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Covariant Equations of Motion for a Charged Particle with a Magnetic Dipole Moment.

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Summary. — The Hamiltonian for a Dirac particle with anomalous magnetic moment in an electromagnetic field is transformed to even form up to terms linear in the coupling constant and without derivatives of the field. The even parts of the position and spin operators are derived by imposing conditions of covariance. Covariant equations of motion and of spin are then deduced; they turn out to have the same form as the classical equations for a composite particle with magnetic dipole moment. (The magnetodynamic effect for a particle in a time-dependent field is shown to contain the vector product of the electric field and the anomalous magnetic moment only.)

1. – Introduction.

A charged dipole particle in an electromagnetic field may be described by means of covariant equations of motion and of spin. These are needed for the interpretation of measurements of magnetic moments, as for instance the $(g-2)$ experiment. Although Dirac particles are concerned in that case, classical equations have generally been used.

Such covariant *classical* equations have often been postulated, *i.e.* either obtained from variational principles *ad hoc* (1) or generalized from nonrelativ-

(1) J. FRENKEL: *Zeits. Phys.*, **37**, 243 (1926); J. I. HORVATH: *Acta Phys. Acad. Sci. Hung.*, **3**, 171 (1954); K. NAGY: *Bull. Acad. Sci. Pol.*, **4**, 341 (1956); F. HALBWACHS, P. HILLION and J. P. VIGIER: *Nuovo Cimento*, **10**, 817 (1958); F. HALBWACHS: *Progr. Theor. Phys.*, **24**, 291 (1960); D. BOHM, P. HILLION, T. TAKABAYASI and J. P. VIGIER: *Progr. Theor. Phys.*, **23**, 496 (1960); A. O. BARUT: *Electrodynamics and Classical Theory of Fields and Particles* (New York, 1964), p. 73.

istic equations [2]. If one tries to *derive* covariant equations one must first define a central point for a composite particle in an external field. For a free composite particle this presents no difficulties [3-5], since one can then impose the Frenkel condition $U_\alpha S^{\alpha\beta} = 0$ (where U^α is the four-velocity and $S^{\alpha\beta}$ the inner angular momentum) together with the condition U^α parallel to P^α (where P^α is the four-momentum). For a composite particle in an external field the Frenkel condition is used by a number of authors [5,6] in deriving equations of motion, although it does not lead to a uniquely defined world-line of central points, as was proved by MØLLER [5]. In fact the equations allow peculiar solutions: even in the field-free case helical motion is possible. (In order to avoid this difficulty the form of the equations is sometimes [7] changed afterwards by means of an «iteration process».) A different condition *viz.* $P_\alpha S^{\alpha\beta} = 0$ has been proposed by NAKANO [8] and TULCZYJEW [9] and was applied extensively by DIXON [10,11]. In Sect. **2** of this article it is proved that this condition leads to one single world-line of centres of energy (or several discrete ones). In Sect. **3** we derive the equations of motion and of spin for the composite dipole particle in an external field, retaining only terms linear in these fields. The set of equations includes an expression for the total momentum in terms of the four-velocity, the inner angular momentum, the electromagnetic dipole tensor and the field. In Sect. **4** the special case of a composite particle with a magnetic dipole moment proportional to the inner angular momentum is considered. DIXON [10,11] already derived part of the equations mentioned; he found the expression for the total momentum only for the case of a normal magnetic moment in a homogeneous field.

An interesting aspect of the equations of motion is that they contain the

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(2) J. FRENKEL: *Zeits. Phys.*, **37**, 243 (1926); H. A. KRAMERS: *Quantentheorie des Elektrons und der Strahlung* (Leipzig, 1938), p. 227; V. BARGMANN, L. MICHEL and V. L. TELEGDI: *Phys. Rev. Lett.*, **2**, 435 (1959); H. BACRY: *Nuovo Cimento*, **26**, 1164 (1962); A. CHAKRABARTI: *Nuovo Cimento*, **43** A, 576 (1966).

(3) A. D. FOKKER: *Relativiteitstheorie* (Groningen, 1929), p. 170.

(4) M. H. L. PRYCE: *Proc. Roy. Soc.*, A **195**, 62 (1949).

(5) C. MØLLER: *Ann. Inst. H. Poincaré*, **11**, 251 (1949).

(6) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **112**, 512 (1939); **116**, 153 (1940); H. J. BHABHA and H. C. CORBEN: *Proc. Roy. Soc.*, A **178**, 273 (1941); J. WEYSSENHOFF and A. RAABE: *Acta Phys. Polon.*, **9**, 7, 19, 26, 34, 46 (1947); H. C. CORBEN: *Nuovo Cimento*, **20**, 529 (1961); *Phys. Rev.*, **121**, 1833 (1961); P. NYBORG: *Nuovo Cimento*, **31**, 1209 (1964); **32**, 1131 (1964); W. G. DIXON: *Journ. Math. Phys.*, **8**, 1591 (1967); J. VLIEGER: *Physica*, **37**, 165 (1967).

(7) E. PLAHTE: *Suppl. Nuovo Cimento*, **4**, 291 (1966); J. VLIEGER and S. EMID: *Physica*, **41**, 368 (1969).

(8) T. NAKANO: *Progr. Theor. Phys.*, **15**, 333 (1956).

(9) W. TULCZYJEW: *Acta Phys. Polon.*, **18**, 393 (1959).

(10) W. G. DIXON: *Nuovo Cimento*, **34**, 317 (1964).

(11) W. G. DIXON: *Nuovo Cimento*, **38**, 1616 (1965).

magnetodynamic effect, associated with the vector product of the electric field and the magnetic moment. Such a term, which occurs already in Frenkel's [1] work, was discussed extensively in recent years [12]; it was found to contain the total magnetic moment. We find however that only the anomalous magnetic moment contributes. For a composite particle, such as an atom or a molecule, this constitutes only a slight difference (see the end of Sect. **3** and ref. [13]). For a single (charged)particle the normal magnetic moment is of the same order as the total magnetic moment, so that then the difference might be quite appreciable. Such a single particle is studied in the remaining part of this paper (Sect. **5**-**8**).

In order to obtain equations of motion for a *Dirac particle* in an electromagnetic field one starts with the Dirac equation including a Pauli anomalous term. The Hamilton operator in the Pauli representation of the Dirac matrices contains even and odd terms. In the field-free case it can be transformed to even form with the help of the Pryce [4]-Foldy-Wouthuysen [14] transformation; positive- and negative-energy solutions may then be distinguished. In the case of minimal coupling with an external field an even form was obtained by FOLDY and WOUTHUYSEN but in nonrelativistic approximation only. The latter restriction was removed by ERIKSEN and KOLSRUD [15], but the transformation operator could only be given in the form of an integral. An explicit transformation operator was given by BLOUNT [16]; he obtained an even relativistic Hamiltonian as a series of terms with space derivatives of the fields of increasing order, so that again positive- and negative-energy solutions may be distinguished. This method is used in Sect. **6** and extended to the case of the Dirac Hamiltonian with Pauli term.

