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Hierarchies of sum rules for squares of spherical Bessel functions

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ABSTRACT
A four-term recurrence relation for squared spherical Bessel functions is shown to yield closed-form expressions for several types of finite weighted sums of these functions. The resulting sum rules, which may contain an arbitrarily large number of terms, are found to constitute three independent hierarchies. Their use leads to an efficient numerical evaluation of these sums.

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1. Introduction

Infinite sums of squares of spherical Bessel functions $j_k(z)$ of the form $\sum_k c_k[j_k(z)]^2$, with various coefficients c_k, have been studied in quite some detail. Several simple examples can be found in standard texts [1,2]. A more extensive list of such sums has been compiled in [3], mainly in the form of expansions of generalized hypergeometric functions. In contrast, information on finite sums of this type, with an arbitrary large but finite number of terms, is less well available. Such finite sums occur in various branches of mathematical physics, for instance in atomic orbital theory [4], in acoustic diffraction problems [5] and more recently in quantum optics [6].

In the present paper, we will determine a collection of closed-form expressions for finite sums of squares of spherical Bessel functions $j_k(z)$ with z-independent coefficients c_k. Our main results are (3.3), (3.9) and (3.20).

2. Lowest order sum rules

The standard relations connecting three spherical Bessel functions $j_k(z)$ of contiguous order can be employed to derive a four-term recurrence relation for their squares:

$$(2k-1)[j_{k-1}(z)]^2-(2k+1)[j_k(z)]^2=\frac{z^2}{2k-1}\{[j_{k-2}(z)]^2-[j_k(z)]^2\}+\frac{z^2}{2k+1}\{[j_{k-1}(z)]^2-[j_{k+1}(z)]^2\}. \tag{2.1}$$

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In fact, the proof follows by elimination of $j_{k-2}(z)$ and $j_{k+1}(z)$ in favour of $j_{k-1}(z)$ and $j_k(z)$ (see [1, formula 10.1.19]). Multiplying this identity by an as yet undetermined coefficient a_k, summing over k and shifting the summation variables, one finds a relation between finite sums:

$$\sum_{k=0}^{\ell} f_k[j_k(z)]^2 = z^2 \sum_{k=0}^{\ell} g_k[j_k(z)]^2 - z^2 \frac{1}{2\ell+3}(a_{\ell+1}+a_{\ell+2})[j_\ell(z)]^2 - z^2 \frac{1}{2\ell+1}(a_{\ell+1}+a_\ell)[j_{\ell+1}(z)]^2 + 2za_{\ell+1}j_\ell(z)j_{\ell+1}(z) + F(z), \quad (2.2)$$

with $\ell \geq 0$, $f_k = (2k+1)(a_{k+1}-a_k)$, and $g_k = (a_{k+2}+a_{k+1})/(2k+3) - (a_k + a_{k-1})/(2k-1)$. The ℓ-dependent terms at the right-hand side follow upon adjusting the upper bounds of the sums. The last term results by completing the sums at their lower bounds; it can easily be found by putting $\ell = 0$. The term with $j_\ell(z)j_{\ell+1}(z)$ arises upon using the equality

$$zj_\ell(z)j_{\ell+1}(z) = \frac{1}{2}z^2 \frac{1}{2\ell+1}\{[j_{\ell+1}(z)]^2 - [j_{\ell-1}(z)]^2\} + \frac{1}{2}(2\ell+1)[j_\ell(z)]^2, \quad (2.3)$$

which is established by elimination of $j_{\ell-1}(z)$ in a similar way as in (2.1).

The identity (2.2) gives a relation between two sums. In general, it does not yield a closed-form expression for any of these. However, one may derive explicit sum rules in two different ways: either by choosing a_k in such a way that the left-hand side vanishes, or such that the sum at the right-hand side drops out. In the first case the coefficients a_k should satisfy the relation $f_k = 0$ or $a_{k+1} = a_k$, so that one may take $a_k = 1$ for all k. In this way one arrives at the sum rule:

$$\sum_{k=0}^{\ell} \frac{1}{(2k-1)(2k+3)}[j_k(z)]^2 = -\frac{1}{4(2\ell+3)}[j_\ell(z)]^2 - \frac{1}{4(2\ell+1)}[j_{\ell+1}(z)]^2 + \frac{1}{4z}j_\ell(z)j_{\ell+1}(z) - \frac{1}{2z}j_1(2z), \quad (2.4)$$

where the last term is found by choosing $\ell = 0$.

