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UNIFORM-FIELD THEORY OF PHASE INSTABILITIES IN ABSORPTIVE OPTICAL BISTABILITY

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Starting from the Maxwell–Bloch theory in the uniform-field approximation we investigate the stability of the phase of the output field for absorptive optical bistability in a Fabry–Pérot cavity. Our main results are analytical. We show that a truncation of the Maxwell–Bloch hierarchy introduces serious anomalies in the instability spectrum. For the full Maxwell–Bloch hierarchy phase instabilities are found to occur only if the ratio $\gamma_\parallel/\gamma_\perp$ of the medium damping coefficients is larger than 1. In that case phase instabilities can be present along the upper branch of the steady-state curve.

1. Introduction

In a recent article [1] we have carried out a linear stability analysis for absorptive optical bistability in a Fabry–Pérot cavity. The linearized Maxwell–Bloch equations were shown to fall apart into a set determining the stability of the amplitude of the output field and another set governing the stability of the phase of this field. Subsequently both infinitely dimensional sets were reduced to two-dimensional systems of linear differential equations. Invoking then the uniform-field approach we derived a solution of these equations for the amplitude case and performed a detailed analysis of the ensuing instability spectrum.

In the present paper we shall focus on the phase of the output field, completing in this way the stability analysis given in ref. [1]. Until recently [2] the phase stability problem for the Fabry–Pérot cavity has not received much attention. Yet it turns out that in order to get a comprehensive view on the instability spectrum in a Fabry–Pérot cavity it is imperative to study not only the stability problem for the amplitude but also that for the phase of the output field [2].

Of course it is true that uniform-field predictions of instabilities cannot be applied when it comes to obtaining agreement between theory and experiment. On the other hand it is important to have some analytical results at our disposal. The treatment we present offers a firm basis for the understanding of the numerical results we have obtained for the general phase stability problem lately [2]. Another reason to do the uniform-field calculations is that for the amplitude case the uniform-field theory has been analyzed extensively [3–5], so that a direct comparison is possible.

2. The phase stability problem in the uniform-field case

The phase stability problem for absorptive optical bistability in a Fabry–Pérot cavity can be formulated as follows [1]:

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{pmatrix}\Delta f\\ \Delta b\end{pmatrix}=\begin{pmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}\begin{pmatrix}\Delta f\\ \Delta b\end{pmatrix}, \tag{2.1}$$

$$\Delta f(\zeta=0)=R^{1/2}\,\Delta b(\zeta=0)\,,\qquad \Delta f(\zeta=1)=R^{-1/2}\,\Delta b(\zeta=1)\,. \tag{2.2}$$

The quantities Δf and Δb are the deviations of the phase of the forward and the backward electric field, respectively. The matrix elements H_{ij} depend on the eigenvalue λ of the stability problem and the stationary forward and backward electric fields f and b.

In general the fields f and b are functions of the spatial variable $\zeta = z/L$; in the uniform-field case, that is when the mirror transmission coefficient $T = 1 - R$ approaches zero at a constant value of the cooperation parameter C, one can write $f = b = x/2 + \mathcal{O}(T)$, with x the square root of the output intensity. Then the stability problem (2.1)–(2.2) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{pmatrix}\Delta f\\ \Delta b\end{pmatrix}=(\tilde{\lambda}^{(0)}+T\tilde{\lambda}^{(1)})\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Delta f\\ \Delta b\end{pmatrix}+CT\begin{pmatrix}A & B\\ -B & -A\end{pmatrix}\begin{pmatrix}\Delta f\\ \Delta b\end{pmatrix}, \tag{2.3}$$

$$\Delta f(0)=(1-\tfrac{1}{2}T)\,\Delta b(0)\,,\qquad \Delta f(1)=(1+\tfrac{1}{2}T)\,\Delta b(1)\,. \tag{2.4}$$