The covariant equations of motion may now be obtained if position and spin operators with covariant properties are defined. If one is interested in expectation values of operators for positive- (or negative-) energy solutions of the Dirac equation, only the even parts of the operators occur. It is known that the even part of the position operator cannot fulfil both of the following conditions, *viz.* 1) covariance (in fact quasi-covariance, see Appendix I, cf. [4,17,18]),

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([12]) P. PENFIELD jr. and H. A. HAUS: *The Electrodynamics of Moving Media* (Cambridge, Mass., 1967), p. 215; *Phys. Lett.*, **26** A, 412 (1968); O. COSTA DE BEAUREGARD: *Compt. Rend.*, **263** B, 1007 (1966); **264** B, 731 (1967); *Phys. Lett.*, **24** A, 177 (1967); W. SHOCKLEY and R. P. JAMES: *Phys. Rev. Lett.*, **18**, 876 (1967); S. COLEMAN and J. H. VAN VLECK: *Phys. Rev.*, **171**, 1370 (1968).

([13]) S. R. DE GROOT and L. G. SUTTORP: *Physica*, **37**, 284, 297 (1967); **39**, 84 (1968); L. G. SUTTORP: *On the Covariant Derivation of Macroscopic Electrodynamics from Electron Theory*, Thesis (Amsterdam, 1968).

([14]) L. L. FOLDY and S. A. WOUTHUYSEN: *Phys. Rev.*, **78**, 29 (1950).

([15]) E. ERIKSEN and M. KOLSRUD: *Suppl. Nuovo Cimento*, **18**, 1 (1960).

([16]) E. I. BLOUNT: *Phys. Rev.*, **126**, 1636 (1962); **128**, 2454 (1962).

([17]) T. F. JORDAN and N. MUKUNDA: *Phys. Rev.*, **132**, 1842 (1963).

([18]) G. LUGARINI and M. PAURI: *Nuovo Cimento*, **47** A, 299 (1967).

2) commutation of Cartesian co-ordinates. If the last condition is chosen, covariance is lost: it leads to the Newton-Wigner position operator ([19]). If however covariant equations are to be obtained the first condition should be imposed. As is shown in Sect. **5** and **7** of this article, one can then *derive* the position and spin operators for the particle in a *unique* way. For the field-free case both of these operators were already found by PRYCE ([4]) (the spin operator also by others ([20,21])); for the case with fields a spin operator of the same form as derived here was proposed as a generalization of the field-free case ([21]). It should be mentioned that some authors try to reconcile conditions 1) and 2). This can only be achieved through an interplay of even and odd parts of the position operator in conditions 1) and 2). One obtains in this way the Dirac position and spin operators ([17,18]). However the even parts alone of these operators violate the covariance condition, and since only these parts occur in the expectation values the latter will not possess covariant properties. Still a different position operator may be proposed ([22,23]) if apart from the requirement of evenness also the second condition is abandoned.

With the knowledge of the Hamiltonian, transformed to even form, and the covariant position and spin operators it is straightforward to find equations of motion and of spin. These equations, obtained in Sect. **8**, turn out to be of the same form as the classical equations derived in the first part of this paper. This justifies their use in the discussion of measurements, such as the $(g-2)$ experiment.

Earlier attempts to derive equations of classical form from quantum theory include Fradkin and Good's ([24]) discussion of the motion of wave packets based on a number of additional assumptions, and valid only for homogeneous fields. Furthermore WKB methods have been used, again for the case of a homogeneous field by RUBINOW-KELLER ([25]) and RAFANELLI-SCHILLER ([26]). DIXON ([11]) employed position and spin operators without specifying the field contributions;

([19]) T. D. NEWTON and E. P. WIGNER: *Rev. Mod. Phys.*, **21**, 400 (1949); K. BARDAKCI and R. ACHARYA: *Nuovo Cimento*, **21**, 802 (1961); P. M. MATHEWS and A. SANKARANARAYANAN: *Progr. Theor. Phys.*, **26**, 499 (1961); **27**, 1063 (1962); W. WEIDLICH and A. K. MITRA: *Nuovo Cimento*, **30**, 385 (1963); U. SCHRÖDER: *Ann. der Phys.*, **14**, 91 (1964); A. GALINDO: *Nuovo Cimento*, **37**, 413 (1965); R. A. BERG: *Journ. Math. Phys.*, **6**, 34 (1965); A. SANKARANARAYANAN and R. H. GOOD jr.: *Phys. Rev.*, **140** B, 509 (1965); P. M. MATHEWS: *Phys. Rev.*, **143**, 985 (1966); M. LUNN: *Journ. Phys.*, A **2**, 17 (1969).

([20]) D. M. FRADKIN and R. H. GOOD jr.: *Nuovo Cimento*, **22**, 643 (1961).

([21]) J. HILGEVOORD and S. A. WOUTHUYSEN: *Nucl. Phys.*, **40**, 1 (1963).

([22]) M. BUNGE: *Nuovo Cimento*, **1**, 977 (1955); M. KOLSRUD: *Phys. Norv.*, **2**, 141, 149 (1967); H. YAMASAKI: *Progr. Theor. Phys.*, **31**, 322, 324 (1964).

([23]) H. C. CORBEN: *Phys. Rev.*, **121**, 1833 (1961).

([24]) D. M. FRADKIN and R. H. GOOD jr.: *Rev. Mod. Phys.*, **33**, 343 (1961).

([25]) S. I. RUBINOW and J. B. KELLER: *Phys. Rev.*, **131**, 2789 (1963).

([26]) K. RAFANELLI and R. SCHILLER: *Phys. Rev.*, **135** B, 279 (1964).

he limited himself to a particle with a normal magnetic moment in a homogeneous field. CORBEN (23), KOLSRUD (27), PLAHTE (28) and YAMASAKI (29) introduce proper time into Dirac theory without solving the well-known difficulties of interpretation pertinent to this notion. Moreover the covariance properties of their equations are not well-defined.

Just as the classical equations, the quantum-mechanical equations of motion contain the magnetodynamic effect. It is found again that only the anomalous magnetic moment (which for a Dirac particle is essentially different from the total magnetic moment) contributes to this effect. If however a noncovariant position operator, such as Newton-Wigner's or the even part of Dirac's operator, is utilized, the incorrect result is found that also the normal magnetic moment (or half of it) contributes to the magnetodynamic effect (30).

2. – Definition of the classical centre of energy.

In classical theory a composite particle, consisting of point charges, in an external electromagnetic field is described by an energy-momentum tensor $t^{\alpha\beta}$ of which the divergence can be expressed in terms of forces f^α acting on the charges:

$$\partial_\beta t^{\alpha\beta} = f^\alpha . \tag{1}$$

A covariant centre of energy may be defined (8-10) by considering those plane surfaces Σ of which the normal n^α is parallel to the total momentum integrated over the surface

$$P^\alpha = -c^{-1}\int_\Sigma t^{\alpha\beta} n_\beta \, \mathrm{d}\Sigma \tag{2}$$

(P^α is assumed to be a timelike vector; metric $g^{00}=-1$, $g^{ii}=1$); in these surfaces one then determines the centre of energy

$$X^\alpha = \int_\Sigma x^\alpha n_\beta t^{\beta\gamma} n_\gamma \, \mathrm{d}\Sigma \Big/ \int_\Sigma n_\varepsilon t^{\varepsilon\zeta} n_\zeta \, \mathrm{d}\Sigma . \tag{3}$$

(27) M. KOLSRUD: *Nuovo Cimento*, **39**, 504 (1965).

(28) E. PLAHTE: *Suppl. Nuovo Cimento*, **4**, 246, 291 (1966); **5**, 944 (1967).

(29) H. YAMASAKI: *Progr. Theor. Phys.*, **39**, 372 (1968).