In the other case the coefficients a_k must fulfill the condition $g_k = 0$ for all k. Solving the ensuing recurrence relation for a_k, by first introducing $b_k = a_{k-1} + a_k$ and evaluating b_k for even and odd k separately, one finds a_k in terms of the initial conditions a_0, a_1 and a_2 as

$$a_k = [\tfrac{1}{2}k + (-1)^{k+1}(\tfrac{1}{2}k^2 - 1)]a_0 + [\tfrac{2}{3}k + \tfrac{1}{3}(-1)^{k+1}k^2]a_1 + [\tfrac{1}{6}k + \tfrac{1}{6}(-1)^k k^2]a_2. \quad (2.5)$$

As a consequence, the coefficient f_k in the sum at the left-hand side of (2.2) gets the form

$$f_k = (2k+1)\{[\tfrac{1}{2} + (-1)^k(k^2 + k - \tfrac{3}{2})]a_0 + [\tfrac{2}{3} + (-1)^k(\tfrac{2}{3}k^2 + \tfrac{2}{3}k + \tfrac{1}{3})]a_1 + [\tfrac{1}{6} + (-1)^{k+1}(\tfrac{1}{3}k^2 + \tfrac{1}{3}k + \tfrac{1}{6})]a_2\}. \quad (2.6)$$

By making specific choices for a_0, a_1 and a_2, one may arrive at sum rules with either alternating coefficients (proportional to $(-1)^k$) or non-alternating coefficients. The latter type shows up by choosing $a_0 = 0$ and $a_2 = 2a_1$. Taking $a_1 = 1$ one gets $a_k = k$ and

$a_{k+1} - a_k = 1$. As a consequence, one arrives at a second sum rule with non-alternating coefficients:

$$\sum_{k=0}^{\ell}(2k+1)[j_k(z)]^2 = -z^2[j_\ell(z)]^2 - z^2[j_{\ell+1}(z)]^2 + 2(\ell+1)zj_\ell(z)j_{\ell+1}(z) + 1, \quad (2.7)$$

where the last term is obtained by taking $\ell = 0$, as before.

Further sum rules, with alternating coefficients, arise by choosing $a_2 = -3a_0 - 4a_1$, while a_0 and a_1 can still be chosen at will. One possible choice is $a_0 = -a_1 = 1$, which implies $a_k = (-1)^k$. The ensuing sum rule following from (2.2) is

$$\sum_{k=0}^{\ell}(-1)^k(2k+1)[j_k(z)]^2 = (-1)^\ell zj_\ell(z)j_{\ell+1}(z) + j_0(2z). \quad (2.8)$$

Finally, one may take a_k to be proportional to a quadratic polynomial in k. Upon putting $a_0 = a_1 = 1$ we get $a_k = (-1)^{k+1}(2k^2 - 1)$. This choice yields the last sum rule of our set:

$$\sum_{k=0}^{\ell}(-1)^k(2k+1)k(k+1)[j_k(z)]^2 = \frac{1}{2}(-1)^\ell z^2[j_\ell(z)]^2 - \frac{1}{2}(-1)^\ell z^2[j_{\ell+1}(z)]^2 + (-1)^\ell\left(\ell^2 + 2\ell + \frac{1}{2}\right)zj_\ell(z)j_{\ell+1}(z) - zj_1(2z). \quad (2.9)$$

In conclusion, by making judicious choices for the coefficients a_k in (2.2) we have derived several independent sum rules for the squares of the spherical Bessel functions: two rules with non-alternating coefficients, namely (2.4) and (2.7), and two closely related ones with alternating coefficients, namely (2.8) and (2.9). In the following we shall see that these sum rules can be used as a basis from which three independent hierarchies of sum rules can be established.

