Since, $y^T c - d = y^T(c - \frac{1}{2}(b+c)) = -\frac{1}{2}(b-c)^T(b-c) < 0$, this implies $y^T(z-c) > 0$. Hence there exists a λ with $0 < \lambda \leq 1$ and

$$\lambda < \frac{2y^T(z-c)}{\|z-c\|^2} \tag{3.19}$$

Then $c + \lambda(z-c)$ belongs to C. Moreover,

$$\begin{aligned} \|(c+\lambda(z-c)) - b\|^2 &= \|\lambda(z-c) - y\|^2 && (3.20)\\ &= \lambda^2\|z-c\|^2 - 2\lambda y^T(z-c) + \|y\|^2 && (3.21)\\ &< \|y\|^2 = \|c-b\|^2 && (3.22) \end{aligned}$$

contradicting the fact that c is a point in C nearest to b. Therefore, for all $z \in C$, $y^T z < d$, and (3.16) holds.

Theorem 3.8 (Von Neumann's Minimax Theorem) *Let $\Gamma_1(f)$ be a finite one-round Blackwell game (i.e. of length 1). Then $\Gamma_1(f)$ is determined, and both players have optimal strategies. [15]*

Proof:
f is (or can be interpreted as) a function $X \times Y \to I\!R$. Without loss of generality we may assume that $X = \{1, \ldots, n\}, Y = \{1, \ldots, m\}$. A strategy σ for player I in $\Gamma_1(f)$ is (or can be interpreted as) a nonnegative vector $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^n x_i = 1$, and similarly, a strategy τ for player II is (or can be interpreted as) a nonnegative vector $(y_1, \ldots, y_m)$ such that $\sum_{j=1}^m y_j = 1$. It is easily seen that

$$\begin{aligned} E(\sigma \text{ vs } \tau \text{ in } \Gamma_1(f)) &= \sum_{i=1}^n \sum_{j=1}^m x_i y_j f(i,j) && (3.23)\\ \mathrm{val}(\sigma \text{ in } \Gamma_1(f)) &= \min_{1 \leq j \leq m} \sum_{i=1}^n x_i f(i,j) && (3.24)\\ \mathrm{val}(\tau \text{ in } \Gamma_1(f)) &= \max_{1 \leq i \leq n} \sum_{j=1}^m y_j f(i,j) && (3.25) \end{aligned}$$

Let $C \subset I\!R^m$ be the convex hull of the vectors $(f(i,1), \ldots, f(i,m))$, $i = 1, \ldots, n$. Then each point $z \in C$ corresponds to (at least) one strategy σ for player I, and the lowest coordinate of z equals the value of σ. Let $C^- = \{z \in I\!R^m \mid \exists z' \in C : z \leq z'\}$. Then player I has a strategy of value $u \in I\!R$ iff the vector $(u, \ldots, u) \in C^-$. It is obvious that C and C^- are both closed convex sets.

Now let $v = \mathrm{val}^{\downarrow}(\Gamma_1(f))$, $\epsilon > 0$. Then there exists no strategy σ for player I of value $v + \epsilon$, i.e. $b = (v+\epsilon, \ldots, v+\epsilon) \notin C^-$. By Lemma 3.7 this implies that there exist a vector $y \in I\!R^m - \{0\}$ and $d \in I\!R$, such that $y^T b > d$ while $y^T z < d$ for any $z \in C^-$. For any $j \leq m$, any $z \in C^-$, and any $r > 0$, $y^T z - y_j r = y^T(z_1, \ldots, z_{j-1}, z_j - r, z_{j+1}, \ldots, z_m) < d$, i.e. $y_j > (y^T z - d)/r$. It follows that for any $j \leq m$, $y_j \geq 0$. Since $y \neq 0$, we may also assume, without loss of generality, that $\sum_{j=1}^{m} y_j = 1$. Then for any $z \in C^-$, $y^T z < d < y^T b = v + \epsilon$. In particular, for $i = 1, \ldots, n$, $\sum_{j=1}^{m} y_j f(i,j) < v + \epsilon$. Thus, y corresponds to a strategy τ of value lower than $v + \epsilon$. Since we can do this construction for any $\epsilon > 0$, it follows that $\mathrm{val}^{\uparrow}(\Gamma_1(f)) = v = \mathrm{val}^{\downarrow}(\Gamma_1(f))$.

The existence of an optimal strategy for player I follows from the observation that C is a closed and bounded subset of $I\!R^m$ (and hence compact), and that the function $\min : I\!R^m \to I\!R$ taking the minimum coordinate is continuous. A similar argument yields the existence of an optimal strategy for player II.

□

Remark 3.9 We will use this result, the determinacy of one-round games, to derive the determinacy of finite, open, G_δ and $G_{\delta\sigma}$ games. Interestingly, the proof of Theorem 3.8 is conceptually quite different from the proofs of determinacy of complex games we will give later; the only connection seems to be that the *result* of the one proof is used in the other proofs. Basically, Game Theory got us this far, now Mathematical Logic will take us farther.

If a game lasts longer than one round, but we know it always stops in a finite number of rounds, and we know that there exists an *upper bound* on that number of rounds, then we can use induction on the natural numbers to prove determinacy.

Theorem 3.10 *Let $\Gamma_n(f)$ be a finite Blackwell game of length n. Then $\Gamma_n(f)$ is determined, and both players have optimal strategies.*

Proof:
We use induction on n, applying Theorem 3.8 for the induction step.
The case $n = 0$ is trivial: if $\Gamma_0(f)$ is a Blackwell game of length 0, then play stops immediately with payoff $f(e)$, and the empty strategies $\sigma = \tau = \emptyset$ are optimal.
So let $n \geq 1$, and suppose that all finite Blackwell games of length $n-1$ are determined. Let $\Gamma_n(f)$ be a finite Blackwell game of length n. For any position p of length 1, $\Gamma_n(f,p)$ is played as a finite Blackwell game of length $n-1$, by Remark 2.8, and hence determined, by the induction hypothesis. Now let $\Gamma_1(g)$ be the finite one-round Blackwell game defined by $g(p) = \mathrm{val}(\Gamma_n(f,p))$. By the Minimax Theorem, this game is also determined, and it has a value v.
Let σ_0 be an optimal strategy for player I in $\Gamma_1(g)$, and let σ_p be an optimal strategy for player I in $\Gamma_n(f,p)$, for all positions p of length 1. We can assume that σ_0 is only defined on e, and that σ_p is only defined on p and those positions that follow p. Then $\sigma = \bigcup_{p \in W_1} \sigma_p \cup \sigma_0$ is a strategy for player I defined on all the positions of the game $\Gamma_n(f)$.
Let τ be a strategy for player II in the game $\Gamma_n(f)$. Then

$$
\begin{aligned}
& E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(f)) \\
&= \int_w f(w) d\mu_{\sigma,\tau}(w) && (3.26)\\
&= \sum_{p' \in W_n} f(p')\mu_{\sigma,\tau}\{w \mid w \supset p'\} && (3.27)\\
&= \sum_{p \in W_1} \left(\sum_{p' \in W_n, p' \supseteq p} f(p') \frac{\mu_{\sigma,\tau}\{w \mid w \supset p'\}}{\mu_{\sigma,\tau}\{w \mid w \supset p\}} \right) \mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.28)\\
&= \sum_{p \in W_1} \left(\sum_{p' \in W_n, p' \supseteq p} f(p')\mu_{\sigma,\tau \text{ in } \Gamma_n(f,p)}\{w \mid w \supset p'\} \right) \mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.29)\\
&= \sum_{p \in W_1} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(f,p))\mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.30)\\
&= \sum_{p \in W_1} E(\sigma_p \text{ vs } \tau \text{ in } \Gamma_n(f,p))\mu_{\sigma_0,\tau}(p) && (3.31)\\
&\geq \sum_{p \in W_1} g(p)\mu_{\sigma_0,\tau}(p) && (3.32)\\
&\geq v && (3.33)
\end{aligned}
$$

So σ has value at least v in $\Gamma_n(f)$. Similarly we can construct a strategy τ for player II in $\Gamma_n(f)$ such that τ has value at most v. It follows that $\Gamma_n(f)$ has value v, and that σ, τ are optimal strategies for players I, II.

□

3.3 The Relativity of Values

In the previous section, we showed that finite games are determined. We did this by combining a strategy for the game up to certain points, with strategies for the games starting from those points. The points in question were the positions of length 1. In this section we generalize this to all sets of stopping positions, using truncated games. The essence of Lemma 3.11 is, that the value of the game for one player is not changed if we decide that at certain points, we will stop the game and pay out the value of the game for that player at that point. The remainder of the section consists of corollaries to this observation. Lemma 3.14 and Corollary 3.15 enable us to link together infinitely many nested truncated subgames, and form the heart of the proofs in the later chapters.