(30) A. CONORT: *Compt. Rend.*, **266** B, 1184 (1968); H. BACRY: *Compt. Rend.*, **267** B, 89 (1968); W. SHOCKLEY: *Phys. Rev. Lett.*, **20**, 343 (1968); W. SHOCKLEY and K. K. THORNBER: *Phys. Lett.*, **27** A, 534 (1968); J. H. VAN VLECK and N. L. HUANG *Phys. Lett.*, **28** A, 768 (1969).

(In the rest frame of P^α this formula reads indeed $\boldsymbol{X}=\int \boldsymbol{x}t^{00}\,\mathrm{d}\boldsymbol{x}/\int t^{00}\,\mathrm{d}\boldsymbol{x}$.) These centres of energy then satisfy the relation

(4) $$P_\alpha S^{\alpha\beta}=0\,,$$

where the inner angular momentum is

(5) $$S^{\;\beta}=-c^{-1}\int_\Sigma \{(x-X)^\alpha t^{\beta\gamma}-(x-X)^\beta t^{\alpha\gamma}\}n_\gamma\,\mathrm{d}\Sigma\,.$$

We shall now prove that for a finite system with a positive definite energy density t^{00} the set of centres of energy defined in this way forms one single world-line (or several discrete ones). Consider such a point X^α determined in a plane surface Σ with normal parallel to P^α. One may ask oneself now if there exists a point $X^\alpha+\delta X^\alpha$ (with $P_\alpha\,\delta X^\alpha=0$) in the infinitesimal neighbourhood of X^α, which is likewise a centre of energy, this time in a plane surface Σ' with normal parallel to the corresponding momentum $P^\alpha+\delta P^\alpha$. In the proper frame of P^α one has from (2), (4) and (5)

(6) $$\int_\Sigma t^{i0}(x^0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0 \qquad\qquad (x^0=X^0)\,,$$

(7) $$\int_\Sigma (\boldsymbol{x}-\boldsymbol{X})\,t^{00}(x^0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0\,.$$

The proper frame of $P^\alpha+\delta P^\alpha$ is connected to the proper frame of P^α by an infinitesimal pure Lorentz transformation

(8) $$x^{0\prime}=x^0+\boldsymbol{\epsilon}\cdot\boldsymbol{x}\,,\qquad \boldsymbol{x}'=\boldsymbol{x}+\boldsymbol{\epsilon}x^0$$

for a certain value of $\boldsymbol{\epsilon}$. The time-space point with co-ordinates $(x^0,\boldsymbol{X}+\delta\boldsymbol{X})$ has in the new frame the co-ordinates $(\hat{x}^{0\prime},\boldsymbol{X}'+\delta\boldsymbol{X})$ which are given by

(9) $$\hat{x}^{0\prime}=x^0+\boldsymbol{\epsilon}\cdot\boldsymbol{X}\,,$$

(10) $$\boldsymbol{X}'+\delta\boldsymbol{X}'=\boldsymbol{X}+\delta\boldsymbol{X}+\boldsymbol{\epsilon}x^0$$

up to terms linear in $\boldsymbol{\epsilon}$ and $\delta\boldsymbol{X}$. Furthermore the space co-ordinates of an arbitrary point in Σ' read in the new frame

(11) $$\hat{\boldsymbol{x}}'=\hat{\boldsymbol{x}}+\boldsymbol{\epsilon}\hat{x}^0\,,$$

where $\hat{\boldsymbol{x}}$ and $\hat{x}^0$ are connected by

$$\hat{x}^{0\prime} = \hat{x}^0 + \boldsymbol{\epsilon}\cdot\hat{\boldsymbol{x}}\ . \tag{12}$$

From (9), (11) and (12) it follows now that

$$\hat{x}^0 = x^0 + \boldsymbol{\epsilon}\cdot(\boldsymbol{X} - \hat{\boldsymbol{x}}')\ , \tag{13}$$

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}' - \boldsymbol{\epsilon} x^0\ . \tag{14}$$

In the proper frame of $P^\alpha + \delta P^\alpha$ the space components of the total momentum vanish (cf. (6)):

$$\int_{\Sigma'} t^{i0\prime}(\hat{x}^{0\prime}, \hat{\boldsymbol{x}}')\,\mathrm{d}\hat{\boldsymbol{x}}' = 0\ . \tag{15}$$

Using the transformation properties of a tensor one gets

$$\int_{\Sigma'} \{t^{i0}(\hat{x}^0, \hat{\boldsymbol{x}}) + \varepsilon^i t^{00}(\hat{x}^0, \hat{\boldsymbol{x}}) + \varepsilon_j t^{ij}(\hat{x}^0, \hat{\boldsymbol{x}})\}\,\mathrm{d}\hat{\boldsymbol{x}}' = 0\ . \tag{16}$$

Introducing (13) and (14) one gets ($\partial_0 \equiv \partial/\partial x^0$):

$$\int \{t^{i0}(x^0, \hat{\boldsymbol{x}}') - x^0\,\boldsymbol{\epsilon}\cdot(\partial/\partial\hat{\boldsymbol{x}}')\,t^{i0}(x^0, \hat{\boldsymbol{x}}') + \\ + \boldsymbol{\epsilon}\cdot(\boldsymbol{X} - \hat{\boldsymbol{x}}')\,\partial_0 t^{i0}(x^0, \hat{\boldsymbol{x}}') + \varepsilon^i t^{00}(x^0, \hat{\boldsymbol{x}}') + \varepsilon_j t^{ij}(x^0, \hat{\boldsymbol{x}}')\}\,\mathrm{d}\hat{\boldsymbol{x}}' = 0\ . \tag{17}$$

After a partial integration (from (9) and the fact that $\hat{x}^{0\prime}$ is constant in the integration it follows that x^0 is constant) one has with (6)

$$\int \{\boldsymbol{\epsilon}\cdot(\boldsymbol{X} - \hat{\boldsymbol{x}}')\,\partial_0 t^{i0}(x^0, \hat{\boldsymbol{x}}') + \varepsilon^i t^{00}(x^0, \hat{\boldsymbol{x}}') + \varepsilon_j t^{ij}(x^0, \hat{\boldsymbol{x}}')\}\,\mathrm{d}\hat{\boldsymbol{x}}' = 0\ . \tag{18}$$

With the equation of motion (1) and a partial integration this becomes

$$\int \{\boldsymbol{\epsilon}\cdot(\boldsymbol{X} - \boldsymbol{x})\boldsymbol{f}(x^0, \boldsymbol{x}) + \boldsymbol{\epsilon} t^{00}(x^0, \boldsymbol{x})\}\,\mathrm{d}\boldsymbol{x} = 0\ . \tag{19}$$

For a finite system the integral is in fact confined to a finite support (Lorentz contraction only improves the argument). As a consequence of (7) the centre of energy $\boldsymbol{X}$ lies also in this domain and therefore $|\boldsymbol{x} - \boldsymbol{X}|$ is bounded. Hence for sufficiently small external electromagnetic fields

$$\left|\boldsymbol{\epsilon}\cdot\int(\boldsymbol{X}-\boldsymbol{x})\boldsymbol{f}(x^0,\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|<\left|\epsilon\int t^{00}\,\mathrm{d}\boldsymbol{x}\right|, \tag{20}$$

so that (19) cannot be satisfied. The conclusion is that no point exists in the infinitesimal neighbourhood of X^α, which is also a centre of energy; in other words the set of centres of energy determines a discrete number of world-lines (*). The condition (4) thus leads to a situation completely different from that following from the condition $U_\alpha S^{\alpha\beta}=0$ (with $U^\alpha=\mathrm{d}X^\alpha/\mathrm{d}s$). As a matter of fact MØLLER proved ([5]) that the latter condition does not suffice to determine a world-line. Moreover he showed that it cannot be supplemented by the requirement that P^α be parallel to U^α (in the general case with external fields).