3. Hierarchies of sum rules

The sum rules (2.4) and (2.7)–(2.9) are part of several hierarchies of sum rules. These hierarchies follow by using (2.2) for suitable a_k as a recurrence relation. As a first example we shall start from (2.4), by choosing the coefficient a_k in (2.2) such that at its left-hand side the sum found in (2.4) shows up. Apart from a trivial factor this implies that a_k should fulfill the relation $a_{k+1} - a_k = 1/(k - \frac{1}{2})_3$, with $(f)_n = f(f+1)\cdots(f+n-1)$ the Pochhammer symbol. Solving for a_k we find $a_k = -1/[2(k - \frac{1}{2})_2]$, where we chose the initial condition as $a_0 = 2$. Inserting this form for a_k in the sum at the right-hand side of (2.2) we obtain as its coefficient $g_k = 3(2k+1)/[2(k - \frac{3}{2})_5]$. Hence, the relation (2.2) yields an expression for a sum with a coefficient proportional to $(2k+1)/(k - \frac{3}{2})_5$, which is the next in a hierarchy of sum rules of which (2.4) was the first. In fact, the above procedure can be repeated. By choosing $a_k^{(p)}$ in (2.2) in such a way that the coefficient in the sum at the left-hand side is $f_k^{(p)} = (2k+1)/(k - p - \frac{1}{2})_{2p+3}$ (for arbitrary integer $p \geq 0$), we arrive at a coefficient

$g_k^{(p)}$ at the right-hand side that is proportional to $(2k+1)/(k-p-\frac{3}{2})_{2p+5}$, for a suitable choice of the initial condition. To achieve this one should take

$$a_k^{(p)} = -\frac{1}{2(p+1)(k-p-\frac{1}{2})_{2p+2}}, \tag{3.1}$$

for $p \geq 0$ and all k. With this choice of $a_k^{(p)}$ the relation (2.2) becomes for $p \geq 0$ and $\ell \geq 0$:

$$\begin{aligned}\sum_{k=0}^{\ell} \frac{2k+1}{(k-p-\frac{3}{2})_{2p+5}}[j_k(z)]^2 = {} & \frac{2(p+1)}{z^2(2p+3)} \sum_{k=0}^{\ell} \frac{2k+1}{(k-p-\frac{1}{2})_{2p+3}}[j_k(z)]^2 \\ & - \frac{1}{(2p+3)(\ell-p+\frac{1}{2})_{2p+3}}[j_\ell(z)]^2 \\ & - \frac{1}{(2p+3)(\ell-p-\frac{1}{2})_{2p+3}}[j_{\ell+1}(z)]^2 \\ & + \frac{2}{z(2p+3)(\ell-p+\frac{1}{2})_{2p+2}} j_\ell(z) j_{\ell+1}(z) \\ & + \frac{1}{(-p-\frac{3}{2})_{2p+4}} \left(\frac{p+1}{z^2} j_0(2z) + \frac{1}{z} j_1(2z) \right), \end{aligned} \tag{3.2}$$

where the last term has been determined by putting $\ell = 0$. Upon using this identity repeatedly we arrive at a *first* hierarchy of sum rules of the form:

$$\begin{aligned}& \sum_{k=0}^{\ell} \frac{2k+1}{(k-p-\frac{1}{2})_{2p+3}}[j_k(z)]^2 \\ & \quad = z^2 A_\ell^{[1],(p)}(z)[j_\ell(z)]^2 + z^2 B_\ell^{[1],(p)}(z)[j_{\ell+1}(z)]^2 + z C_\ell^{[1],(p)}(z) j_\ell(z) j_{\ell+1}(z) \\ & \qquad + \frac{1}{(-p-\frac{1}{2})_{2p+2}} \sum_{k=0}^{p} (-1)^k \frac{(p-k+1)_k}{z^{k+1}} j_{k+1}(2z), \end{aligned} \tag{3.3}$$

for any $p \geq 0$ and $\ell \geq 0$. The coefficients at the right-hand side are polynomials in $1/z^2$:

$$A_\ell^{[1],(p)}(z) = B_{\ell+1}^{[1],(p)}(z) = -\frac{1}{2} \sum_{k=0}^{p} \frac{(p-k+1)_k}{(p-k+\frac{1}{2})_{k+1}(\ell-p+k+\frac{3}{2})_{2p-2k+1}} \frac{1}{z^{2k+2}}, \tag{3.4}$$

$$C_\ell^{[1],(p)}(z) = \sum_{k=0}^{p} \frac{(p-k+1)_k}{(p-k+\frac{1}{2})_{k+1}(\ell-p+k+\frac{3}{2})_{2p-2k}} \frac{1}{z^{2k+2}}. \tag{3.5}$$

The last term in (3.3) is obtained from (3.2) by using the recurrence relations for $j_k(2z)$. For $p=0$ one recovers the sum rule (2.4), which we have used as our starting-point. For small $p>0$ and arbitrary ℓ the coefficients of the squared spherical Bessel functions at the right-hand side of (3.3) are small-degree polynomials in $1/z$ that are easily evaluated. For general $p \geq 0$ the sum rules (3.3), with spherical Hankel functions instead of spherical

Bessel functions, were of crucial importance in the analysis of the modified atomic decay rates in [6].