Lemma 3.11 *Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be a truncated subgame of $\Gamma(f)$, truncated at a set of positions H, equivalent for player I [for player II]. Then* $\mathrm{val}^{\downarrow}(\Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma_H(g))$ *[*$\mathrm{val}^{\uparrow}(\Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma_H(g))$*]. Furthermore, for any $\epsilon > 0$, any ϵ-optimal strategy for player I [player II] in $\Gamma_H(g)$ (if it is undefined on all positions at or after positions in H) can be extended to an ϵ-optimal strategy for player I [player II] in $\Gamma(f)$.*

Proof concept:
The proof is similar to the induction step of the proof of Theorem 3.10. Basically, we find ϵ-optimal strategies, tie them together, and then calculate how well the combination strategy performs against opposing strategies.
Proof:
Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be a truncated subgame of$\Gamma(f)$, truncated at a set of positions H, equivalent for player I. This means that for any $p \in H$, $g(p) = \mathrm{val}^{\downarrow}(\Gamma(f,p))$, and for any $w \in W$ that does not hit any position in H, $g(w) = f(w)$.
Let $\epsilon > 0$, and let σ_0 be an ϵ-optimal strategy for player I in the game $\Gamma_H(g)$. If $v = \mathrm{val}^{\downarrow}(\Gamma_H(g))$, and $u = \mathrm{val}(\sigma_0 \text{ in } \Gamma_H(g))$, then $0 \leq v - u < \epsilon$. So pick $\delta > 0$ such that $\delta < \epsilon - (v-u)$, and pick for each $p \in H$ a δ-optimal strategy σ_p for player I in $\Gamma(f,p)$. We can assume that for any $p \in H$, σ_p is defined exactly on p and those positions that are after p, and that σ_0 is defined exactly on those positions that are not at or after *any* position in H. It follows that $\sigma = \bigcup_{p \in H} \sigma_p \cup \sigma_0$ is a well-defined strategy for player I in the game $\Gamma(f)$.
Now let τ be a strategy for player II in $\Gamma(f)$. Then for each $p \in H$,

$$\begin{aligned}
\int_{w \supset p} f(w) d\mu_{\sigma,\tau}(w) &= \left(\frac{\int_{w \supset p} f(w) d\mu_{\sigma,\tau}(w)}{\mu_{\sigma,\tau}\{w \mid w \supset p\}} \right) \mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.34)\\
&= \left(\int_{w \supset p} f(w) d\mu_{\sigma,\tau \text{ in } \Gamma(f,p)}(w) \right) \mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.35)\\
&= E(\sigma \text{ vs } \tau \text{ in } \Gamma(f,p)) \mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.36)\\
&= E(\sigma_p \text{ vs } \tau \text{ in } \Gamma(f,p)) \mu_{\sigma_0,\tau}\{w \mid w \supset p\} && (3.37)\\
&\geq \left(\mathrm{val}^{\downarrow}(\Gamma(f,p)) - \delta \right) \mu_{\sigma_0,\tau}\{w \mid w \supset p\} && (3.38)\\
&= (g(p) - \delta) \mu_{\sigma_0,\tau}\{w \mid w \supset p\} && (3.39)
\end{aligned}$$

and consequently,

$$\begin{aligned}
&E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))\\
&= \int_w f(w) d\mu_{\sigma,\tau}(w) && (3.40)\\
&= \sum_{p \in H} \left(\int_{w \supset p} f(w) d\mu_{\sigma,\tau}(w) \right) + \int_{w \text{ avoids } H} f(w) d\mu_{\sigma,\tau}(w) && (3.41)\\
&\geq \sum_{p \in H} (g(p) - \delta) \mu_{\sigma_0,\tau}\{w \mid w \supset p\} + \int_{w \text{ avoids } H} g(w) d\mu_{\sigma_0,\tau}(w) && (3.42)\\
&\geq \left(\sum_{p \in H} g(p) \mu_{\sigma_0,\tau}\{w \mid w \supset p\} + \int_{w \text{ avoids } H} g(w) d\mu_{\sigma_0,\tau}(w) \right) - \delta && (3.43)\\
&= \int_w g(w) d\mu_{\sigma_0,\tau}(w) - \delta && (3.44)\\
&= u - \delta && (3.45)\\
&> v - \epsilon && (3.46)
\end{aligned}$$

So σ is an extension of σ_0 of value greater than $v-\epsilon$. Since this can be done for any $\epsilon > 0$, and any ϵ-optimal strategy σ_0, this implies that $\mathrm{val}^{\downarrow}(\Gamma(f)) \geq v$. Similarly, let $\epsilon > 0$, and let σ be a strategy for player I in $\Gamma(f)$. Then we can find counterstrategies τ_0, τ_p for $p \in H$ for player II in $\Gamma_H(g)$, $\Gamma(f,p)$ respectively, such that $E(\sigma \text{ vs } \tau_0 \text{ in } \Gamma_H(g)) < \mathrm{val}^{\downarrow}(\Gamma_H(g)) + \epsilon/2$, and $E(\sigma \text{ vs } \tau_p \text{ in } \Gamma(f,p)) < \mathrm{val}^{\downarrow}(\Gamma(f,p)) + \epsilon/2$ for any $p \in H$. We can assume that for any $p \in H$, τ_p is defined exactly on those positions that are at or after p, and that τ_0 is defined exactly on those positions that are not at or after any *any* position in H. It follows that $\tau = \bigcup_{p\in H} \tau_p \cup \tau_0$ is a well-defined strategy for player II in the game $\Gamma(f)$. Then for each $p \in H$,

$$\begin{aligned}
\int_{w\supset p} f(w)d\mu_{\sigma,\tau}(w) &= \left(\frac{\int_{w\supset p} f(w)d\mu_{\sigma,\tau}(w)}{\mu_{\sigma,\tau}\{w \mid w \supset p\}}\right)\mu_{\sigma,\tau}\{w \mid w\supset p\} && (3.47)\\
&= \left(\int_{w\supset p} f(w)d\mu_{\sigma,\tau \text{ in } \Gamma(f,p)}(w)\right)\mu_{\sigma,\tau}\{w \mid w \supset p\} && (3.48)\\
&= E(\sigma \text{ vs } \tau \text{ in } \Gamma(f,p))\mu_{\sigma,\tau}\{w \mid w\supset p\} && (3.49)\\
&= E(\sigma \text{ vs } \tau_p \text{ in } \Gamma(f,p))\mu_{\sigma,\tau_0}\{w \mid w\supset p\} && (3.50)\\
&\leq \left(\mathrm{val}^{\downarrow}(\Gamma(f,p)) + \epsilon/2\right)\mu_{\sigma,\tau_0}\{w \supset p\} && (3.51)\\
&= (g(p)+\epsilon/2)\mu_{\sigma,\tau_0}\{w \mid w\supset p\} && (3.52)
\end{aligned}$$

and consequently,

$$\begin{aligned}
&E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))\\
&\quad= \int_w f(w)d\mu_{\sigma,\tau}(w) && (3.53)\\
&\quad= \sum_{p\in H}\left(\int_{w\supset p} f(w)d\mu_{\sigma,\tau}(w)\right) + \int_{w \text{ avoids } H} f(w)d\mu_{\sigma,\tau}(w) && (3.54)\\
&\quad\leq \sum_{p\in H}(g(p)+\epsilon/2)\mu_{\sigma,\tau_0}\{w \mid w\supset p\} + \int_{w \text{ avoids } H} g(w)d\mu_{\sigma,\tau_0}(w) && (3.55)\\
&\quad\leq \sum_{p\in H} g(p)\mu_{\sigma,\tau_0}\{w \mid w\supset p\} + \int_{w \text{ avoids } H} g(w)d\mu_{\sigma,\tau_0}(w) + \epsilon/2 && (3.56)\\
&\quad= \int_w g(p)d\mu_{\sigma,\tau_0}(w) + \epsilon/2 && (3.57)\\
&\quad< v + \epsilon/2 + \epsilon/2 && (3.58)\\
&\quad= v + \epsilon && (3.59)
\end{aligned}$$

Since this can be done for any strategy σ and any $\epsilon > 0$, it follows that $\mathrm{val}^{\downarrow}(\Gamma(f)) \leq v$. Hence

$$\mathrm{val}^{\downarrow}(\Gamma(f)) = v = \mathrm{val}^{\downarrow}(\Gamma_H(g)) \tag{3.60}$$

An ϵ-optimal strategy for player I in $\Gamma_H(g)$ can be extended to an ϵ-optimal strategy for player I in $\Gamma(f)$ as in the first part of the proof.

The proof for the upper value is analogous.

□

Corollary 3.12 *Let $\Gamma(f)$ be a Blackwell game, and let $\Gamma_H(g)$ be a equivalent truncated subgame of $\Gamma(f)$ (truncated at H). If $\Gamma_H(g)$ is determined, then $\Gamma(f)$ is determined, and $\mathrm{val}(\Gamma(f)) = \mathrm{val}(\Gamma_H(g))$. Furthermore, any ϵ-optimal strategy for player I or player II in $\Gamma_H(g)$ can be extended to an ϵ-optimal strategy for player I or player II in $\Gamma(f)$.*

Proof:
$\Gamma_H(g)$ is a truncated subgame of $\Gamma(f)$, equivalent for player I and for player II. Applying Lemma 3.11 yields that $\mathrm{val}^{\downarrow}(\Gamma(f)) = \mathrm{val}^{\downarrow}(\Gamma_H(g))$ and $\mathrm{val}^{\uparrow}(\Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma_H(g))$, and that ϵ-optimal strategies for player I or player II in $\Gamma_H(g)$ can be extended to ϵ-optimal strategies for player I or player II in $\Gamma(f)$. If $\Gamma_H(g)$ is also determined, then $\mathrm{val}^{\downarrow}(\Gamma_H(g)) = \mathrm{val}^{\uparrow}(\Gamma_H(g))$, which concludes the proof.

□

Corollary 3.13 *Let $\Gamma(f), \Gamma_H(g)$ be Blackwell games. If for any $p \in H$, $g(p) \leq \mathrm{val}^{\downarrow}(\Gamma(f,p))$, and for any $w \in W$ that does not hit any position in H, $g(w) \leq f(w)$, then $\mathrm{val}^{\downarrow}(\Gamma_H(g)) \leq \mathrm{val}^{\downarrow}(\Gamma(f))$.*
Similarly for the value and the upper value, and for $\geq$ instead of $\leq$.

Proof:
Define the payoff function h by

$$\begin{aligned} h(p) &= \mathrm{val}^{\downarrow}(\Gamma(f,p)) \text{ for } p \in H && (3.61)\\ h(w) &= f(w) \text{ if } w \text{ does not hit any position in } H && (3.62) \end{aligned}$$

Then by Lemma's 3.1 and 3.11, $\mathrm{val}^{\downarrow}(\Gamma(g)) \leq \mathrm{val}^{\downarrow}(\Gamma(h)) = \mathrm{val}^{\downarrow}(\Gamma(f))$. The other proofs are completely similar.