3. – Classical equations of motion for a charged particle with electromagnetic dipole moments in an external field.

For a composite particle, which satisfies the energy-momentum law (1), the derivative of P^α with respect to the proper time s of the world-line $X^\alpha(s)$ is given by

$$\frac{\mathrm{d}P^\alpha}{\mathrm{d}s}\,\mathrm{d}s\equiv-c^{-1}\,\mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Sigma(s)}t^{\alpha\beta}n_\beta\,\mathrm{d}\Sigma=\int_{\Sigma(s)}^{\Sigma(s+\mathrm{d}s)}c^{-1}f^\alpha\,\mathrm{d}V\,, \tag{21}$$

where the integration in the first integral is extended over the surface $\Sigma(s)$ orthogonal to $P^\alpha(s)$ (or $n^\alpha(s)$) and containing the centre of energy $X^\alpha(s)$; the last integral, which is obtained from Gauss's theorem and (1), is extended over the domain bounded by the surfaces $\Sigma(s)$ and $\Sigma(s+\mathrm{d}s)$. The volume element $\mathrm{d}V$ may be written as

$$\mathrm{d}V=-U_\alpha n^\alpha\{1-\dot n_\beta(x-X)^\beta/U_\gamma n^\gamma\}\,\mathrm{d}s\,\mathrm{d}\Sigma\,, \tag{22}$$

where $U^\alpha=\mathrm{d}X^\alpha/\mathrm{d}s$ and $\dot n^\alpha=\mathrm{d}n^\alpha/\mathrm{d}s$. Since $n^\alpha=P^\alpha/\sqrt{-P^2}$ and $P_\alpha(x-X)^\alpha=0$ one has $\dot n^\alpha(x-X)^\alpha=(\dot P_\alpha/\sqrt{-P^2})(x-X)^\alpha$ and thus (21) becomes with (22)

$$\mathrm{d}P^\alpha/\mathrm{d}s=F^\alpha\,, \tag{23}$$

(*) One may prove from (6) and (7) that for sufficiently small fields the world-line is timelike so that a proper time s along the world line may be introduced.

where F^α is the total four-force expressed in terms of the force density $f^\alpha(x)$:

$$F^\alpha(s) \equiv -\int\limits_{\Sigma(s)} \frac{f^\alpha U_\beta P^\beta}{c\sqrt{-P^2}} \left\{1 - \frac{\dot{P}_\gamma (x-X)^\gamma}{U_\varepsilon P^\varepsilon}\right\} \mathrm{d}\Sigma . \tag{24}$$

If the energy-momentum tensor is symmetric one has from (1) the angular-momentum law

$$\partial_\gamma\{(x-X)^\alpha t^{\beta\gamma} - (x-X)^\beta t^{\alpha\gamma}\} = (x-X)^\alpha f^\beta - (x-X)^\beta f^\alpha . \tag{25}$$

With the help of this equation one finds for the derivative of the inner angular momentum $S^{\alpha\beta}(5)$ with respect to the proper time s:

$$\mathrm{d}S^{\alpha\beta}/\mathrm{d}s = D^{\alpha\beta} - (U^\alpha P^\beta - U^\beta P^\alpha) , \tag{26}$$

where the total torque

$$D^{\alpha\beta}(s) \equiv -\int\limits_{\Sigma(s)} \{(x-X)^\alpha f^\beta - (x-X)^\beta f^\alpha\} \frac{U_\gamma P^\gamma}{c\sqrt{-P^2}} \left\{1 - \frac{\dot{P}_\varepsilon (x-X)^\varepsilon}{U_\zeta P^\zeta}\right\} \mathrm{d}\Sigma \tag{27}$$

is expressed in terms of the force density $f^\alpha(x)$.

For the case of a composite particle, consisting of (spinless) point particles with charges e_i, masses m_i and positions $R_i^\alpha(s_i)$ with s_i the proper time, interacting with each other and with an external field the total energy-momentum tensor reads

$$\begin{aligned} t_{(i)}^{\alpha\beta}(x) = {} & c\sum_i \int m_i \frac{\mathrm{d}R_i^\alpha}{\mathrm{d}s_i}\frac{\mathrm{d}R_i^\beta}{\mathrm{d}s_i}\delta^{(4)}\{R_i(s_i)-x\}\,\mathrm{d}s_i + \\ & + F^{\alpha\gamma}(x)F^\beta_{\cdot\gamma}(x) - \tfrac{1}{4}F^{\gamma\varepsilon}(x)F_{\gamma\varepsilon}(x)g^{\alpha\beta} + \\ & + \sum_i \{F^{\alpha\gamma}(x)f^\beta_{(i)\cdot\gamma}(x) + f^{\alpha\gamma}_{(i)}(x)F^\beta_{\cdot\gamma}(x) - \tfrac{1}{2}f^{\gamma\varepsilon}_{(i)}(x)F_{\gamma\varepsilon}(x)g^{\alpha\beta}\} + \\ & \qquad + \sum_{i,j,i\neq j}\{f^{\alpha\gamma}_{(i)}(x)f^\beta_{(j)\cdot\gamma}(x) - \tfrac{1}{4}f^{\gamma\varepsilon}_{(i)}(x)f_{(j)\gamma\varepsilon}(x)g^{\alpha\beta}\} \end{aligned} \tag{28}$$

(diagonal terms in the double sum are left out to avoid self-interaction terms). The field $f^{\alpha\beta}_{(i)}$ ($f^{0k}_{(i)} = \boldsymbol{e}^k_{(i)}$, $f^{kl}_{(i)} = \varepsilon^{klm}\boldsymbol{b}_{(i)m}$), which is generated by particle i, satisfies the equations

$$\partial_\beta f^{\alpha\beta}_{(i)} = e_i\int (\mathrm{d}R_i^\alpha/\mathrm{d}s_i)\,\delta^{(4)}\{R_i(s_i)-x\}\,\mathrm{d}s_i , \tag{29}$$

$$\partial_\alpha f_{(i)\beta\gamma} + \partial_\beta f_{(i)\gamma\alpha} + \partial_\gamma f_{(i)\alpha\beta} = 0 , \tag{30}$$

whereas the external field $F^{\alpha\beta}$ ($F^{0k} = E^k$, $F^{kl} = \varepsilon^{klm}B_m$) satisfies homogeneous equations of the same form. The conservation law $\partial_\beta t^{\alpha\beta}_{(i)} = 0$ leads with the help of (29)-(30) to an energy-momentum law of the form (1) with the energy-momentum tensor

$$t^{\alpha\beta} = c\sum_i \int m_i \frac{\mathrm{d}R_i^\alpha}{\mathrm{d}s_i}\frac{\mathrm{d}R_i^\beta}{\mathrm{d}s_i}\,\delta^{(4)}\{R_i(s_i) - x\}\,\mathrm{d}s_i + \sum_{i,j,i\neq j}\{f^{\alpha\gamma}_{(i)}(x) f^{\beta}_{(j)\gamma}(x) - \tfrac{1}{4} f^{\gamma\varepsilon}_{(i)}(x) f_{(j)\gamma\varepsilon}(x) g^{\alpha\beta}\}, \tag{31}$$

(which is seen to be symmetric) and the (Lorentz) force density

$$f^\alpha(x) = F^{\alpha\beta}(x)\, j_\beta(x)/c\,, \tag{32}$$

where $j^\alpha(x)$ is the four-current, given by

$$j^\alpha(x)/c = \sum_i e_i \int (\mathrm{d}R_i^\alpha/\mathrm{d}s_i)\,\delta^{(4)}\{R_i(s_i) - x\}\,\mathrm{d}s_i\,. \tag{33}$$

In this expression the parameters along the world-lines i need not be chosen as the proper times s_i. It will be convenient to use instead the parameter s in the force density. (This parametrization may in fact be induced with the help of the surfaces $\Sigma(s)$.)