We neglected terms of $\mathcal{O}(T^2)$ and expanded the eigenvalue $\tilde{\lambda} = \lambda L/c$ in powers of T. The matrix in the second line of (2.3) is given by [1]

$$A=\frac{1+\lambda_\perp^{-1}}{4x^2}\,G+\tfrac{1}{2}(1-\lambda_\perp^{-1})D_N D_{\lambda,N} D_{\lambda,0}^{-1}+\lambda_\perp^{-1}D_0\,, \tag{2.5}$$

$$B=-\,\frac{1+\lambda_\perp^{-1}}{4x^2}\,G-\tfrac{1}{2}(1-\lambda_\perp^{-1})D_N D_{\lambda,N} D_{\lambda,0}^{-1}+\lambda_\perp^{-1}D_1\,, \tag{2.6}$$

with

$$G=-1+\frac{(1+4x^2)D_0}{\lambda_p-1}+\frac{D_{\lambda,0}^{-1}}{\lambda_p^{-1}-1}+\frac{2x^2(1+\lambda_p)}{1-\lambda_p}\,D_N D_{\lambda,N} D_{\lambda,0}^{-1}\,, \tag{2.7}$$

and the definitions $\lambda_p = \lambda_\perp \lambda_\parallel$ with $\lambda_i = 1 + \gamma_i^{-1}\lambda$ for $i = \perp, \parallel$. The symbols $\gamma_\perp$ and $\gamma_\parallel$ denote the medium damping coefficients. The quantities $\{D_m\}_{m=0}^{\infty}$ are the slowly varying envelopes of the harmonics of the population inversion. In deriving (2.5)–(2.7) we assumed that $D_m = 0$ if $m > N$, with N a positive integer.

The envelopes $\{D_m\}_{m=0}^{N}$ are determined by x according to the relations [1]

$$D_m = \frac{(-1)^{m+1} C_{N-m}}{(1+2x^2)C_N - 2x^2 C_{N-1}} , \tag{2.8}$$

$$C_m(x^2) = \frac{1}{u} T_m(u) + \frac{u-1}{u} U_m(u)$$
$$= \cos(m\theta) + \frac{\cos\theta - 1}{\sin\theta} \sin(m\theta) , \tag{2.9}$$

with $m = 0, 1, 2, \ldots, N$ and $u = \cos\theta = 1 + 1/(2x^2)$. From (2.8) the fields $\{D_{\lambda,m}\}_{m=0}^{N}$ can be obtained by making the substitution $x^2 \to \lambda_p^{-1} x^2$. At the right-hand side of (2.9) the Chebyshev polynomials of the first and the second kind occur [6].

If we solve the stability problem (2.3)–(2.4) in lowest order of T we find $\tilde{\lambda}^{(0)} = \pi n\mathrm{i}$, with n an integer. In first order we obtain for the resonant mode

$$\tilde{\lambda}_0^{(1)} = -\tfrac{1}{2} + CA(\lambda \to 0) + CB(\lambda \to 0) , \tag{2.10}$$

and for the side modes

$$\tilde{\lambda}_n^{(1)} = -\tfrac{1}{2} + CA(\lambda \to \pi n\mathrm{i}) , \tag{2.11}$$

with non-vanishing integer n. After substitution of (2.5)–(2.7) into (2.10) and taking the limit $\lambda \to 0$ one arrives at

$$\tilde{\lambda}_0^{(1)} = -\frac{1}{2} - C\,\frac{1 + D_0}{2x^2} = -\frac{y}{2x} , \tag{2.12}$$

where y is the square root of the input intensity [1]. From the last equality we see that the resonant mode is always stable against phase fluctuations.

3. Influence of truncation on phase instabilities

The influence of standing-wave effects on the side-mode instability spectrum can be assessed by analyzing this spectrum for various values of the truncation parameter N. We have performed such a program for amplitude instabilities in ref. [1]. The subject of the present section is to carry out a similar analysis for phase instabilities.