A rather different hierarchy follows by starting from the sum rule (2.7) and choosing $g_k = 2k+1$ in (2.2). Solving for a_k one finds $a_k = \frac{1}{2}(k-1)_3$ for all k, when a suitable choice of initial conditions is made. Subsequently, f_k is obtained as $f_k = \frac{3}{2}(2k+1)(k)_2$, so that (2.2), with (2.7) inserted at the right-hand side, leads to a sum rule for ℓ-dependent sums with a coefficient $(2k+1)(k)_2$. The procedure can be generalized by taking

$$a_k^{(p)} = \frac{(k-p-1)_{2p+3}}{2(p+1)}, \tag{3.6}$$

for $p \geq 0$ and all k, and hence

$$f_k^{(p)} = \frac{2p+3}{2(p+1)}(2k+1)(k-p)_{2p+2}, \quad g_k^{(p)} = (2k+1)(k-p+1)_{2p}. \tag{3.7}$$

In this way we get from (2.2) a relation between sums of similar form:

$$\begin{aligned}
&\frac{2p+3}{2(p+1)} \sum_{k=p+1}^{\ell} (2k+1)(k-p)_{2p+2}[j_k(z)]^2 \\
&= z^2 \sum_{k=p}^{\ell} (2k+1)(k-p+1)_{2p}[j_k(z)]^2 - \frac{z^2}{2(p+1)}(\ell-p+1)_{2p+2}[j_\ell(z)]^2 \\
&\quad - \frac{z^2}{2(p+1)}(\ell-p)_{2p+2}[j_{\ell+1}(z)]^2 + \frac{z}{p+1}(\ell-p)_{2p+3}\, j_\ell(z) j_{\ell+1}(z),
\end{aligned} \tag{3.8}$$

for $\ell \geq p \geq 0$. The last term in (2.2) is found to be 0 in this case, as follows by putting $\ell = p$. By employing this identity recursively and using (2.7), one arrives at a *second* hierarchy of sum rules with non-alternating coefficients, on a par with (3.3):

$$\begin{aligned}
\sum_{k=p}^{\ell} (2k+1)(k-p+1)_{2p}[j_k(z)]^2 &= z^2 A_\ell^{[2],(p)}(z)[j_\ell(z)]^2 + z^2 B_\ell^{[2],(p)}(z)[j_{\ell+1}(z)]^2 \\
&\quad + z C_\ell^{[2],(p)}(z) j_\ell(z) j_{\ell+1}(z) + \frac{p!}{(\frac{3}{2})_p} z^{2p},
\end{aligned} \tag{3.9}$$

for all $\ell \geq p \geq 0$. The coefficients at the right-hand side are polynomials in z^2:

$$A_\ell^{[2],(p)}(z) = B_{\ell+1}^{[2],(p)}(z) = -\frac{1}{2} \sum_{k=0}^{p} \frac{(p-k+1)_k(\ell-p+k+2)_{2p-2k}}{(p-k+\frac{1}{2})_{k+1}} z^{2k}, \tag{3.10}$$

$$C_\ell^{[2],(p)}(z) = \sum_{k=0}^{p} \frac{(p-k+1)_k(\ell-p+k+1)_{2p-2k+1}}{(p-k+\frac{1}{2})_{k+1}} z^{2k}. \tag{3.11}$$

For $p=0$ the sum rule (3.9) reduces to (2.7), which served as the basis of the hierarchy. For small $p>0$ the z-dependent polynomials occurring at the right-hand side of (3.9) have

got a small degree, as in (3.3). Sum rules closely related to (3.9), with spherical Hankel functions as before, have been used in the quantum optics problem in [6]. Comparing the two hierarchies (3.3) and (3.9) we see that the coefficients in the weighted sums differ considerably. Whereas the coefficient in (3.3) contains an odd number of factors in the denominator, the Pochhammer symbol in (3.9) leads to an even number of factors in the numerator, at least if the common factor $2k+1$ is left out of consideration. Furthermore, the factors in (3.3) are half-integer (so that no singularities can arise), and in (3.9) they are all integer. In both cases the number of factors increases with p.