With the explicit forms (32) and (33) the equations of motion (23)-(24) and (26)-(27) are completely specified. The total force (24) becomes with (32), (33) and a multipole expansion

$$F^\alpha(s) = -F^{\alpha\beta}(X)\sum_i e_i \int \frac{\mathrm{d}R_{i\beta}}{\mathrm{d}s}\,\delta^{(4)}\{R_i(s) - x\}\,\frac{U_\gamma P^\gamma}{c\sqrt{-P^2}}\cdot \left\{1 - \frac{\dot P_\varepsilon (x-X)^\varepsilon}{U_\zeta P^\zeta}\right\}\mathrm{d}\Sigma\,\mathrm{d}s - \partial_\gamma F^{\alpha\beta}(X)\sum_i e_i \int (x-X)^\gamma \frac{\mathrm{d}R_{i\beta}}{\mathrm{d}s}\cdot \delta^{(4)}\{R_i(s) - x\}\,\frac{U_\varepsilon P^\varepsilon}{c\sqrt{-P^2}}\left\{1 - \frac{\dot P_\zeta (x-X)^\zeta}{U_\eta P^\eta}\right\}\mathrm{d}\Sigma\,\mathrm{d}s\,, \tag{34}$$

where only terms with the fields and their first derivatives have been retained (terms with higher derivatives would contain quadrupoles and higher moments, which are neglected). Substituting (22) with $n^\alpha = P^\alpha/\sqrt{-P^2}$ and introducing the internal variable $r_i^\alpha = (R_i - X)^\alpha$ and the total time derivative $\mathrm{d}/\mathrm{d}s \equiv U^\alpha \partial_\alpha$

we obtain

$$(35)\qquad F^{\alpha}(s) = (e/c)F^{\alpha\beta}(X)\,U_\beta + \\ + c^{-1}\partial^{\gamma}F^{\alpha\beta}(X)\sum_i e_i\left\{r_{i\gamma}U_\beta - r_{i\beta}U_\gamma + \frac{1}{2}\left(r_{i\gamma}\frac{\mathrm{d}r_{i\beta}}{\mathrm{d}s} - r_{i\beta}\frac{\mathrm{d}r_{i\gamma}}{\mathrm{d}s}\right)\right\} + \\ + c^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\{F^{\alpha\beta}(X)\sum_i e_i r_{i\beta}\}\,,$$

where the electric quadrupole moment $\sum_i e_i r_i^\alpha r_i^\beta$ has been neglected. This expression contains the electromagnetic dipole moment tensor ([31]):

$$(36)\qquad M^{\alpha\beta} = c^{-1}\sum_i e_i\left\{r_i^\alpha U^\beta - r_i^\beta U^\alpha + \frac{1}{2}\left(r_i^\alpha\frac{\mathrm{d}r_i^\beta}{\mathrm{d}s} - r_i^\beta\frac{\mathrm{d}r_i^\alpha}{\mathrm{d}s}\right)\right\}.$$

Furthermore one has from (36) and the orthogonality relation $r_{i\alpha}P^\alpha = 0$ the identity

$$(37)\qquad \sum_i e_i r_i^\alpha = cM^{\alpha\beta}P_\beta/U_\gamma P^\gamma\,,$$

where again the quadrupole moment has been discarded. With (35), (36), (37) and the homogeneous field equation for $F^{\alpha\beta}$ the equation of motion (23) takes the form

$$(38)\qquad \mathrm{d}P^\alpha/\mathrm{d}s = (e/c)F^{\alpha\beta}(X)\,U_\beta + \tfrac{1}{2}\{\partial^\alpha F^{\beta\gamma}(X)\}\,M_{\beta\gamma} + (\mathrm{d}/\mathrm{d}s)\{F^{\alpha\beta}(X)\,M_{\beta\gamma}P^\gamma/U_\varepsilon P^\varepsilon\}\,.$$

In the same fashion we obtain from (26) and (27) as the equation for the inner angular momentum

$$(39)\qquad \mathrm{d}S^{\alpha\beta}/\mathrm{d}S = F^{\alpha\gamma}(X)\,M_\gamma^{\cdot\beta} - F^{\beta\gamma}(X)\,M_\gamma^{\cdot\alpha} - \\ - F^{\alpha\gamma}(X)\,M_{\gamma\varepsilon}P^\varepsilon U^\beta/U_\zeta P^\zeta + F^{\beta\gamma}(X)\,M_{\gamma\varepsilon}P^\varepsilon U^\alpha/U_\zeta P^\zeta + P^\alpha U^\beta - P^\beta U^\alpha\,.$$

The eqs. (38) and (39) with the supplementary condition (4) are the equations of motion and of spin for a classical composite particle with charge and electromagnetic dipole moments in an external field. In order to discuss them we first consider the field-free case. Then the equations reduce to

$$(40)\qquad \mathrm{d}P^\alpha/\mathrm{d}s = 0\,,\qquad \mathrm{d}S^{\alpha\beta}/\mathrm{d}s = P^\alpha U^\beta - P^\beta U^\alpha\,.$$

By differentiating the condition (4) one finds with (40)

$$(41)\qquad P_\alpha P^\alpha U^\beta - P_\alpha U^\alpha P^\beta = 0\,.$$

([31]) S. R. de Groot and L. G. Suttorp: *Physica*, **31**, 1713 (1965).

Hence the four-vectors P^α and U^α are parallel, so that

$$P^\alpha = M U^\alpha \qquad (M \equiv -c^{-2} P_\alpha U^\alpha) . \tag{42}$$

Now (40) reduces to

$$\mathrm{d} U^\alpha/\mathrm{d}s = 0 , \qquad \mathrm{d}S^{\alpha\beta}/\mathrm{d}s = 0 , \tag{43}$$

since $\mathrm{d}M/\mathrm{d}s$ vanishes as follows from (42).