□

Lemma 3.14 *Let $(\Gamma_{H_i}(g_i))_{i \in I\!N}$ be a nested series of truncated games equivalent for player I [player II]. Then all the games have the same lower value [upper value]. Furthermore, we can find a strategy for player I [player II] that is ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$.*

Proof concept:
Basically, we apply Lemma 3.11 a number of times and use induction.
Proof:
For all $i \in I\!N$, $\Gamma_{H_i}(g_i)$ is a truncated subgame of $\Gamma_{H_{i+1}}(g_{i+1})$, equivalent for player I. By Lemma 3.11, this means that for all $i \in I\!N$, $\mathrm{val}^{\downarrow}(\Gamma_{H_i}(g_i)) = \mathrm{val}^{\downarrow}(\Gamma_{H_{i+1}}(g_{i+1}))$. It follows that for all $i \in I\!N$, $\mathrm{val}^{\downarrow}(\Gamma_{H_i}(g_i)) = \mathrm{val}^{\downarrow}(\Gamma_{H_0}(g_0))$. Now let $\epsilon > 0$, and let σ_0 be an ϵ-optimal strategy for player I in $\Gamma_{H_0}(g_0)$.

Assume that σ_0 is only defined on non-stopping positions of $\Gamma_{H_0}(g_0)$. We can inductively define ϵ-optimal strategies σ_i in the games $\Gamma_{H_i}(g_i)$ such that for all $i \in I\!N$, σ_{i+1} is an extension of σ_i. For suppose we have extended σ_0 to an ϵ-optimal strategy σ_i in $\Gamma_{H_i}(g_i)$, defined only on non-stopping positions. Then σ_i is undefined on positions at or after positions in H, and thus by Lemma 3.11 we can extend σ_i to an ϵ-optimal strategy σ'_{i+1} in $\Gamma_{H_{i+1}}(g_{i+1})$. By Remark 2.26, any stopping position in $\Gamma_{H_{i+1}}(g_{i+1})$ is also a stopping position in $\Gamma_{H_i}(g_i)$. So the restriction σ_{i+1} of σ'_{i+1} to non-stopping positions in $\Gamma_{H_{i+1}}(g_{i+1})$ is also an extension of σ_i. Now, if p is a position, and $\sigma_i(p)$, $\sigma_j(p)$ are defined, $i < j$, then $\sigma_i(p) = \sigma_{i+1}(p) = \ldots = \sigma_j(p)$. Therefore we can define σ by

$$\sigma(p) = \sigma_i(p) \text{ if } \sigma_i \text{ is defined on } p \tag{3.63}$$

Then for any $i \in I\!N$, for any non-stopping position p of $\Gamma_{H_i}(g_i)$, $\sigma(p)$ is defined and $\sigma(p) = \sigma_i(p)$. Hence, σ is ϵ-optimal in all the games. σ is defined on all the non-stopping positions of all the games $\Gamma_{H_i}(g_i)$, but if necessary we can extend σ to a strategy σ' defined on all positions, by choosing a default probability distribution for those positions on which σ is not defined.

□

Corollary 3.15 *Let $(\Gamma_{H_i}(g_i))_{i\in I\!N}$ be a nested series of equivalent truncated subgames. If $\Gamma_{H_0}(g_0)$ is determined, then all the games are determined, and all the games have the same value. Furthermore, we can find strategies for player I and player II that are ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$.*

Proof:
If $(\Gamma_{H_i}(g_i))_{i\in I\!N}$ is a nested series of equivalent truncated subgames, then the series is equivalent for player I and for player II. Applying Lemma 3.14 twice yields that all the games have the same lower value and all the games have the same upper value, and that there exist strategies for player I and player II, defined on all the non-stopping positions of all the games $\Gamma_{H_i}(g_i)$, that are ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$. If $\Gamma_{H_0}(g_0)$ is also determined, then $\mathrm{val}^{\downarrow}(\Gamma_{H_0}(g_0)) = \mathrm{val}^{\uparrow}(\Gamma_{H_0}(g_0))$, which concludes the proof.

□

Remark 3.16 If all games involved are finite, then we can extend optimal strategies with optimal strategies to optimal strategies, i.e. drop the ϵ in Lemma's 3.11 and 3.14. The proofs are almost exactly the same, and are left as an exercise for the reader.

Remark 3.17 A consequence of Lemma 3.11 is, that the lower value of a game at a position p can be calculated from the lower values of the game starting from the positions directly following p, as the value of the one-round game with those lower values as payoffs. In addition to a value, Von Neumann's Minimax Theorem produces, for any position p, a probability distribution on the possible moves that is an optimal strategy for player I in the aforementioned one-round game, i.e. for any position p, Von Neumann's Minimax Theorem produces the answer to the question 'what should I do right *now*'. Consider the strategy assigning this probability distribution to each position. The principle behind this strategy can be expressed as 'at any moment, do what seems best at that moment'. In the case of a finite Blackwell game, this strategy is optimal (and in fact, this strategy is the one produced by our proof of Theorem 3.10). The same is true in the case of an open game (i.e. a Blackwell game where the payoff function is the indicator function of an open set), but only for the second player. In general, however, this does not hold. Even though any single probability distribution is optimal in the corresponding one-round game, it is not necessarily optimal in the whole game, only if the part of the strategy following that round, is optimal as a whole. See also Example 4.3.

Example 3.18 The following example shows, that there is no algorithm to find optimal or ϵ-optimal strategies that uses as input only the values a game has at each of its positions. Consider the following two Blackwell games. Player II generates a sequence of 0's and 1's. In the game $\Gamma(S_1)$, player II wins if he generates infinitely many 1's. In the game $\Gamma(S_2)$, player II wins if he generates only finitely many 1's. Player I has no influence over the outcome of the game. Payoff is 1 if player I wins, 0 if player II wins.

It is clear that player II can win from any position, in both games. Hence both games have lower and upper value 0, starting from all positions. But any strategy for player II that is good in $\Gamma(S_1)$, will be bad in $\Gamma(S_2)$, and vice versa, since the two sets of winning positions are complimentary. Hence there cannot be any method of finding optimal or good strategies that merely uses the values of a game.

Another interesting aspect is, that here we have two sets of winning positions S_1 and S_2, such that $\Gamma(S_1)$, $\Gamma(S_2)$ have value 0 (in all positions), and $\Gamma(S_1 \cup S_2)$ has value 1 (in all positions).

Chapter 4

Open Games

In the first section of this chapter we prove determinacy of the game $\Gamma(O)$, where $O \subseteq W$ is an open set. This game can be described as 'There are positions in which player I has won (payoff 1), and player II wins (payoff 0) if no such position is ever reached'. As any payoff other than 0 is made at a finite time and ends the game, open games are nearly as simple as finite games. The proof given is mainly a translation of the standard proof in terms of my definitions, meant as an illustration of the techniques involved in the other proofs.
In the second section we look at a generalization of open games, where the payoff is the supremum of the payoffs at the positions hit. In addition to determinacy, we derive a result for these and open games comparable to the compactness of W.

4.1 Open Games Part The First

Theorem 4.1 *Let O be an open set. Then $\Gamma(O)$ is determined.*

Proof concept:
In an open game, there are certain positions in which player I has won. Player II wins if he can successfully avoid those positions. Suppose for a moment, that we were dealing with a game of Perfect Information, in which both players waited their turn to move. And suppose player I has no winning strategy from the current position. Then, if it is player I's turn to move, there is no move she can make to get to a position from which she has a winning strategy, and if it is player II's turn to move, there is a move he can make to get to a position from which player I has no winning strategy. Otherwise, player I would have a winning strategy from this position, contradicting our hypothesis. Now, if player II keeps playing those moves, then player I never gets to a position from which she can win, and hence she never gets to a position in which she *has won*. Thus, this is a winning strategy for player II.

Going back to our original Blackwell games and translating this proof in terms of values and probability distributions, a good strategy for player II seems to be to play, in each position, to minimize the expectation of the lower value in the next position. In order to tie all these local strategies together, we define a collection of finite truncated subgames $\Gamma_n(g_n), n \in \mathbb{N}$, each of which is equivalent for player I to $\Gamma(O)$. Each of these subgames is finite, and hence determined, and a good strategy for player II in each of these subgames minimizes the expectation of the lower value of $\Gamma(O)$ in each of the stopping positions. We then show that these subgames form a nested series of equivalent finite truncated subgames. This allows us to find a strategy for player II that is optimal in each of the truncated subgames. This strategy is produced by Corollary 3.15, and consists of optimal strategies in the truncated subgames, all glued together. It also functions as a strategy in the game $\Gamma(O)$, and has a value in $\Gamma(O)$ equal to the lower value of $\Gamma(O)$, proving the determinacy of $\Gamma(O)$.

Proof:

Put $v = \mathrm{val}^{\downarrow}(\Gamma(O))$.

Define for any $n \in \mathbb{N}$, the payoff functions $g_n : W_n \to [0,1]$ by

$$g_n(p) = \mathrm{val}^{\downarrow}(\Gamma(O,p)) \tag{4.1}$$

Now the games $\Gamma_n(g_n)$ are truncated subgames of $\Gamma(O)$, and are equivalent to $\Gamma(O)$ for player I, by Definition 2.24. Thus, by Lemma 3.11, for any $n \in \mathbb{N}$ and any position p of length $\leq n$,

$$\mathrm{val}^{\downarrow}(\Gamma_n(g_n,p)) = \mathrm{val}^{\downarrow}(\Gamma(O,p)) \tag{4.2}$$

Furthermore, the games $\Gamma_n(g_n,p)$ are finite, and hence determined, by Theorem 3.10. It follows that, for any $n \in \mathbb{N}$ and any position p,

$$\mathrm{val}(\Gamma_n(g_n,p)) = \mathrm{val}^{\downarrow}(\Gamma(O,p)) \tag{4.3}$$

Combining (4.1), (4.2), (4.3) yields that for any $n, m \in \mathbb{N}, n \leq m$, and any position $p \in W_n$,

$$g_n(p) = \mathrm{val}(\Gamma_m(g_m,p)) \tag{4.4}$$

Hence the games $\Gamma_n(g_n)$ form a nested series of equivalent truncated subgames. Consequently, by Corollary 3.15, all the games have the same value. Since $\Gamma_0(g_0)$ is the trivial game that stops immediately, this value is equal to the payoff $g_0(e) = \mathrm{val}^{\downarrow}(\Gamma(O)) = v$. Also, we can find a strategy for player II that is ϵ-optimal in all the games $\Gamma_n(g_n)$, and since all the games $\Gamma_n(g_n)$ are finite, by Remark 3.16 we can even find a strategy that is optimal in all the games $\Gamma_n(g_n)$. So let τ be such a strategy. It follows that for any $n \in \mathbb{N}$, and any strategy σ for player I in $\Gamma_n(g_n)$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(g_n)) \leq v \tag{4.5}$$