Insertion of (2.5) into (2.11) yields

$$\tilde{\lambda}_n^{(1)} = -\frac{1}{2} + C\,\frac{1+\lambda_\perp^{-1}}{4x^2}\left[-1 + \frac{(1+4x^2)D_0}{\lambda_p - 1} + \frac{D_{\lambda,0}^{-1}}{\lambda_p^{-1} - 1}\right]$$

$$+ C\lambda_\perp^{-1}D_0 + \frac{C}{2}\left[\frac{(1+\lambda_\perp^{-1})(1+\lambda_p)}{1-\lambda_p} + 1 - \lambda_\perp^{-1}\right]D_N D_{\lambda,N} D_{\lambda,0}^{-1}\,, \quad (3.1)$$

where λ_j must be taken equal to $1 + i\pi\tau_j$ with $\tau_j = nc/(L\gamma_j)$ for $j = \perp, \|$ and $n \neq 0$. If we choose $N = 1$ in (3.1) and calculate the real part of $\tilde{\lambda}_n^{(1)}$ we can prove immediately that it is always negative. Thus in lowest-order truncation phase instabilities do not occur, at least in the uniform-field approach. It has been shown earlier [3] that there are amplitude instabilities for $N = 1$.

From numerical work one learns that for values of N greater than unity the side modes do generate phase instabilities. This means that there is a qualitative difference between the cases $N = 1$ and $N > 1$. In order to analyze the latter case in more detail one can compute the boundaries of instability regions in the $(x, \tau_\perp)$-plane at fixed values of the cooperation parameter C and the ratio $d = \gamma_\|/\gamma_\perp$. The outcome of such a computation is most surprising: for $N > 1$ the result (3.1) gives rise to instability domains which are needle-shaped and which seem to be of infinite length. The last mentioned property is unphysical and asks for a careful examination of the right-hand side of (3.1).

On the basis of numerical explorations we propose to evaluate the eigenvalue $\tilde{\lambda}_n^{(1)}$ in the following limit:

$$\tau_\perp \to \infty\,, \qquad x \to \infty\,, \qquad \pi\tau_\perp/x = a\,, \quad (3.2)$$

with a and the ratio d fixed. For our purposes we must know the asymptotic behaviour of the envelopes D_0 and D_N. For $x \gg 1$ the Chebyshev form C_m can be written as

$$C_m(x^2) = 1 + \tfrac{1}{2}m(m+1)x^{-2} + \mathcal{O}(x^{-4})\,, \quad (3.3)$$

for $m = 0, 1, 2, \ldots, N$, so that we have with (2.8)

$$\lim_{x\to\infty} D_0 = \lim_{x\to\infty} (-1)^N D_N = \frac{-1}{2N+1}\,. \quad (3.4)$$

Using this result and (2.8), with x^2 replaced by $\lambda_p^{-1}x^2$, we can calculate each term of the right-hand side of (3.1) in the regime (3.2). This brings us to

$$\tilde{\lambda}_n^{(1)} = -\frac{1}{2} + C\lambda_{\rm p}^{-1}\,\frac{(2N-1)C_{\lambda,N} - (2N+1)C_{\lambda,N-1}}{2(2N+1)C_{\lambda,N}} - \frac{C\lambda_\perp^{-1}}{2N+1} + \frac{C\lambda_\perp^{-1}}{(2N+1)C_{\lambda,N}}\,, \tag{3.5}$$

where $C_{\lambda,m} = C_m(\lambda_{\rm p}^{-1}x^2)$. Clearly the third contribution of the right-hand side can be dropped. In the remaining terms one should insert

$$C_{\lambda,m} = C_m\Big(-\frac{d}{a^2}\Big) - \mathrm{i}\,\frac{d(d+1)}{a^3x}\,C_m'\Big(-\frac{d}{a^2}\Big) + \mathcal{O}(x^{-2})\,, \tag{3.6}$$

an expansion, which follows directly from

$$\lambda_{\rm p}^{-1}x^2 = -\frac{d}{a^2} - \mathrm{i}\,\frac{d(d+1)}{a^3x} + \mathcal{O}(x^{-2})\,. \tag{3.7}$$

The prime with C_m indicates that one must take the derivative with respect to the argument.