In the case *with* fields differentiation of (4) and substitution of (38) and (39) leads to

$$\begin{aligned} P^\alpha U_\beta P^\beta = U^\alpha P_\beta P^\beta + M^{\alpha\beta} F_{\beta\gamma} P^\gamma - U^\alpha P^\beta F_{\beta\gamma} M^{\gamma\varepsilon} P_\varepsilon / U_\zeta P^\zeta - (e/c) S^{\alpha\beta} F_{\beta\gamma} U^\gamma - \\ - \tfrac{1}{2} S^{\alpha\beta} (\partial_\beta F_{\gamma\varepsilon}) M^{\gamma\varepsilon} - S^{\alpha\beta} (\mathrm{d}/\mathrm{d}s)(F_{\beta\gamma} M^{\gamma\varepsilon} P_\varepsilon / U_\zeta P^\zeta) . \end{aligned} \tag{44}$$

Hence now P^α is not parallel to U^α. If (44) is multiplied by U_α one obtains the equality

$$\begin{aligned} P_\alpha P^\alpha = -c^{-2}(U_\alpha P^\alpha)^2 + c^{-2}(U_\alpha + c^2 P_\alpha / U_\varepsilon P^\varepsilon) M^{\alpha\beta} F_{\beta\gamma} P^\gamma - \\ - c^{-2} U_\alpha S^{\alpha\beta} \{(e/c) F_{\beta\gamma} U^\gamma + \tfrac{1}{2} (\partial_\beta F_{\gamma\varepsilon}) M^{\gamma\varepsilon} + (\mathrm{d}/\mathrm{d}s)(F_{\beta\gamma} M^{\gamma\varepsilon} P_\varepsilon / U_\zeta P^\zeta)\} . \end{aligned} \tag{45}$$

According to (42) and the condition (4) all terms on the right-hand side but the first are at least of second order in the fields. Hence if one wants to confine oneself to terms linear in the fields the equality (44) may be written as

$$\begin{aligned} P^\alpha = M U^\alpha - c^{-2} M^{\alpha\beta} F_{\beta\gamma} U^\gamma - c^{-4} U^\beta F_{\beta\gamma} M^{\gamma\varepsilon} U_\varepsilon U^\alpha + \\ + (e/Mc^3) S^{\alpha\beta} F_{\beta\gamma} U^\gamma + (1/2Mc^2) S^{\alpha\beta} (\partial_\beta F_{\gamma\varepsilon}) M^{\gamma\varepsilon} - (1/Mc^4) S^{\alpha\beta} (\mathrm{d}/\mathrm{d}s)(F_{\beta\gamma} M^{\gamma\varepsilon} U_\varepsilon) , \end{aligned} \tag{46}$$

so that now the total momentum P^α is expressed in terms of U^α, $M \equiv -c^{-2} P_\alpha U^\alpha$ (cf. (42)) and $S^{\alpha\beta}$, e, $M^{\alpha\beta}$, $F^{\alpha\beta}$.

The equations of motion and spin (38) and (39) get a simplified form if —as in (46)—only terms linear in the fields are retained. One obtains then, with (46)

$$\mathrm{d}P^\alpha/\mathrm{d}s = (e/c) F^{\alpha\beta} U_\beta + \tfrac{1}{2} (\partial^\alpha F^{\beta\gamma}) M_{\beta\gamma} - c^{-2} (\mathrm{d}/\mathrm{d}s)(F^{\alpha\beta} M_{\beta\gamma} U^\gamma) , \tag{47}$$

$$\begin{aligned} \mathrm{d}S^{\alpha\beta}/\mathrm{d}s = P^\alpha U^\beta - P^\beta U^\alpha + F^{\alpha\gamma} M_\gamma^{\cdot\beta} - F^{\beta\gamma} M_\gamma^{\cdot\alpha} + \\ + c^{-2} F^{\alpha\gamma} M_{\gamma\varepsilon} U^\varepsilon U^\beta - c^{-2} F^{\beta\gamma} M_{\gamma\varepsilon} U^\varepsilon U^\alpha . \end{aligned} \tag{48}$$

An alternative expression for the total momentum P^α may be obtained with the help of the relation

$$(49)\qquad -c^{-2}(\mathrm{d}S^{\alpha\beta}/\mathrm{d}s)\,U_\beta = (e/Mc^3)S^{\alpha\beta}F_{\beta\gamma}U^\gamma + (1/2Mc^2)S^{\alpha\beta}(\partial_\beta F_{\gamma\varepsilon})M^{\gamma\varepsilon} - (1/Mc^4)S^{\alpha\beta}(\mathrm{d}/\mathrm{d}s)(F_{\beta\gamma}M^{\gamma\varepsilon}U_\varepsilon)\,,$$

which follows from (48) with (46). Substitution of this relation into (46) gives

$$(50)\qquad P^\alpha = MU^\alpha - c^{-2}M^{\alpha\beta}F_{\beta\gamma}U^\gamma - c^{-4}U^\beta F_{\beta\gamma}M^{\gamma\varepsilon}U_\varepsilon U^\alpha - c^{-2}(\mathrm{d}S^{\alpha\beta}/\mathrm{d}s)\,U_\beta\,.$$

Thus if only terms linear in the fields are taken into account the equations of motion and spin are given by (47) and (48) with the condition (4) and the expression (46) or (50) for the total momentum.

Equations like (47), (48) and (50) but with the condition $U_\alpha S^{\alpha\beta} = 0$ instead of (4) have been discussed earlier [5,6]. Owing to the use of this different subsidiary condition it is then not possible to go back to (46) starting from (47), (48) and (50); this fact is connected with the appearance of the unwanted helical solutions, even in the field-free case (*).

4. – Equations for a composite particle with magnetic dipole moment proportional to the inner angular momentum.

Let us consider the special case of a composite charged particle without electric dipole moment and with a magnetic dipole moment proportional to the inner angular momentum; then

$$(51)\qquad M^{\alpha\beta} = \varkappa S^{\alpha\beta}\,.$$

The eqs. (47) and (48) with (4) and (46) read then, if one retains again only terms linear in the fields,

$$(52)\qquad \mathrm{d}P^\alpha/\mathrm{d}s = (e/c)F^{\alpha\beta}U_\beta + \tfrac{1}{2}(\partial^\alpha F^{\beta\gamma})M_{\beta\gamma}\,,$$

$$(53)\qquad \mathrm{d}S^{\alpha\beta}/\mathrm{d}s = P^\alpha U^\beta - P^\beta U^\alpha + F^{\alpha\gamma}M_\gamma^{\cdot\beta} - F^{\beta\gamma}M_\gamma^{\cdot\alpha}\,,$$

$$(4)\qquad P_\alpha S^{\alpha\beta} = 0\,,$$

$$(54)\qquad P^\alpha = MU^\alpha - c^{-2}M^{\alpha\beta}F_{\beta\gamma}U^\gamma + (e/Mc^3)S^{\alpha\beta}F_{\beta\gamma}U^\gamma + (1/2Mc^2)S^{\alpha\beta}(\partial_\beta F_{\gamma\varepsilon})M^{\gamma\varepsilon}\,.$$

(*) In ref. [13], equations like (47), (48) and (50) and the condition $U_\alpha S^{\alpha\beta} = 0$ were derived for a composite particle on the basis of an explicit construction of a central point and with the use of the Darwin approximation for the intra-atomic fields. Helical motions of macroscopic dimensions are then excluded.

From these equations one may prove that the square of the inner angular momentum $S_{\alpha\beta}\ S^{\alpha\beta}$ and the quantity

$$M^* \equiv M + \tfrac{1}{2}c^{-2}F_{\alpha\beta}M^{\alpha\beta} \tag{55}$$

are conserved. In fact multiplying (53) with $S_{\alpha\beta}$ and using (4) one gets

$$(\mathrm{d}/\mathrm{d}s)(S_{\alpha\beta}S^{\alpha\beta}) = 4S_{\alpha\beta}F^{\alpha\gamma}M_{\gamma}^{\cdot\beta}\,, \tag{56}$$

which vanishes as follows if (51) is introduced.