We now apply the result obtained to the game $\Gamma(O)$. As with any open set there is associated with O a set H of positions, such that

$$O = \{w \in W \mid \exists n \in \mathbb{N} : w_{|n} \in H\} \tag{4.6}$$

Define for $n \in \mathbb{N}$ the payoff functions $f_n : W_n \to [0,1]$ by

$$\begin{aligned} f_n(p) &= 1 \text{ if } \exists p' \in H : p' \subseteq p && (4.7)\\ &= 0 \text{ if } \neg\exists p' \in H : p' \subseteq p && (4.8) \end{aligned}$$

The functions f_n are well-defined, and are easily seen to satisfy

$$\begin{array}{lll} \forall n \in \mathbb{N} : & f_n \leq g_n & (4.9)\\ \forall w \in W : I_O(w) = \lim_{n\to\infty} f_n(w) & & (4.10) \end{array}$$

Now let σ be any strategy for player I in $\Gamma(f)$. Then

$$\begin{aligned} &E(\sigma \text{ vs } \tau \text{ in } \Gamma(O))\\ &= \lim_{n\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(f_n)) && (4.11)\\ &\leq \lim_{n\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(g_n)) && (4.12)\\ &\leq v && (4.13) \end{aligned}$$

So

$$\text{val}^{\uparrow}(\Gamma(f)) \leq \text{val}(\tau \text{ for player II in } \Gamma(f)) \leq v \tag{4.14}$$

Therefore,

$$\text{val}^{\uparrow}(\Gamma(f)) = \text{val}^{\downarrow}(\Gamma(f)) = v \tag{4.15}$$

□

Example 4.2 Let r be a real number, $0 \leq r \leq 1$. We will construct an open Blackwell game of value r. Set $X = Y = \{0,1\}$, and define $\phi : W \to [0,1]$ by

$$\phi((x_1, y_1, x_2, y_2, \ldots)) = \sum_{i=1}^{\infty} 2^{-i}(x_i \oplus y_i) \tag{4.16}$$

where $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$. Now, it is easy to see that for $w = (x_1, y_1, x_2, y_2, \ldots) \in W$,

$$\phi(w) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} 2^{-i}(x_i \oplus y_i) + 2^{-n}\right) \tag{4.17}$$

and hence,

$$\phi(w) < r \Leftrightarrow \exists n : \sum_{i=1}^{n} 2^{-i}(x_i \oplus y_i) + 2^{-n} < r \tag{4.18}$$

So $O = \{w \in W \mid 0 \leq \phi(w) < r\}$ is an open subset of W.

Now, let σ, τ be two strategies, and suppose that one of those strategies is the strategy that assigns the $\frac{1}{2}$-$\frac{1}{2}$ probability distribution on X or Y, respectively. Then for any $i \in \mathbb{N}$, $x_i \oplus y_i$ has equal chances of being 0 or 1. It follows that the distribution of $\phi(w)$ on $[0,1]$ is the uniform distribution on $[0,1]$. Hence $\mu_{\sigma,\tau}(O) = \mu_{\text{Lebesgue}}([0,r\rangle) = r$. So both players have a strategy of value r, and the value of this open game is r.

Example 4.3 Consider the following Blackwell game. Each round, both players say either 'Now' or 'Not yet'. If both players say 'Not yet', then play continues. Otherwise, the game halts: player II wins (payoff 0) if both players said 'Now', while player I wins (payoff 1) if only one of the players said 'Now'. If play continues indefinitely, and neither player ever says 'Now', then payoff is 0, i.e. player II wins.

This is clearly an open game. An interpretation of this game is, that player II tries to guess on which round player I will say 'Now', and tries to match her. If player II guesses wrong, i.e. says 'Now' too soon or not soon enough, then player I wins, if player II guesses right, then he wins.

A strategy of value $1 - \frac{1}{n}$ for player I is, to pick randomly a number i between 1 and n, and say 'Now' on round i. Translated to the standard format for strategies, this becomes:

say 'Now' $\frac{1}{n}$ of the time on round 1,
otherwise say 'Now' $\frac{1}{n-1}$ of the time on round 2,
otherwise say 'Now' $\frac{1}{n-2}$ of the time on round 3,

$\vdots$

otherwise say 'Now' 100% of the time on round n.

Hence, the value of this game is 1. In fact, the value of this game at any position in which game has not yet ended is 1. But there exists no optimal strategy of value 1. For suppose there exists such a strategy, of value 1. Then on any round (in which play has not yet ended), the chance that player I will say 'Now' in that round is 0%. For otherwise, the strategy would not score 100% against the counterstrategy that player II says 'Now' on that round. But then, player I will never say 'Now', and this strategy will lose against the counterstrategy that player II never says 'Now'. So any strategy for player I has value strictly less than 1, although there are strategies with values arbitrarily close to 1.

The 'locally optimal strategy' of Remark 3.17 is also the strategy of never saying 'Now'. Hence, this game also serves as an example of a game where the 'locally optimal strategy' performs very badly.

4.2 Generalizing Open Games

Theorem 4.4 *Let $u : P \to I\!R$ be a bounded function, and let $f : W \to I\!R$ be the payoff function defined by $f(w) = \sup_{j \in I\!N} u(w_{|j})$. Then $\Gamma(f)$ is determined, and*

$$\mathrm{val}(\Gamma(f)) = \lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n)) \tag{4.19}$$

where $f_n(w) = \sup_{j \le n} u(w_{|j})$.

Proof concept:

Showing that $\lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n))$ exists and is no greater than the lower value of $\Gamma(f)$ is not difficult. To show that it is no less than the upper value, we approximate $\Gamma(f)$ with a collection of finite auxiliary games $\Gamma_n(g_n)$ such that the payoff at the stopping positions is an *estimate* of the value of the game at that point. We then show that these auxiliary games form a nested series of equivalent finite truncated subgames. This allows us to find a strategy that is optimal in each of the truncated subgames. This strategy is also a strategy in the game $\Gamma(f)$, and has a value in $\Gamma(f)$ equal to $\lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n))$.

Proof:

Without loss of generality we may assume that the function u has range $[0, 1]$. For any $p \in P$, and any $n \in I\!N$, the game $\Gamma_n(f_n, p)$ is finite (of length $\leq n$), and thus determined. It is easily seen that $f_0 \leq f_1 \leq f_2 \leq \ldots \leq f \leq 1$. Consequently, for any $p \in P$,

$$\mathrm{val}(\Gamma_0(f_0)) \leq \mathrm{val}(\Gamma_1(f_1, p)) \leq \mathrm{val}(\Gamma_2(f_2, p)) \leq \ldots \leq \mathrm{val}^{\downarrow}(\Gamma(f, p)) \leq 1 \quad (4.20)$$

$\lim_{k\to\infty} \mathrm{val}(\Gamma_k(f_k, p))$ exists for all $p \in P$, since all monotone non-descending bounded sequences converge. Furthermore, for all $p \in P$,

$$\lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n, p)) \leq \mathrm{val}^{\downarrow}(\Gamma(f, p)) \quad (4.21)$$

Define for any $n \in I\!N$ the payoff function $g_n : W_n \to [0, 1]$ by

$$g_n(p) = \lim_{k\to\infty} \mathrm{val}(\Gamma_k(f_k, p)) \text{ for } p \in W_n \quad (4.22)$$

Then for all $p \in W_n$, $g_n(p) \geq \mathrm{val}(\Gamma_n(f_n, p)) = f_n(p)$.

Furthermore, the games $\Gamma_n(g_n)$ form a nested series of equivalent truncated subgames. For fix $n \in I\!N$, $p \in W_n$. Define for $k \in I\!N$, $h_{n+1,k} : W_{n+1} \to I\!R$ by $h_{n+1,k}(p') = \mathrm{val}(\Gamma_k(f_k, p'))$ for $p' \in W_{n+1}$. Now, W_{n+1} is finite, and for $p' \in W_{n+1}$, $\lim_{k\to\infty} h_{n+1,k}(p') = g_{n+1}(p')$. Hence by Corollary 3.5,

$$\lim_{k\to\infty} \mathrm{val}(\Gamma_{n+1}(h_{n+1,k}, p)) = \mathrm{val}(\Gamma_{n+1}(\lim_{k\to\infty} h_{n+1,k}, p)) = \mathrm{val}(\Gamma_{n+1}(g_{n+1}, p)) \quad (4.23)$$

Now, by Corollary 3.12, for all $k \in I\!N$, $\mathrm{val}(\Gamma_k(f_k, p)) = \mathrm{val}(\Gamma_{n+1}(h_{n+1,k}, p))$. It follows that

$$g_n(p) = \lim_{k\to\infty} \mathrm{val}(\Gamma_k(f_k, p)) = \mathrm{val}(\Gamma_{n+1}(g_{n+1}, p)) \quad (4.24)$$

Since $(\Gamma_n(g_n))_{n\in I\!N}$ is a nested series of equivalent truncated subgames, by Corollary 3.15 the games $\Gamma_n(g_n)$ all have the same value, say v. Also, we can find a strategy for player II that is ϵ-optimal in all the games $\Gamma_n(g_n)$, and since all the games $\Gamma_n(g_n)$ are finite, by Remark 3.16 we can even find a strategy that is optimal in all the games $\Gamma_n(g_n)$. So let τ be such a strategy. Then for any strategy σ, and any $n \in I\!N$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(g_n)) \leq \mathrm{val}(\Gamma_n(g_n)) = v \quad (4.25)$$

Now let σ be any strategy for player I in $\Gamma(f)$. Then

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma(f))$$

$$\begin{aligned}
&= \lim_{n\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(f_n)) && (4.26)\\
&\leq \lim_{n\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_n(g_n)) && (4.27)\\
&\leq \lim_{k\to\infty} \text{val}(\Gamma_k(f_k)) && (4.28)\\
&= v && (4.29)
\end{aligned}$$

So

$$\text{val}^{\uparrow}(\Gamma(f)) \leq \text{val}(\tau \text{ for player II in } \Gamma(f)) \leq v \tag{4.30}$$

But also

$$v = \lim_{k\to\infty} \text{val}(\Gamma_k(f_k)) \leq \text{val}^{\downarrow}(\Gamma(f)) \tag{4.31}$$

Therefore,

$$\text{val}^{\uparrow}(\Gamma(f)) = \text{val}^{\downarrow}(\Gamma(f)) = \lim_{k\to\infty} \text{val}(\Gamma_k(f_k)) \tag{4.32}$$

□

Remark 4.5 Applying Theorem 4.4 to the indicator function of an open set, we obtain a proof of the determinacy of open Blackwell games that is slightly different from the proof given in Theorem 4.1. This was done on purpose. The proof of Theorem 4.1 is mainly an illustration of techniques. Theorem 4.4 is somewhat stronger, allowing us to derive an additional result.