At this point it is easily seen that as long as the form $C_{\lambda,N}$ remains finite in the limit (3.2) the eigenvalue $\tilde{\lambda}_n^{(1)}$ converges to $-\frac{1}{2}$. In the case that the equality

$$C_N\Big(-\frac{d}{a^2}\Big) = 0 \tag{3.8}$$

is satisfied the limiting behaviour of the expression (3.5) for $\tilde{\lambda}_n^{(1)}$ must be considered separately. It appears that if $C_{\lambda,N}$ is of order x^{-1} the last term of (3.5) gives a finite contribution in the limit (3.2). This implies that $\tilde{\lambda}_n^{(1)}$ does not converge to $-\frac{1}{2}$. Instead the limiting expression becomes

$$\tilde{\lambda}_n^{(1)} = -\frac{1}{2} + \frac{Ca^2}{d(d+1)(2N+1)C_N'(-d/a^2)}\,, \tag{3.9}$$

where $C_{\lambda,m} = C_m(\lambda_{\rm p}^{-1}x^2)$. Clearly the third contribution of the right-hand side can be dropped. In the remaining terms one should insert

$$a = 2d^{1/2}\sin\phi_k\,, \tag{3.10}$$

with

$$\phi_k = \frac{2k+1}{2N+1}\,\frac{\pi}{2}\,, \tag{3.11}$$

for $k = 0, 1, 2, \ldots, N-1$. Obviously the final result for $\tilde{\lambda}_n^{(1)}$ follows from substitution of (3.10) into (3.9). The derivative in (3.9) can be calculated from (2.9) and reads

$$\frac{d}{a^4}\, C_N'\left(-\frac{d}{a^2}\right) = -\frac{N+1}{2d}\,\frac{\sin(N\theta)}{\sin\theta} + \frac{(1-\cos\theta)[\cos\theta\sin(N\theta) - N\sin\theta\cos(N\theta)]}{2d\sin^3\theta}\,, \tag{3.12}$$

with $\cos\theta = 1 - a^2/(2d)$. The substitution can be done directly now and leads to

$$\tilde{\lambda}_n^{(1)} = -\frac{1}{2} + \frac{2(-1)^{k+1} C\cos^2\phi_k}{(d+1)(2N+1)^2\sin\phi_k}\,. \tag{3.13}$$

The new contribution to $\tilde{\lambda}_n^{(1)}$ is positive for odd k.

The analysis of (3.1) has taught us that in the limit (3.2) side-mode instabilities are present for sufficiently high C if the constraint (3.10) is fulfilled for odd k. Thus indeed expression (3.1) may lead to instability domains of infinite length. For a given value of N there can be present $[N/2]$ infinitely long "needles" in the $(x, \tau_\perp)$-plane. The stability threshold of the needles is given by

$$C = \frac{d+1}{4}\,(2N+1)^2\,\frac{\sin\phi_k}{\cos^2\phi_k}\,, \tag{3.14}$$

with k odd. The threshold values for C increase monotonously with k. At $d = 1$ the value for $k = 1$ attains a minimum for $N = 3$, namely $C = 24.99$. The corresponding needle lies along the line $\tau_\perp = 0.3969x$. As a check we have done numerical work for $N = 3$ and reproduced these figures. The needle instabilities are observed to retreat towards infinity if C is lowered to its threshold value.

For a truncation parameter N much greater than unity the criterion (3.14) reduces to

$$C = \tfrac{3}{4}\pi(d+1)N + \mathcal{O}(1)\,, \tag{3.15}$$

where we took $k = 1$. From this result it is evident that the occurrence of infinitely long instability domains is an artifact resulting from the truncation of the Maxwell–Bloch hierarchy. We may thus conclude with the statement that, while for the amplitude case truncation introduces serious quantitative errors [1], its consequences for the phase instability spectrum are even more radical: after truncation this spectrum exhibits unphysical properties.