Furthermore the time derivative of $M \equiv -c^{-2}U_\alpha P^\alpha$ becomes with (52) and (54)

$$\mathrm{d}M/\mathrm{d}s = -\tfrac{1}{2}c^{-2}(\mathrm{d}F_{\alpha\beta}/\mathrm{d}s)M^{\alpha\beta} + c^{-4}(\mathrm{d}U_\alpha/\mathrm{d}s)\cdot \\ \cdot\{M^{\alpha\beta}F_{\beta\gamma}U^\gamma - (e/Mc)S^{\alpha\beta}F_{\beta\gamma}U^\gamma - (1/2M)S^{\alpha\beta}(\partial_\beta F_{\gamma\varepsilon})M^{\gamma\varepsilon}\}\,. \tag{57}$$

Since $\mathrm{d}U^\alpha/\mathrm{d}s$ vanishes in the field-free case, this equality becomes up to first order in the fields

$$\mathrm{d}M/\mathrm{d}s + \tfrac{1}{2}c^{-2}(\mathrm{d}F_{\alpha\beta}/\mathrm{d}s)M^{\alpha\beta} = 0\,. \tag{58}$$

Finally if (53) is multiplied by $F_{\alpha\beta}$ one finds with (51) and (54)

$$F_{\alpha\beta}\mathrm{d}M^{\alpha\beta}/\mathrm{d}s = 0\,, \tag{59}$$

if again only linear field terms are retained. From (58) and (59) it follows indeed that $M + \frac{1}{2}c^{-2}F_{\alpha\beta}M^{\alpha\beta}$ is conserved.

With the definition of a « classical gyromagnetic factor » g by means of

$$\varkappa = ge/2M^*c \tag{60}$$

we can write the eqs. (52) and (53) with (54) and (55), up to terms linear in the fields, as

$$M^*\mathrm{d}U^\alpha/\mathrm{d}s = (e/c)F^{\alpha\beta}U_\beta + (ge/4M^*c)(\partial^\alpha F^{\beta\gamma})S_{\beta\gamma} + (e/2M^*c^3)(\mathrm{d}/\mathrm{d}s)\cdot \\ \cdot\{\tfrac{1}{2}gF^{\beta\gamma}S_{\beta\gamma}U^\alpha + (g-2)S^{\alpha\beta}F_{\beta\gamma}U^\gamma\} - (ge/4M^{*2}c^3)S^{\alpha\beta}(\mathrm{d}/\mathrm{d}s)\{(\partial_\beta F_{\gamma\varepsilon})S^{\gamma\varepsilon}\}\,, \tag{61}$$

$$\mathrm{d}S^{\alpha\beta}/\mathrm{d}s = (ge/2M^*c)(F^{\alpha\gamma}S_\gamma^{\cdot\beta} - F^{\beta\gamma}S_\gamma^{\cdot\alpha}) - (g-2)(e/2M^*c^3)\cdot \\ \cdot(S^{\alpha\gamma}F_{\gamma\varepsilon}U^\varepsilon U^\beta - S^{\beta\gamma}F_{\gamma\varepsilon}U^\varepsilon U^\alpha) + (ge/4M^{*2}c^3)\{S^{\alpha\gamma}(\partial_\gamma F_{\varepsilon\zeta})S^{\varepsilon\zeta}U^\beta - S^{\beta\gamma}(\partial_\gamma F_{\varepsilon\zeta})S^{\varepsilon\zeta}U^\alpha\}. \tag{62}$$

In the right-hand sides of (61) and (62) the leading terms with the inner angular momentum $S^{\alpha\beta}$ contain the first derivatives of the field and the field itself respectively. Hence if the fields are sufficiently homogeneous the last terms in (61) and (62) may be discarded.

The space parts of (61) and (62) without the last terms may be written in three-dimensional notation. We write $U^\alpha = (\gamma c, \gamma \boldsymbol{v})$ with $\boldsymbol{v} = c\boldsymbol{\beta}$; from (4), (54) and the Lorentz transformation of the antisymmetric tensor $S^{\alpha\beta}$ it follows that the space-space and space-time components $S^{ij} = \varepsilon^{ijk} S_k$ and $S^{i0} = T^i$ may be expressed in terms of the rest-frame inner angular momentum $S_k^{(0)} = \frac{1}{2}\varepsilon_{ijk} S^{(0)ij}$ as $\boldsymbol{S} = \gamma \boldsymbol{S}^{(0)} - \gamma^2(\gamma+1)^{-1}\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{S}^{(0)}$ and $\boldsymbol{T} = \gamma\boldsymbol{\beta}\wedge\boldsymbol{S}^{(0)}$ for the field-free parts. Hence in the right-hand side of (61) and (62) one may substitute $\boldsymbol{T} = \boldsymbol{\beta}\wedge\boldsymbol{S}$, since terms quadratic in the fields were not taken into account. The result is finally

$$(63)\qquad \gamma M^* \mathrm{d}(\gamma \boldsymbol{v})/\mathrm{d}t = \gamma e(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B}) + (ge/2M^*c)\{(\nabla\boldsymbol{E})\cdot(\boldsymbol{\beta}\wedge\boldsymbol{S}) + (\nabla\boldsymbol{B})\cdot\boldsymbol{S}\} + \\ + (e/2M^*c^2)\gamma^2(\mathrm{d}/\mathrm{d}t)[g\boldsymbol{S}\cdot(\boldsymbol{B}-\boldsymbol{\beta}\wedge\boldsymbol{E})\boldsymbol{\beta} - (g-2)\{\boldsymbol{S}\wedge(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) - \boldsymbol{S}\wedge\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{E}\}],$$

$$(64)\qquad \gamma\,\mathrm{d}\boldsymbol{S}/\mathrm{d}t = (ge/2M^*c)\{\boldsymbol{S}\wedge\boldsymbol{B} + (\boldsymbol{\beta}\wedge\boldsymbol{S})\wedge\boldsymbol{E}\} + \\ + (g-2)(e/2M^*c)\{\gamma^2\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B}) - \boldsymbol{S}\boldsymbol{\beta}\cdot\boldsymbol{E} - \gamma^2\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{S}\boldsymbol{\beta}\cdot\boldsymbol{E}\}.$$

(Time and space differentiations in the right-hand sides of (61)-(64) operate in effect only on the fields, because quadratic field terms are discarded.)

Equation (63) contains the «magnetodynamic effect» *i.e.* the time derivative of the so-called «hidden momentum». Its leading term is $(e/2M^*c^2)\cdot(g-2)\boldsymbol{S}\wedge\boldsymbol{E}$ and contains thus only the anomalous magnetic moment. For an atom or a molecule g is of the order of several thousands and hence the anomalous magnetic moment is approximately equal to the total magnetic moment. For a single (charged) particle however the anomalous magnetic moment is essentially different from the total magnetic moment. The study of such a particle which is described by Dirac theory is therefore of particular interest. It will be shown in the following Sections that equations of the same form as the classical ones may be derived for the Dirac particle.

5. – The free Dirac particle: covariant position and spin operators.

It will be useful to study first some aspects of the free particle; in particular covariantly defined position and spin operators. These will be generalized to the case with fields in Sect. **7**.