Corollary 4.6 *Let $O = \bigcup_i O_i$ be the union of open sets. Then* $\text{val}(\Gamma(O)) = \lim_{n\to\infty} \text{val}(\Gamma(\bigcup_{i\leq n} O_i))$.

Proof:
Define the set of positions H, and the compact sets $O'_j \subseteq O$, by

$$\begin{aligned}
H &= \{p \in P \mid \forall w' \in W : w' \subset p \Rightarrow w' \in O\} && (4.33)\\
O'_j &= \{w \in W \mid w_{|j} \in H\} && (4.34)
\end{aligned}$$

Then

$$\begin{aligned}
O &= \{w \in W \mid \exists p \in H : p \subset w\} && (4.35)\\
O'_j &= \{w \in W \mid \exists p \in H : p \subset w \wedge \text{len}(p) \leq j\} && (4.36)
\end{aligned}$$

and hence for all $w \in W$,

$$\begin{aligned}
I_O(w) &= \sup_{j\in\mathbb{N}} I_H(w_{|j}) && (4.37)\\
I_{O'_j}(w) &= \sup_{j\leq n} I_H(w_{|j}) && (4.38)
\end{aligned}$$

so applying Theorem 4.4, we find that

$$\text{val}(\Gamma(O)) = \lim_{j\to\infty} \text{val}(\Gamma_j(O'_j)) \tag{4.39}$$

For each $j \in \mathbb{N}$, O'_j is a closed and compact set covered by the open sets $(O_i)_{i \in \mathbb{N}}$. So by compactness there is for each $j \in \mathbb{N}$ a $n_j \in \mathbb{N}$ such that $O'_j \subseteq \bigcup_{i=1}^{n_j} O_i$. Then for all $n \geq n_j$,

$$\mathrm{val}(O'_j) \leq \mathrm{val}(\bigcup_{i=1}^{n} O_i) \leq \mathrm{val}(O) \tag{4.40}$$

The corollary follows immediately.

□

Remark 4.7 In the case of open and generalized open games, there is an optimal strategy for player II. This strategy can be described as 'at every position player II plays the optimal one-round strategy, looking at the values the game has *for player I* from all positions directly following that one', as can be seen for open games in the proof of Theorem 4.1. This optimal strategy is equal to the 'locally optimal strategy' described in Remark 3.17. However, for player I there does not always exist an optimal strategy, as Example 4.3 shows.

Chapter 5

G_δ- and $G_{\delta\sigma}$-Games

In this chapter, we extend the results of the last chapter to the next two Borel levels, proving determinacy of the game $\Gamma(f)$ in the cases where f is the indicator function of a G_δ or $G_{\delta\sigma}$ set. Structurally, this proof is similar to a proof by Davis for $G_{\delta\sigma}$ games of perfect information [5], although the main lemma is proved in an entirely different manner.

Davis' proof of determinacy is based upon the idea of 'imposing restrictions' on the range of moves players II can make (i.e. declaring a loss if that player makes one of the 'forbidden' moves), in such a way that (a) if player I did not have a win before, she does not get a win now, and (b) a particular G_δ set is now certain to be avoided. By applying this to all the G_δ subsets of a $G_{\delta\sigma}$ set, and using compactness, he shows that if player I cannot force the resulting sequence to be in one of the G_δ sets, player II can force the resulting sequence to be outside all of them.

The union of all the sequences in which one of the 'forbidden' moves is played, is an open set that contains the G_δ set in question. One way of looking at Davis' proof is, that he enlarges each of the G_δ sets to an open set without increasing the (lower) value of the game, in order to apply determinacy of open games.

In the first section we show that the game for a single G_δ set has a value, and that this set can be 'enlarged' to an open set without increasing the value of the game by more than an arbitrarily small amount. In the next section, this is generalized to the case where there is a payoff function for those sequences that are not in the G_δ set (although only the lower value is considered, not the upper value). In the section after that, this is applied to the G_δ subsets of a $G_{\delta\sigma}$ set, using Corollary 4.6 on the value of the game for a countable union of open sets instead of compactness, to arrive at the determinacy of $G_{\delta\sigma}$ sets.

5.1 Ye Olde G_δ Set

Theorem 5.1 *Let D be a G_δ set, i.e. let D be the intersection of countably many open sets $D = \bigcap_i O_i$. Then $\Gamma(D)$ is determined, and*

$$\mathrm{val}(\Gamma(D)) = \inf_{O \supseteq D, O \text{ open}} \mathrm{val}(\Gamma(O)) \tag{5.1}$$

Proof concept:
We define a collection of auxiliary games $\Gamma_{H_i}(g_i)$ of the game $\Gamma(D)$, in which the amount player I gets at a stopping position p is the aforementioned estimate for the value of $\Gamma(D)$ at position p, namely $\inf_{O \supseteq D, O \text{ open}} \mathrm{val}(\Gamma(O,p))$. We then show that these auxiliary games form a nested series of equivalent finite truncated subgames. This allows us to find a strategy that is ϵ-optimal in each of the truncated subgames. This strategy is also a strategy in the game $\Gamma(D)$, and has the required value.

Proof:
Put $v = \inf_{O \supseteq D, O \text{ open}} \mathrm{val}(\Gamma(O))$.
For any G_δ set D we can find a set of positions H, such that $D = \{w \in W \mid \#\{p \in H \mid p \subset w\} = \infty\}$. We may assume that $e \in H$.
Define for any $i \in I\!\!N$,

$$H_i := \{p \in H \mid \text{there are exactly } i \text{ positions } p' \text{ in } H \text{ strictly preceding } p\} \tag{5.2}$$

Define for any $i \in I\!\!N$ the payoff functions g_i, h_i by

$$g_i(p) = \inf_{O \supseteq D} \mathrm{val}(\Gamma(O,p)) \text{ for } p \in H_i \tag{5.3}$$

$$g_i(w) = 0 \text{ if } w \text{ does not hit any position in } H_i \tag{5.4}$$

$$h_i(p) = 1 \text{ for } p \in H_i \tag{5.5}$$

$$h_i(w) = 0 \text{ if } w \text{ does not hit any position in } H_i \tag{5.6}$$

The functions g_i are well-defined, because by Theorem 4.1 the games $\Gamma(O,p)$ are determined for all open sets O and positions p.

First, the games $\Gamma_{H_i}(g_i)$ form a nested series of equivalent truncated subgames. For let $i \in I\!\!N$ and fix $p \in H_i$. Let $O \supseteq D$, then for any $p' \in H_{i+1}$ such that $p' \supseteq p$, $\mathrm{val}(\Gamma(O,p')) \geq g_{i+1}(p')$, and for any $w \supset p$ that does not hit any position in H_{i+1}, $I_O(w) \geq 0 = g_{i+1}(w)$. Hence by Corollary 3.13, for any $O \supseteq D$, $\mathrm{val}(\Gamma(O,p)) \geq \mathrm{val}^{\uparrow}(\Gamma_{H_{i+1}}(g_{i+1},p))$. Therefore,

$$g_i(p) \geq \mathrm{val}^{\uparrow}(\Gamma_{H_{i+1}}(g_{i+1},p)) \tag{5.7}$$

On the other hand, for any $\epsilon > 0$ we can find, for each $p' \in H_{i+1}$, an open set $O_{p'} \supseteq D$ such that

$$\mathrm{val}(\Gamma(O_{p'},p')) \leq g_{i+1}(p') + \epsilon \tag{5.8}$$

Setting $O = \{w \mid \exists p' \in H_{i+1} : p' \subset w \wedge w \in O_{p'}\}$, we have that for all $p' \in H_{i+1}$, $\mathrm{val}(\Gamma(O, p')) = \mathrm{val}(\Gamma(O_{p'}, p')) \leq g_{i+1}(p') + \epsilon$, and for any $w \in W$ that does not hit any position in H_{i+1}, $I_O(w) = 0 = g_{i+1}(w)$. Hence by Corollary 3.13,

$$g_i(p) \leq \mathrm{val}(\Gamma(O, p)) \leq \mathrm{val}^{\downarrow}(\Gamma_{H_{i+1}}(g_{i+1}, p)) + \epsilon \tag{5.9}$$

This holds for any $\epsilon > 0$, therefore

$$g_i(p) = \mathrm{val}(\Gamma_{H_{i+1}}(g_{i+1}, p)) \tag{5.10}$$

Finally, for any $i \in \mathbb{N}$, and any play w that does not hit any positions in H_i, w does not hit any positions in H_{i+1} either, and $g_i(w) = 0 = g_{i+1}(w)$.