4. Phase instabilities for the complete Maxwell–Bloch hierarchy

To investigate whether phase instabilities can occur in the physical case $N \to \infty$ we start again from (3.1). In the limit $N \to \infty$ the envelope $D_{\lambda,N}$ vanishes. The envelopes D_0^{-1} and $D_{\lambda,0}^{-1}$ converge towards $-U = -(1+4x^2)^{1/2}$ and $-U_\lambda = -(1+4\lambda_p^{-1}x^2)^{1/2}$, respectively, with $\mathrm{Re}\, U_\lambda > 0$ [1]. Hence expression (3.1) becomes

$$\tilde{\lambda}_n^{(1)(\infty)} = -\frac{1}{2} - C\,\frac{1+\lambda_\perp^{-1}}{4x^2}\left(1 + \frac{U}{\lambda_p - 1} + \frac{U_\lambda}{\lambda_p^{-1} - 1}\right) - C\lambda_\perp^{-1}U^{-1} . \qquad (4.1)$$

Writing $\pi\tau_\perp = \alpha$ the real part of the right-hand side is found to equal

$$\mathrm{Re}\,\tilde{\lambda}_n^{(1)(\infty)} = -\frac{1}{2} - \frac{C}{4x^2}\left[\frac{2+\alpha^2}{1+\alpha^2} - \frac{d_2\,\mathrm{Re}\,U_\lambda}{d_1} - \frac{4(d-1)(1+\alpha^2)x^2 + d_4}{(1+\alpha^2)d_1 U} + \frac{2d(d+1)d_3x^2}{d_1 d_5\,\mathrm{Re}\,U_\lambda}\right], \qquad (4.2)$$

with the abbreviations $d_1 = (d+1)^2 + \alpha^2$, $d_2 = d^2 + d + \alpha^2 + 2$, $d_3 = 2d^2 + 2d + \alpha^2$, $d_4 = d^2 + d\alpha^2 + 3d$ and $d_5 = (d^2+\alpha^2)(1+\alpha^2)$. The real part of U_λ is

$$\mathrm{Re}\,U_\lambda = \left[\frac{1}{2} + \frac{2d(d-\alpha^2)x^2}{d_5} + \frac{1}{2}\left(1 + \frac{8d(d-\alpha^2)x^2 + 16d^2x^4}{d_5}\right)^{1/2}\right]^{1/2} . \qquad (4.3)$$

We shall prove now that the right-hand side of (4.2) is always negative for $0 < d \leqslant 1$. Since it can be shown easily that $\mathrm{Re}\,\tilde{\lambda}_n^{(1)(\infty)} < 0$ for $\alpha = 0$, we choose $\alpha > 0$. We shall consider the expression

$$\Lambda = \frac{2+\alpha^2}{1+\alpha^2} - \frac{d_2\,\mathrm{Re}\,U_\lambda}{d_1} - \frac{4(d-1)(1+\alpha^2)x^2 + d_4}{(1+\alpha^2)d_1 U} + \frac{2d(d+1)d_3x^2}{d_1 d_5\,\mathrm{Re}\,U_\lambda} . \qquad (4.4)$$

It is sufficient to demonstrate that Λ is nonnegative for $0 < d \leqslant 1$.

Employing the trivial result $d - \alpha^2 \leqslant d_5^{1/2}$ one obtains from (4.3) the following inequality:

$$\mathrm{Re}\,U_\lambda \leqslant \left[1 + \frac{2dx^2(d - \alpha^2 + d_5^{1/2})}{d_5}\right]^{1/2} =: V . \qquad (4.5)$$

Because of the fact that $d_5^{1/2} \geqslant \alpha^2$ the expression between the square brackets is always positive. After replacing Re U_λ by V in (4.4) we find

$$\Lambda \geq \frac{2+\alpha^2}{1+\alpha^2} - \frac{d_4}{(1+\alpha^2)d_1 U} - \frac{d_2}{d_1 V} + \frac{2x^2}{d_1} F , \tag{4.6}$$

with

$$F = \frac{2(1-d)}{U} - \frac{dd_2}{d_5^{1/2} V} + \frac{d(d+1)d_3 - d(d-\alpha^2)d_2}{d_5 V} . \tag{4.7}$$