Finally, we can build a hierarchy of sum rules with alternating coefficients by starting from (2.8) and (2.9). When we choose a_k to have the somewhat elaborate form

$$a_k^{(p)} = (-1)^k \frac{k}{2} \sum_{m=0}^{p} c_m^{(p)} \frac{1}{(m+1)(m+2)} (k-m-1)_{2m+3}, \tag{3.12}$$

with

$$c_m^{(p)} = \frac{1}{2^{2p-2m} m!(p-m)!} (2m-p+2)_{2p-2m}, \tag{3.13}$$

the coefficients f_k and g_k are found as

$$f_k^{(p)} = (-1)^{k+1}(2k+1) \sum_{m=0}^{p+2} c_m^{(p+2)} (k-m+1)_{2m}, \tag{3.14}$$

$$g_k^{(p)} = (-1)^{k}(2k+1) \sum_{m=0}^{p} c_m^{(p)} (k-m+1)_{2m}, \tag{3.15}$$

for all $p \geq 0$ and $k \geq 0$. In deriving (3.14) we have used the recurrence relation $c_m^{(p+2)} = c_{m-2}^{(p)}/[m(m-1)] + c_{m-1}^{(p)}(2m+1)/(2m)$ for $m \geq 2$ and $p \geq 0$. Since the expressions (3.14) and (3.15) are closely analogous, with $f_k^{(p)} = -g_k^{(p+2)}$, one may use (2.2) to derive a recurrence relation for sums of a similar type:

$$\begin{aligned}
&\sum_{k=0}^{\ell} (-1)^k (2k+1) \left[\sum_{m=0}^{p+2} c_m^{(p+2)} (k-m+1)_{2m} \right] [j_k(z)]^2 \\
&\quad = -z^2 \sum_{k=0}^{\ell} (-1)^k (2k+1) \left[\sum_{m=0}^{p} c_m^{(p)} (k-m+1)_{2m} \right] [j_k(z)]^2 \\
&\qquad + \frac{1}{2}(-1)^\ell z^2 \left[\sum_{m=0}^{p} c_m^{(p)} \frac{1}{m+1} (\ell-m+1)_{2m+2} \right] [j_\ell(z)]^2 \\
&\qquad - \frac{1}{2}(-1)^\ell z^2 \left[\sum_{m=0}^{p} c_m^{(p)} \frac{1}{m+1} (\ell-m)_{2m+2} \right] [j_{\ell+1}(z)]^2 \\
&\qquad + (-1)^\ell (\ell+1) \left[\sum_{m=0}^{p} c_m^{(p)} \frac{1}{(m+1)(m+2)} (\ell-m)_{2m+3} \right] z j_\ell(z) j_{\ell+1}(z).
\end{aligned} \tag{3.16}$$

Once again the last term in (2.2) drops out, as follows by taking $\ell = 0$. Starting from the sum rules (2.8) and (2.9) and using the recurrence relation separately for even and odd values of p we may obtain explicit sum rules for all p. For any $p \geq 0$ and $\ell \geq 0$ we get:

$$\begin{aligned}
&\sum_{k=0}^{\ell}(-1)^k(2k+1)\left[\sum_{m=0}^{p} c_m^{(p)}(k-m+1)_{2m}\right][j_k(z)]^2 \\
&\quad = \frac{1}{2}(-1)^\ell\left[\sum_{m=1}^{[p/2]}(-1)^{m+1}z^{2m}\sum_{n=0}^{p-2m} c_n^{(p-2m)}\frac{1}{n+1}(\ell-n+1)_{2n+2}\right. \\
&\qquad \left. + \delta_p^o(-1)^{(p-1)/2}z^{p+1}\right][j_\ell(z)]^2 \\
&\qquad + \frac{1}{2}(-1)^\ell\left[\sum_{m=1}^{[p/2]}(-1)^{m}z^{2m}\sum_{n=0}^{p-2m} c_n^{(p-2m)}\frac{1}{n+1}(\ell-n)_{2n+2}\right. \\
&\qquad \left. + \delta_p^o(-1)^{(p+1)/2}z^{p+1}\right][j_{\ell+1}(z)]^2 \\
&\qquad + (-1)^\ell\left[(\ell+1)\sum_{m=1}^{[p/2]}(-1)^{m+1}z^{2m-1}\sum_{n=0}^{p-2m} c_n^{(p-2m)}\frac{1}{(n+1)(n+2)}(\ell-n)_{2n+3}\right. \\
&\qquad \left. + \delta_p^e(-1)^{p/2}z^{p+1} + \delta_p^o(-1)^{(p-1)/2}z^p(\ell+1)^2\right]j_\ell(z)j_{\ell+1}(z) \\
&\qquad + \delta_p^e(-1)^{p/2}z^p j_0(2z) + \delta_p^o(-1)^{(p-1)/2}z^{p-1}\left[\frac{1}{2}j_0(2z) - zj_1(2z)\right].
\end{aligned} \tag{3.17}$$

Here δ_p^e equals 1 for even p, and 0 for odd p, while δ_p^o is defined analogously, with even and odd interchanged. The upper bounds of the summations contain the 'entier' function $[x]$ which is the largest integer $\leq x$. For $p=0$ the sum rule (3.17) reduces to (2.8), whereas for $p=1$ a linear combination of (2.8) and (2.9) is recovered.

By taking suitable linear combinations of the sum rules (3.17) for various values of p, we may obtain expressions for sums with the simple coefficients $(-1)^k(2k+1)(k-p+1)_{2p}$. In fact, we may use the identity for $q \geq m$:

$$\sum_{p=m}^{q} f_p^{(q)} c_m^{(p)} = \delta_{m,q} \quad \text{with} \quad f_p^{(q)} = (-1)^{p+q}\frac{q!(2q-p)!}{2^{2q-2p}p!(q-p)!}, \tag{3.18}$$

which can be proved by employing a relation (due to Dzhrbashyan [7]) for a terminating generalized hypergeometric function ${}_3F_2(1)$ with unit argument (see also [3], formula 3.13.3(9)). We now take the sum $\sum_{p=0}^{q} f_p^{(q)}$ of (3.17) and use (3.18) at the left-hand side. In the first term at the right-hand side we interchange the order of the summations in such a way that the sum over p can be carried out first. Substitution of (3.13) then leads to an

expression that can be evaluated with the help of [7]:

$$\begin{aligned}
&\sum_{p=2m+n}^{q}(-1)^p\frac{(2q-p)!}{p!(q-p)!(p-2m-n)!}(2n-p+2m+2)_{2p-4m-2n}\\
&\quad=(-1)^n\frac{(2q-2m-n)!}{(2m+n)!(q-2m-n)!}\\
&\qquad\times{}_3F_2(-q+2m+n,n+2,-n-1;-2q+2m+n,2m+n+1;1)\\
&\quad=(-1)^n\frac{2^{2q-4m-2n}}{(m-1)!}(q-2m-n+1)_{m-1}\left(m+n+\frac{3}{2}\right)_{q-2m-n}.
\end{aligned}\tag{3.19}$$

Treating the second and third terms in (3.17) in a similar way and relabelling q as p, we find a *third* hierarchy of sum rules:

$$\begin{aligned}
&\sum_{k=p}^{\ell}(-1)^k(2k+1)(k-p+1)_{2p}[j_k(z)]^2\\
&\quad=z^2A_\ell^{[3],(p)}(z)[j_\ell(z)]^2+z^2B_\ell^{[3],(p)}(z)[j_{\ell+1}(z)]^2\\
&\qquad+zC_\ell^{[3],(p)}(z)j_\ell(z)j_{\ell+1}(z)+(-1)^pp!z^pj_p(2z),
\end{aligned}\tag{3.20}$$

for $p\geq 0$ and $\ell\geq p$. The coefficients at the right-hand side are polynomials in z^2 that are given as follows:

$$\begin{aligned}
A_\ell^{[3],(p)}(z)=B_{\ell+1}^{[3],(p)}(z)&=\frac{1}{2}(-1)^{p+\ell+1}p!\sum_{m=0}^{[(p-1)/2]}\sum_{n=0}^{p-2m-1}(-1)^{m+n}\frac{1}{m!n!}\\
&\quad\times\left(m+n+\frac{3}{2}\right)_{p-2m-n-1}(p-2m-n)_m(\ell-n+2)_{2n}z^{2m},
\end{aligned}\tag{3.21}$$

$$\begin{aligned}
C_\ell^{[3],(p)}(z)&=(-1)^{p+\ell}p!(\ell+1)\sum_{m=0}^{[p/2]}\sum_{n=0}^{p-2m}(-1)^{m+n}\frac{1}{m!n!}\\
&\quad\times\left(m+n+\frac{1}{2}\right)_{p-2m-n}(p-2m-n+1)_m(\ell-n+2)_{2n-1}z^{2m},
\end{aligned}\tag{3.22}$$

with $(f)_{-1}=\Gamma(f-1)/\Gamma(f)=1/(f-1)$ for $f\neq 1$. For $p=0$ and $p=1$ the sum rule (3.20) yields (2.8) and (2.9), respectively. The z-dependent polynomials in (3.20) are easily evaluated for small $p>1$ and arbitrary ℓ, as their degree is small in that case.

The three hierarchies of sum rules (3.3), (3.9) and (3.20) for squares of spherical Bessel functions are the main results of this paper. They have been derived in a systematic way from the four-term recurrence relation (2.1).

4. Discussion and conclusion

The derivation of the hierarchies for finite sums of squares of spherical Bessel functions shows that these hierarchies appear to be uniquely defined as generalizations of the lowest order sum rules from Section 2. The latter followed from the fundamental recurrence relation (2.2) for squares of spherical Bessel functions.

The finite sums of Bessel functions found above converge as the upper limit ℓ tends to ∞, since $[j_k(z)]^2$ goes to 0 quite fast for $k \to \infty$ at fixed z. For infinite ℓ the sum rules (3.3), (3.9) and (3.20) are consistent with those found before (see [3], formulas 9.4.4(13) and 9.4.7(9)). For finite ℓ the sum rules derived above are all new, to the best of our knowledge.

Hierarchies of sum rules similar to those given in (3.3), (3.9) and (3.20) may be established for sums over products $f_\ell(z)g_\ell(z)$, with $f_\ell(z)$ and $g_\ell(z)$ equal to $j_\ell(z)$, $y_\ell(z)$, $h_\ell^{(1)}(z)$ or $h_\ell^{(2)}(z)$, independently, since the recurrence relation (2.1) holds true for any product of these functions. It should be noted that the term $F(z)$ in (2.2), and hence the terms independent of ℓ in (3.3), (3.9) and (3.20) get a different form upon switching to general functions $f_\ell(z)$ and $g_\ell(z)$. Furthermore, the product $j_\ell(z)j_{\ell+1}(z)$ must be replaced by $\frac{1}{2}[f_\ell(z)g_{\ell+1}(z) + f_{\ell+1}(z)g_\ell(z)]$. In [6] the sum rules (3.3) and (3.9), with spherical Hankel functions instead of $j_\ell(z)$, have been used to determine indefinite integrals over squares of these functions. The general form of these indefinite integrals is $\int^z \mathrm{d}u u^{-n} h_{\ell_1}^{(1)}(u) h_{\ell_2}^{(i)}(u)$, for real $z > 0$, integer n, non-negative integers ℓ_1, ℓ_2 and $i = 1,2$.

The identities (3.3), (3.9) and (3.20) yield an efficient way to evaluate the sums at their left-hand sides numerically, in particular for small p and large ℓ. Comparing for instance the evaluation times of both sides of (3.20) for $p = 0$, $\ell = 50$ and $z = 50$ with the help of the numerical software contained in Mathematica, one finds that calculating the right-hand side is more than 10 times faster than calculating the left-hand side. Such an increase in the efficiency of the numerical evaluation proved to be advantageous in producing the plots in [6] for a range of values of z.

Remarkably enough, the derivation presented above shows the close connection between the three hierarchies (3.3), (3.9) and (3.20): all three follow, on an equal footing, from the single fundamental recurrence relation (2.1) for the squared spherical Bessel functions.

Disclosure statement

No potential conflict of interest was reported by the authors.

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