Let $\epsilon > 0$.
Since $(\Gamma_{H_i}(g_i))_{i \in \mathbb{N}}$ is a nested series of equivalent truncated subgames, by Corollary 3.15 all the games are determined and all have the same value, namely $\mathrm{val}(\Gamma_{H_0}(g_0)) = g_0(e) = v$, and there exists a strategy σ for player I that is ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$, i.e. for any strategy τ, and any $i \in \mathbb{N}$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(g_i)) \geq \mathrm{val}(\Gamma_{H_i}(g_i)) - \epsilon = v - \epsilon \tag{5.11}$$

Now let τ be any strategy for player II in $\Gamma(D)$. Then

$$\begin{aligned} & E(\sigma \text{ vs } \tau \text{ in } \Gamma(D)) \\ & \quad = \lim_{i \to \infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(h_i)) && (5.12) \\ & \quad \geq \lim_{i \to \infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(g_i)) && (5.13) \\ & \quad \geq v - \epsilon && (5.14) \end{aligned}$$

So σ is a strategy for player I of value at least $v - \epsilon$. This implies that $\mathrm{val}^{\downarrow}(\Gamma(D)) \geq v - \epsilon$. This construction can be done for any $\epsilon > 0$, hence

$$\mathrm{val}^{\downarrow}(\Gamma(D)) \geq v \tag{5.15}$$

For any $O \supseteq D$, $\mathrm{val}^{\uparrow}(\Gamma(D)) \leq \mathrm{val}(\Gamma(O))$, hence

$$\mathrm{val}^{\uparrow}(\Gamma(D)) \leq \inf_{O \supseteq D, O \text{ open}} \mathrm{val}(\Gamma(O)) = v \tag{5.16}$$

Hence $\mathrm{val}(\Gamma(D)) = v$.

□

5.2 Ye New and Improved G_δ-set

Theorem 5.2 *Let $f : W \to [0,1]$ be a measurable function and let D be a G_δ set. Then*

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) = \inf_{O \supseteq D, O \textit{ open}} \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O))) \tag{5.17}$$

Proof concept:
Basically, this proof is the same as the proof of Theorem 5.1, except that values are replaced by lower values, and the games $\Gamma(O)$ are replaced by the games $\Gamma(\max(f, I_O))$.
Proof:
Put $v = \inf_{O \supseteq D, O \text{ open}} \text{val}^\downarrow(\Gamma(\max(f, I_O)))$
For any G_δ set D we can find a set of positions H, such that $D = \{w \in W \mid \#\{p \in H \mid p \subset w\} = \infty\}$. We may assume that $e \in H$.
Define for any $i \in I\!N$,

$$H_i := \{p \in H \mid \text{there are exactly } i \text{ positions } p' \text{ in } H \text{ strictly preceding } p\} \tag{5.18}$$

Define for any $i \in I\!N$ the payoff functions g_i, h_i by

$$\begin{aligned} g_i(p) &= \inf_{O \supseteq D, O \text{ open}} \text{val}^\downarrow(\Gamma(\max(f, I_O), p)) \text{ for } p \in H_i && (5.19)\\ g_i(w) &= f(w) \text{ if } w \text{ does not hit any position in } H_i && (5.20)\\ h_i(p) &= 1 \text{ for } p \in H_i && (5.21)\\ h_i(w) &= f(w) \text{ if } w \text{ does not hit any position in } H_i && (5.22) \end{aligned}$$

First, the games $\Gamma_{H_i}(g_i)$ form a nested series of truncated subgames equivalent for player I.
For let $i \in I\!N$, and fix $p \in H_i$. Let $O \supseteq D$, then for any $p' \in H_{I+1}$ such that $p' \supseteq p$, $\text{val}^\downarrow(\Gamma(\max(f, I_O), p')) \geq g_{i+1}(p')$, and for any $w \supset p$ that does not hit any position in H_{i+1}, $\max(f, I_O)(w) \geq f(w) = g_{i+1}(w)$. Hence by Corollary 3.13, for any $O \supseteq D$, $\text{val}^\downarrow(\Gamma(\max(f, I_O), p)) \geq \text{val}^\downarrow(\Gamma_{H_{i+1}}(g_{i+1}, p))$. Therefore,

$$g_i(p) \geq \text{val}^\downarrow(\Gamma_{H_{i+1}}(g_{i+1}, p)) \tag{5.23}$$

On the other hand, for any $\epsilon > 0$ we can find, for each $p' \in H_{i+1}$, an open set $O_{p'} \supseteq D$ such that

$$\text{val}^\downarrow(\Gamma(\max(f, I_{O_{p'}}), p')) \leq g_{i+1}(p') + \epsilon \tag{5.24}$$

Setting $O = \{w \mid \exists p' \in H_{i+1} : p' \subset w \wedge w \in O_{p'}\}$, we have that for all $p' \in H_{i+1}$, $\text{val}^\downarrow(\Gamma(\max(f, I_O), p')) = \text{val}^\downarrow(\Gamma(\max(f, I_{O_{p'}}), p')) \leq g_{i+1}(p') + \epsilon$, and for any $w \in W$ that does not hit any position in H_{i+1}, $\max(f, I_O)(w) = f(w) = g_{i+1}(w)$. Hence by Corollary 3.13,

$$g_i(p) \leq \text{val}^\downarrow(\Gamma(\max(f, I_O), p)) \leq \text{val}^\downarrow(\Gamma_{H_{i+1}}(g_{i+1}, p)) + \epsilon \tag{5.25}$$

This holds for any $\epsilon > 0$, therefore

$$g_i(p) = \text{val}^\downarrow(\Gamma_{H_{i+1}}(g_{i+1}, p)) \tag{5.26}$$

Finally, for any $i \in I\!N$, and any play w that does not hit any positions in H_i, w does not hit any positions in H_{i+1} either, and $g_i(w) = f(w) = g_{i+1}(w)$.

Let $\epsilon > 0$.
Since $(\Gamma_{H_i}(g_i))_{i \in I\!N}$ is a nested series of truncated subgames equivalent for player I, by Lemma 3.14 all the games have the same lower value, namely $\mathrm{val}^{\downarrow}(\Gamma_{H_0}(g_0)) = g_0(e) = v$, and there exists a strategy σ for player I that is ϵ-optimal in all the games $\Gamma_{H_i}(g_i)$, i.e. for any strategy τ, and any $i \in I\!N$,

$$E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(g_i)) \geq \mathrm{val}^{\downarrow}(\Gamma_{H_i}(g_i)) - \epsilon = v - \epsilon \tag{5.27}$$

Now let τ be any strategy for player II in $\Gamma(\max(f, I_D))$. Then

$$\begin{aligned} & E(\sigma \text{ vs } \tau \text{ in } \Gamma(\max(f, I_D))) \\ & \quad = \lim_{i\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(h_i)) && (5.28) \\ & \quad \geq \lim_{i\to\infty} E(\sigma \text{ vs } \tau \text{ in } \Gamma_{H_i}(g_i)) && (5.29) \\ & \quad \geq v - \epsilon && (5.30) \end{aligned}$$

So σ is a strategy for player I of value at least $v - \epsilon$. This implies that $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \geq v - \epsilon$. This construction can be done for any $\epsilon > 0$, hence

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \geq v \tag{5.31}$$

For any $O \supseteq D$, $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \leq \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O)))$, hence

$$\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) \leq \inf_{O \supseteq D, O \text{ open}} \mathrm{val}^{\downarrow}(\Gamma(\max(f, I_O))) = v \tag{5.32}$$

Hence $\mathrm{val}^{\downarrow}(\Gamma(\max(f, I_D))) = v$.

□

Corollary 5.3 *Let $f : W \to [0,1]$ be a measurable function and let D be a G_δ set. Suppose that $\Gamma(\max(f, I_D))$ has lower value v. Then for any $\epsilon > 0$, there exist an open set O, $D \subseteq O$, such that $\Gamma(\max(f, I_O))$ has lower value at most $v + \epsilon$.*

Proof:
This is merely a reformulation of (the non-trivial part of) Theorem 5.2

□

Corollary 5.4 *Let S be a measurable set, and let D be a G_δ set. Suppose that $\Gamma(S \cup D)$ has lower value v. Then for any $\epsilon > 0$, there exist an open set O, $D \subseteq O$, such that $\Gamma(S \cup O)$ has lower value at most $v + \epsilon$.*

Proof:
Take $f = I_S$ and apply Corollary 5.3.

□

5.3 $G_{\delta\sigma}$-sets

Theorem 5.5 *Let $S = \bigcup_i D_i$ be a $G_{\delta\sigma}$ set. Then $\Gamma(S)$ is determined.*

Proof concept:
As in the proof of Davis [5], each of the G_δ sets composing the $G_{\delta\sigma}$ set is enlarged to an open set using Corollary 5.4, in such a way that at all times the lower value is not increased by more than ϵ (compared to the original game), where ϵ is arbitrarily small. The resulting union of open sets is itself open, and hence determined, and furthermore Corollary 4.6 allows us to conclude that the total increase of the lower value is still not more than ϵ. This means that the upper value of the original game is also only at most ϵ more than the lower value.
Note that, unlike the previous proofs, this proof does not produce a strategy of the required value.
Proof:
Put $v = \mathrm{val}^\downarrow(\Gamma(S))$. Let $\epsilon > 0$. Using Corollary 5.4, we can find inductively open sets $O_i \supseteq D_i$ such that for all $j \in I\!N$,

$$\mathrm{val}^\downarrow(\Gamma(S \cup \bigcup_{i \le j+1} O_i)) \le \mathrm{val}^\downarrow(\Gamma(S \cup \bigcup_{i \le j} O_i)) + \epsilon/2^j \tag{5.33}$$

Then for all $j \in I\!N$,

$$\mathrm{val}^\downarrow(\Gamma(S \cup \bigcup_{i \le j} O_i)) \le v + \epsilon \tag{5.34}$$

and hence, for all $j \in I\!N$,

$$\mathrm{val}(\Gamma(\bigcup_{i \le j} O_i)) \le v + \epsilon \tag{5.35}$$

Then by Corollary 4.6,

$$\mathrm{val}(\Gamma(\bigcup_i O_i)) \le v + \epsilon \tag{5.36}$$

Since $S = \bigcup_i D_i \subseteq \bigcup_i O_i$,

$$\mathrm{val}^\uparrow(\Gamma(S)) \le \mathrm{val}^\uparrow(\Gamma(\bigcup_i O_i)) = \mathrm{val}(\Gamma(\bigcup_i O_i)) \le v + \epsilon \tag{5.37}$$

This is true for any ϵ, hence $\mathrm{val}^\uparrow(\Gamma(S)) = v$.

□

Remark 5.6 Theorem 5.1 shows that any G_δ set can be enlarged to an open set such that the value of the Blackwell game on that set is not increased by more than an arbitrarily small amount. The proof of Theorem 5.5 shows the same for $G_{\delta\sigma}$ sets.