The sum of the first three terms at the right-hand side of (4.6) attains its minimum value for $x = 0$. Since this value amounts to zero it follows that this expression is nonnegative. Next we focus on F. With the help of the inequality

$$U^{-1} \geq \frac{d}{d_5^{1/2} V} \tag{4.8}$$

we can eliminate U in (4.7) and derive in this way

$$d^{-1} d_5 V F \geq \alpha^4 + \alpha^2(d^2 + d + 3) + d^3 + 3d^2 - d_5^{1/2}(\alpha^2 + d^2 + 3d) , \tag{4.9}$$

for $0 < d \leq 1$. Since the right-hand side is nonnegative for these values of d the proof is complete.

If d is greater than unity the system can become unstable against phase fluctuations. For the proof of this statement we assume $x \gg 1$ and $\alpha = \mathcal{O}(1)$. The relation (4.3) then reduces to

$$\text{Re } U_\lambda = \left[\frac{2d(d - \alpha^2 + d_5^{1/2})}{d_5} \right]^{1/2} x + \mathcal{O}(1) . \tag{4.10}$$

The root with square brackets will be called $\bar{V}$. After substitution of (4.10) into (4.4) the leading term of Λ can be calculated. It reads

$$\Lambda_0 = \frac{2(1-d)x}{d_1} + \frac{2dx}{d_1 d_5 \bar{V}} [(d+1)d_3 - d_2(d - \alpha^2 + d_5^{1/2})] . \tag{4.11}$$

Using in the first term of the right-hand side the inequality

$$(1-d) \leq \frac{2d(1-d)d_5^{1/2}}{d_5 \bar{V}} , \tag{4.12}$$

valid for $d > 1$, we arrive at

Table I
Values for the cooperation parameter C at which side-mode instabilities appear (first row) and at which upper-branch instabilities appear (second row).

d					
1.0	1.2	1.4	1.6	1.8	2.0
∞	2930	890	471	311	231
∞	3090	941	501	332	248

$$\Lambda_0 \leqslant \frac{2dx}{d_1 d_5 \bar{V}} [\alpha^4 + \alpha^2(d^2 + d + 3) + d^3 + 3d^2 - d_5^{1/2}(\alpha^2 + d^2 + 3d)] . \tag{4.13}$$

The factor between the square brackets has been encountered already in (4.9). For $d > 1$ this factor is negative. As a consequence, Re $\tilde{\lambda}_n^{(1)(\infty)}$ is positive for $x \gg 1$, $d > 1$, $\alpha = \mathcal{O}(1)$ and C sufficiently large.

We have numerically calculated the values for the cooperation parameter C at which instabilities come into being in the $(x, \tau_\perp)$-plane for various d. The results are displayed in table I; indeed the critical value for C diverges as d approaches unity.

In conclusion, we compare our uniform-field predictions on instabilities in absorptive optical bistability with those for a unidirectional ring cavity. It is well known that in the absence of a backward electric field phase instabilities do not occur [1, 7], while fluctuations in the amplitudes of the fields can be responsible for so-called positive-slope instabilities [8]. Now, if one incorporates standing-wave effects in the theory the picture becomes markedly different. First of all, amplitude instabilities are restricted to the negative-slope part of the bistability curve [1, 5]. Secondly, phase instabilities are no longer absent; for suitable values of the cooperation parameter they are even predicted along a part of the upper branch of the bistability curve. The values of C at which these positive-slope instabilities emerge are listed in table I. In assessing the relevance of positive-slope instabilities in the phase one should bear in mind that in the full nonlinear Maxwell–Bloch equations the phases of the fields are coupled to the amplitudes. Therefore, one expects that an instability which arises in the phase of the output signal also affects the output intensity as the instability grows in time.

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