A viable proposition is, that this holds for any Borel-measurable set. This is true in the case of games of Perfect Information. Such a game, on a Borel-set S, is determined and has value 0 or 1. If it has value 0 then player II has a winning strategy. The set of plays that cannot occur if player II uses that strategy, is open, and the game on that set has value 0 as well.

It is also true in the case of measures. For any measurable set S, there exist open sets $O \supseteq S$ whose measure is an arbitrarily small amount larger. The proof for measures is by induction on Borel complexity. Unfortunately, it does not go through for values of Blackwell games (or Borel games, for that matter), because if $S_1 \subseteq O_1$, $S_2 \subseteq O_2$, and

$$\mathrm{val}(O_1) \leq \mathrm{val}(S_1) + \epsilon_1, \mathrm{val}(O_2) \leq \mathrm{val}(S_2) + \epsilon_2 \tag{5.38}$$

then it does not always follow that

$$\mathrm{val}(O_1 \cup O_2) \leq \mathrm{val}(S_1 \cup S_2) + \epsilon_1 + \epsilon_2. \tag{5.39}$$

Chapter 6

Variations on a Theme

6.1 Payoff Variations

In the case of games of Perfect Information, games with multiple-valued payoff functions are not more complicated to analyze than games with two-valued payoff functions. For example, if f is a bounded Borel function, then the set $S_v = \{w \in W \mid f(w) \geq v\}$ is a Borel set for each value v, and hence the game (of Perfect Information) for that set is determined. A winning strategy in that game is a strategy of value v in the game of Perfect Information with payoff function f. Thus, for every value v, at least one player has a strategy of that value. It follows that the game of Perfect Information with payoff function f is determined.

Unfortunately this reasoning does not hold in the case of Blackwell Games. In games of Perfect Information, either one player wins, or the other wins. In Blackwell games, there is no such dichotomy, and the probabilistic results we get do not combine well. However, there are some specific results we can obtain.

Theorem 6.1 *Let f be a continuous function. Then $\Gamma(f)$ is determined.*

Proof:
By Remark 2.3 W is compact. Since f is continuous, this implies that $f[W]$ is compact, and hence bounded. Define $u : P \to I\!\!R$ by $u(p) := \inf_{w \supset p} f(w)$. Then u is well-defined and bounded, and by the continuity of f, $f(w) = \sup_{n \in I\!\!N} u(w_{|n})$ for all $w \in W$. Applying Theorem 4.4 yields the corollary.

□

Theorem 6.2 *Let H be a set of positions, such that no position in H precedes another position in H. Let $u : H \to [0,1]$ be a mapping, and let $f : W \to [0,1]$ be the payoff function defined by $f_n(w) = u(w_j)$ iff $w_j \in H$ for some $j \in I\!\!N$, $f_n(w) = 0$ otherwise. Then $\Gamma(f)$ is determined, and*

$$\mathrm{val}(\Gamma(f)) = \lim_{n\to\infty} \mathrm{val}(\Gamma_n(f_n)) \tag{6.1}$$

where $f_n(w) = u(w_j)$ iff $w_j \in H$ for some $j \leq n$, 0 otherwise.

Proof:
Extend the domain of u to P by setting, for $p \notin H$, $u(p) := 0$, and apply Theorem 4.4.

$\square$

Remark 6.3 Theorem 6.2 holds if all positions in H are assigned a nonnegative payoff (or if all positions in H are assigned a nonpositive payoff). If both positive and negative payoffs are assigned, the equality no longer holds, and while determinacy can still be proven, to my knowledge the only proof is as a corollary to a much more general result.

Theorem 6.4 *Let $u : P \to I\!R$ be a bounded function, and let $f : W \to I\!R$ be the payoff function defined by $f(w) = \limsup_{n \in I\!N} u(w_{|n})$. Then $\Gamma(f)$ is determined.*

Proof:
This follows from a result by Maitra and Sudderth [8] [9].

$\square$

Corollary 6.5 *Let H be a set of positions, such that no position in H precedes another position in H. Let $u : H \to I\!R$ be a bounded function, and let $f : W \to [0, 1]$ be the payoff function defined by $f_n(w) = u(w_j)$ iff $w_j \in H$ for some $j \in I\!N$, $f_n(w) = 0$ otherwise. Then $\Gamma(f)$ is determined.*

Proof:
Extend the domain of u to P by setting, for $p \notin H$, $u(p) := u(p')$ if $p' \subseteq p, p' \in H$, $u(p) = 0$ otherwise. Then apply Theorem 6.4.

$\square$

6.2 Pure Variations

We can view a pure strategy for player II, as a function assigning to each position $p \in P$ an element of Y. Let us denote the set of pure strategies for player II by T. Then $T = Y^P$. Given a probability distribution π on T, we can play a given Blackwell game by using π to pick a pure strategy, and then play according to that pure strategy. This 'strategy' corresponds to an ordinary mixed strategy, and vice versa.

Theorem 6.6 *For any probability distribution π on the set T of pure strategies we can find a mixed strategy τ for player II, and conversely for any mixed* *strategy τ for player II we can find a probability distribution π on T, such that in any Blackwell game $\Gamma(f)$, playing against any strategy σ,*

$$\int_{\tau' \in T} E(\sigma \text{ vs } \tau' \text{ in } \Gamma(f)) d\pi(\tau') = E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{6.2}$$

Proof:
Given a probability distribution π on T, we can define the strategy τ by setting, for $p = (x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in P$, $y_n \in Y$:

$$\tau(p)(y_n) = \frac{\pi\{\tau' \in T \mid \tau'(p_{|(i-1)}) = y_i \text{ for } i = 1, 2, \ldots, n-1\}}{\pi\{\tau' \in T \mid \tau'(p_{|(i-1)}) = y_i \text{ for } i = 1, 2, \ldots, n-1, n\}} \tag{6.3}$$

Given a strategy τ for player II, we can define the probability distribution π as the unique measure induced by setting, for different $p_1, p_2, \ldots, p_n \in P$, $y_1, y_2, \ldots, y_n \in Y$,

$$\pi\{\tau' \in T \mid \tau'(p_i) = y_i \text{ for } i = 1, 2, \ldots, n\} = \prod_{i=1}^{n} \tau(p_i)(y_i) \tag{6.4}$$

In both cases, τ and π now satisfy, for any position $p = (x_1, y_1, \ldots, x_n, y_n) \in P$,

$$\prod_{i=1}^{n} \tau(p_{|(i-1)})(y_i) = \pi\{\tau' \in T \mid \tau'(p_{|(i-1)}) = y_i \text{ for } i = 1, 2, \ldots, n\} \tag{6.5}$$

Now fix a strategy σ for player I. For any position $p = (x_1, y_1, \ldots, x_n, y_n) \in P$,

$$\begin{aligned}
&\int_{\tau' \in T} \mu_{\sigma,\tau'}\{w \mid w \supset p\} d\pi(\tau') \\
&= \int_{\tau' \in T} \left(\prod_{i=1}^{n} \sigma(p_{|(i-1)})(x_i) \bullet \tau'(p_{|(i-1)})(y_i) \right) d\pi(\tau') && (6.6) \\
&= \prod_{i=1}^{n} \sigma(p_{|(i-1)})(x_i) \bullet \int_{\tau' \in T} \prod_{i=1}^{n} \tau'(p_{|(i-1)})(y_i) d\pi(\tau') && (6.7) \\
&= \prod_{i=1}^{n} \sigma(p_{|(i-1)})(x_i) \bullet \pi\{\tau' \in T \mid \tau'(p_{|(i-1)}) = y_i \text{ for } i = 1, 2, \ldots, n\} && (6.8) \\
&= \prod_{i=1}^{n} \sigma(p_{|(i-1)})(x_i) \bullet \prod_{i=1}^{n} \tau(p_{|(i-1)})(y_i) && (6.9) \\
&= \mu_{\sigma,\tau}\{w \mid w \supset p\} && (6.10)
\end{aligned}$$

□

Taking the integral over T commutes with countable (bounded) summation, and since π is a probability distribution it commutes with subtraction from 1 as well. Since the sets $\{w \mid w \supset p\}, p \in P$ generate the Borel σ-algebra, by Sierpinski's Lemma it follows that for all Borel-sets S,

$$\int_{\tau' \in T} \mu_{\sigma,\tau'}(S) d\pi(\tau') = \mu_{\sigma,\tau}(S) \tag{6.11}$$

and hence for any bounded measurable function $f : W \to I\!\!R$,

$$\int_{\tau' \in T} \int_{w \in W} f(w) d\mu_{\sigma,\tau'}(w) d\pi(\tau') = \int_{w \in W} f(w) d\mu_{\sigma,\tau}(w) \tag{6.12}$$

which is equivalent to (6.2).

□

Corollary 6.7 *Let $\Gamma(f)$ be a Blackwell game, and let σ be a strategy for player I. Then player II has pure ϵ-optimal counterstrategies τ' against σ.*

Proof:
Let τ be an ϵ-optimal counterstrategy against σ, i.e. $E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) < \text{val}(\sigma \text{ in } \Gamma(f)) + \epsilon$. By Theorem 6.6, τ can be interpreted as a probability distribution π on the space of pure strategies, such that

$$\int_{\tau' \in T} E(\sigma \text{ vs } \tau' \text{ in } f) d\pi(\tau') = E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{6.13}$$

This implies that there exists a pure strategy τ' such that

$$E(\sigma \text{ vs } \tau' \text{ in } f) \leq E(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{6.14}$$

□

6.3 Axiomatic Variations

For Games of Perfect Information, there exists the Axiom of Determinacy, which states that any Game of Perfect Information with finite choice-of-moves and payoff function $f = I_S$ is determined. AD has many interesting consequences, such as the existence of an ultrafilter on $\aleph_1$, the existence of a complete measure on $I\!\!R$, the non-existence of a sequence of $\aleph_1$ reals, and the negation of the Axiom of Choice. We can formulate an analogue of AD with respect to Blackwell Games, and look at the consequences of that axiom. But AD is an axiom about games on all subsets of W, not just the Borel measurable subsets, and indeed, AD with respect only to games on Borel measurable subsets is provable from CAC [7], and hence not a very strong statement. In order to formulate a proper analogue, we first need to redefine the expectations and values of Blackwell Games, to include games with arbitrary, possibly non-measurable bounded payoff functions.

Definition 6.8 Let $\Gamma(f)$ be a Blackwell Game, where f is bounded but not necessarily Borel measurable. Let σ and τ be strategies for players I, II. σ and τ determine a probability measure $\mu_{\sigma,\tau}$ on W, induced by setting

$$\mu_{\sigma,\tau}\{w \mid w \supset p\} = \prod_{i=1}^{n} \left(\sigma(p_{|(i-1)})(x_i) \bullet \tau(p_{|(i-1)})(y_i)\right) \tag{6.15}$$

for any position $p = (x_1, y_1, \ldots, x_n, y_n) \in P$.
Instead of the expected income of player I, if she plays according to σ and player II plays according to τ, we now have the *lower* and *upper expected income* :

$$E^-(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \sup_{g \leq f, g \text{ measurable}} \int g(w) d\mu_{\sigma,\tau}(w) \tag{6.16}$$

$$E^+(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) = \inf_{g \geq f, g \text{ measurable}} \int g(w) d\mu_{\sigma,\tau}(w) \tag{6.17}$$

Lower value and upper value are redefined in the obvious way:

$$\mathrm{val}^{\downarrow}(\Gamma(f)) = \sup_{\sigma} \inf_{\tau} E^-(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{6.18}$$

$$\mathrm{val}^{\uparrow}(\Gamma(f)) = \inf_{\tau} \sup_{\sigma} E^+(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \tag{6.19}$$

Remark 6.9 Note that in the case that f is measurable, these definitions reduce to the old definitions.

Now we can axiomatize the determinacy of Blackwell Games:

Definition 6.10 The *Axiom of Determinacy of Blackwell Games* (AD-Bl) is the statement
for every pair of non-empty finite sets X, Y, and every bounded function f on $W = (X \times Y)^{\mathbb{N}}$, the Blackwell Game $\Gamma(f)$ is determined, i.e. $\mathrm{val}^{\downarrow}(\Gamma(f)) = \mathrm{val}^{\uparrow}(\Gamma(f))$.

Some consequences of AD-Bl are:

Theorem 6.11 *Assuming AD-Bl, it follows that there exists a complete measure on $\mathbb{R}$.*

Proof:
It suffices to show that there exists a complete measure on $R = \{0,1\}^{\mathbb{N}}$. Set $X = Y = \{0,1\}$, and define $\phi : W \to R$ by

$$\phi(x_1, y_1, x_2, y_2, \ldots) = (x_1 \oplus y_1, x_2 \oplus y_2, \ldots) \tag{6.20}$$

where $0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$. Now, for any two strategies σ, τ, the function ϕ induces a measure $\phi(\mu_{\sigma,\tau})$ on R, by setting for $S \subseteq R$,

$$\phi(\mu_{\sigma,\tau})(S) = \mu_{\sigma,\tau}(\phi^{-1}[S]) \tag{6.21}$$

If either σ or τ is the strategy which assigns to each position p the $\frac{1}{2}$-$\frac{1}{2}$ probability distribution on X or Y, respectively, then this induced measure is equal to the usual Lebesgue measure μ_L on R.

Now let S be a subset of $R = \{0,1\}^{\mathbb{N}}$, and consider the Blackwell Game $\Gamma(f)$, where $f = I_{\phi^{-1}[S]}$. For any two strategies σ, τ, the upper and lower expected income of player I are now equal to the inner and outer measure of $\phi^{-1}[S]$ under the measure $\mu_{\sigma,\tau}$, or the inner and outer measure of S under the induced measure $\phi(\mu_{\sigma,\tau})$ For any strategy σ there exists a strategy τ such that $E^-(\sigma \text{ vs } \tau \text{ in } \Gamma(f)) \leq \mu_L^{\text{inner}}(S)$, namely the strategy assigning the $\frac{1}{2}$-$\frac{1}{2}$ probability distribution to each position. This implies that $\text{val}^{\downarrow}(\Gamma(f)) \leq \mu_L^{\text{inner}}(S)$. Similarly, $\text{val}^{\uparrow}(\Gamma(f)) \geq \mu_L^{\text{outer}}(S)$. From the determinacy of $\Gamma(f)$, it now follows that, for any set $S \subseteq R$,

$$\mu_L^{\text{inner}}(S) = \mu_L^{\text{outer}}(S) \tag{6.22}$$

and hence, the Lebesgue measure can be extended to a complete measure on R.

□

Theorem 6.12 *AD-Bl is not consistent with AC*

Proof:
This follows from the previous Theorem, since AC implies the existence of a non-measurable set.

□

Remark 6.13 The only places in this paper where we use Choice are the proofs of Lemma's 3.11 and 3.14 where we choose and combine countably many strategies, and the proof of Theorem 5.5, where we inductively enlarge countably many G_δ sets to open sets. For the first purpose, CAC, the Countable Axiom of Choice, suffices, although the proof of Lemma 3.14 needs some rewriting. In order to prove Theorem 5.5 without using a stronger axiom such as DC, the Principle of Dependent Choice, we can adapt a proof given by Blackwell [3] to prove Corollary 5.4 in a more constructive manner, i.e. given a G_δ set D and associated set of positions H, we can use the adapted proof to construct an open set $O \supseteq D$ that is at most ϵ 'better' for player I, in a 'canonical' manner. CAC suffices to find, for a given $G_{\delta\sigma}$ set S, component sets D_i and associated sets of positions H_i, which then can be inductively enlarged without using more Choice. Hence, CAC suffices to prove Borel-determinacy up to complexity $G_{\delta\sigma}$ at least.

Theorem 6.14 *Assuming AD-Bl, it follows that $\aleph_1$ is a real-valued measurable cardinal, i.e. there exists a σ-complete nonatomic measure on $\aleph_1$.*

Proof:
This measure can be constructed in a manner similar to my own elementary

construction of an ultrafilter using AD [17]. The complete proof, however, is outside the scope of this paper.

□

Remark 6.15 An open problem is that of the relationship between AD and AD-Bl, whether AD follows from AD-Bl, or vice versa, or even whether AD-Bl follows from a stronger version of AD such as $AD_{I\!R}$. From a given (binary) game of Perfect Information, we can easily construct a Blackwell game that is 'equivalent', and assuming AD-Bl we can find an ϵ-optimal strategy for that equivalent Blackwell-game. However, to derive AD from AD-Bl, we need to have a pure strategy, and even though we can interpret the strategy we have as a probability distribution on pure strategies (Theorem 6.6), there is no guarantee that any of these pure strategies will function separately against *all* counterstrategies.

6.4 Half-Finite Variations

Theorem 6.16 *Let X be a finite set, Y an infinite set, or vice versa, and $f : X \times Y \to I\!R$ a bounded function. Then the 'Blackwell Game' $\Gamma_1(f)$ is determined.*

Proof:
Without loss of generality we may assume that X is finite, Y is infinite, and $f : X \times Y \to [0,1]$. Define for $n > 0$, the payoff function $f_n : X \times Y \to \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ by

$$f_n(x,y) = \frac{\lfloor f(x,y) \bullet n \rfloor}{n} \tag{6.23}$$

and the equivalence relation $\sim_n$ on Y by

$$y \sim_n y' \Leftrightarrow \forall x \in X : f_n(x,y) = f_n(x,y') \tag{6.24}$$

For any $n > 0$, the set of equivalence classes $Y/\sim_n$ is finite, of cardinality at most $(n+1)^{\#X}$. Consider the game where player I picks an $x \in X$, and simultaneously player II picks a $\bar{y} \in Y/\sim_n$, with payoff being equal to $f_n(x,y)$ for $y \in \bar{y}$. This is a one-round Blackwell game with finite choice of moves, and hence determined by Von Neumann's Minimax Theorem. But it is easily seen that strategies for this game translate to strategies of equal value for the game $\Gamma_1(f_n)$ and vice versa. We conclude that each game $\Gamma_1(f_n)$ is determined. By Lemma's 3.1 and 3.2, we have that for $n > 0$,

$$\mathrm{val}(\Gamma_1(f_n)) \leq \mathrm{val}^{\downarrow}(\Gamma_1(f)) \leq \mathrm{val}^{\uparrow}(\Gamma_1(f)) \leq \mathrm{val}(\Gamma_1(f_n)) + \frac{1}{n} \tag{6.25}$$

Therefore $\mathrm{val}^{\downarrow}(\Gamma_1(f)) = \mathrm{val}^{\uparrow}(\Gamma_1(f))$.

□

Theorem 6.17 *If we allow either player I or II to have an infinite selection of moves (i.e. X or Y is infinite, but not both), and S is an open or G_δ set, then $\Gamma(S)$ is determined.*

Proof:
Most of the proofs in Chapters 3, 4, 5 merely use the result of determinacy of one-round games, and are not concerned with the internal details of that proof. Using Theorem 6.16 instead of Von Neumann's Min-Max Theorem, by inspection all the results given in Chapters 3, 4 and 5 are still valid, with the exceptions of Corollary 3.5, Theorem 4.4 and Corollary 4.6, that depend on the finiteness of the sets W_n and compactness of W, and no longer hold, and the proof of Theorem 5.5 (the determinacy of $G_{\delta\sigma}$-games), which uses Corollary 4.6.

□

Example 6.18 It is essential that either X or Y is finite. If, for example, we take $X = Y = I\!N$, $S = \{(x, y) \in I\!N \times I\!N \mid x > y\}$, then the game $\Gamma_1(S)$ is not determined. For let σ be any strategy for player I, and let $\epsilon > 0$. We can interpret σ as a probability distribution on $I\!N$. As $\sum_{i=0}^{\infty} \sigma(i) = 1$, there is an $N \in I\!N$ such that $\sum_{i=0}^{N} \sigma(i) > 1 - \epsilon$. For player II, the strategy of playing $N+1$ beats σ with probability greater than $1-\epsilon$. It follows that σ has value less than ϵ for any $\epsilon > 0$, i.e. σ has value 0. Similarly, any strategy τ for player II has value 1. Hence, the game $\Gamma_1(S)$ is not determined.

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