The wave function for the free particle satisfies Dirac's equation

$$(65)\qquad H^0_{\mathrm{op}}\psi = -(\hbar/i)\partial\psi/\partial t\,, \qquad H^0_{\mathrm{op}} \equiv c\boldsymbol{\alpha}\cdot\boldsymbol{p}_{\mathrm{op}} + \beta mc^2\,.$$

The Hamiltonian H^0_{op} contains « odd » matrices (*i.e.*, which couple the upper and lower components of ψ in the Pauli representation of the Dirac matrices). Pryce ([4]) and Foldy-Wouthuysen ([14]) showed that a transformation with the unitary operator

$$U_{op} = \frac{E_{op} + mc^2 + c\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}_{op}}{\sqrt{2E_{op}(E_{op} + mc^2)}}\,, \qquad (E_{op} \equiv \sqrt{c^2\boldsymbol{p}^2_{op} + m^2c^4}) \tag{66}$$

brings the Hamiltonian in the « even » form

$$\hat{H}^0_{op} \equiv U_{op} H^0_{op} U^\dagger_{op} = \beta E_{op}\,. \tag{67}$$

Since this expression is even the wave equations for upper and lower components are not coupled. For that reason one can distinguish between positive- and negative-energy solutions. If one calculates the expectation value of a physical quantity for a positive- or a negative-energy solution only the even part of the corresponding operator plays a role.

In particular if one wants to define a position operator only its even part is of importance. This even part $\boldsymbol{X}_{op}$ is completely determined if we impose a number of conditions. In the first place from the transformation properties of translation (*viz.* $[P^i_{op}, X^j_{op}] = (\hbar/i)\delta^{ij}$ with the generator $\boldsymbol{P}_{op}$ equal to the operator $\boldsymbol{p}_{op}$), rotation (*viz.* $[M^i_{op}, X^j_{op}] = i\hbar\varepsilon^{ijk}X_{op,k}$ with the generator $\boldsymbol{M}_{op} = \boldsymbol{x}\wedge\boldsymbol{p}_{op} + \frac{1}{2}\hbar\boldsymbol{\sigma}$) and inversion ((A.21) for a polar vector) it follows that in the P-FW picture $\boldsymbol{X}_{op}$ has the form

$$\hat{\boldsymbol{X}}_{op} \equiv U_{op}\boldsymbol{X}_{op}U^\dagger_{op} = \boldsymbol{x} + \{f_1(E_{op}) + \beta f_2(E_{op})\}\boldsymbol{p}_{op} + \{f_3(E_{op}) + \beta f_4(E_{op})\}\boldsymbol{\sigma}\wedge\boldsymbol{p}_{op}\,. \tag{68}$$

Moreover the transformation character under pure Lorentz transformations is determined by the commutation rule ([4,17,18])

$$[N^i_{op}, X^j_{op}] = \tfrac{1}{2}c^{-1}\{X^i_{op}, [H^0_{op}, X^j_{op}]\}\,, \tag{69}$$

where $\boldsymbol{N}_{op} = \frac{1}{2}c^{-1}\{\boldsymbol{x}_D, H^0_{op}\}$ with $\boldsymbol{x}_D$ the Dirac co-ordinate is the generator of a pure Lorentz transformation (see Appendix I). If the form (68) is substituted into (69) one finds a number of differential equations for the form factors $f_i(E_{op})$. If the solutions are inserted into (68) we obtain finally as the position operator in the P-FW picture (see Appendix II):

$$\hat{\boldsymbol{X}}_{op} = \boldsymbol{x} + \hbar\boldsymbol{\sigma}\wedge\boldsymbol{p}_{op}/2m(E_{op} + mc^2) \tag{70}$$

and—with the help of (66)—in the Dirac picture,

$$\boldsymbol{X}_{op} = \boldsymbol{x} + (i\hbar/2mc)\beta(\boldsymbol{\alpha} - c^2\boldsymbol{\alpha}\cdot\boldsymbol{p}_{op}\boldsymbol{p}_{op}/E^2_{op})\,. \tag{71}$$

In this way the even part of the position operator has been obtained uniquely by imposing its transformation character. A position operator of this form has been proposed by Pryce [4] starting from classical considerations.

The spin angular momentum $\frac{1}{2}\hbar\boldsymbol{\Sigma}_{\text{op}}$ is obtained by subtracting the orbital angular momentum $\boldsymbol{X}_{\text{op}}\wedge\boldsymbol{p}_{\text{op}}$ from the total angular momentum $\boldsymbol{x}\wedge\boldsymbol{p}_{\text{op}}+\frac{1}{2}\hbar\boldsymbol{\sigma}$; it reads in the P-FW picture

$$\hat{\boldsymbol{\Sigma}}_{\text{op}}=(E_{\text{op}}/mc^2)\boldsymbol{\sigma}-\boldsymbol{p}_{\text{op}}\boldsymbol{p}_{\text{op}}\cdot\boldsymbol{\sigma}/m(E_{\text{op}}+mc^2) \tag{72}$$

and in the Dirac picture

$$\boldsymbol{\Sigma}_{\text{op}}=\boldsymbol{\sigma}-i\beta\boldsymbol{\alpha}\wedge\boldsymbol{p}_{\text{op}}/mc\,. \tag{73}$$

Since (72) is even, the spin defined in this way is conserved, as follows from (67). Expression (73) is the space-space part of the tensor [4,20,21]

$$\Sigma^{\mu\nu}_{\text{op}}=\sigma^{\mu\nu}+(1/mc)(\gamma^\mu p^\nu_{\text{op}}-\gamma^\nu p^\mu_{\text{op}})\,, \tag{74}$$

where $\boldsymbol{\gamma}=-i\beta\boldsymbol{\alpha}$, $\gamma^0=-i\beta$, $\sigma^{\mu\nu}=-\frac{1}{2}i[\gamma^\mu,\gamma^\nu]$ and $p^0_{\text{op}}=\boldsymbol{\alpha}\cdot\boldsymbol{p}_{\text{op}}+\beta mc$. Its space-time components $\Sigma^{i0}_{\text{op}}\equiv\boldsymbol{T}^i_{\text{op}}$ read in the P-FW picture $\hat{\boldsymbol{T}}_{\text{op}}=\beta\boldsymbol{p}_{\text{op}}\wedge\boldsymbol{\sigma}/mc$. The velocity operator in the P-FW picture is $\hat{\boldsymbol{V}}_{\text{op}}\equiv(i/\hbar)[\hat{H}^0_{\text{op}},\hat{\boldsymbol{X}}_{\text{op}}]=\beta c^2\boldsymbol{p}_{\text{op}}/E_{\text{op}}$ as follows from (67) and (70). Hence one has the relation $c^{-1}\hat{\boldsymbol{V}}_{\text{op}}\wedge\hat{\boldsymbol{\Sigma}}_{\text{op}}=\hat{\boldsymbol{T}}_{\text{op}}$, which is the quantum-mechanical counterpart of (4) with (42).

6. – Transformation to even form of the Hamiltonian for a Dirac particle in an external field.

The Dirac-Hamiltonian for a particle in an electromagnetic field $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{B}(\boldsymbol{r},t)$, with potentials $\varphi(\boldsymbol{r},t)$, $\boldsymbol{A}(\boldsymbol{r},t)$, reads

$$H_{\text{op}}\;=c\boldsymbol{\alpha}\cdot\boldsymbol{\pi}_{\text{op}}+\beta mc^2+e\varphi+H_{a,\text{op}}\,, \tag{75}$$

$$H_{a,\text{op}}\equiv\tfrac{1}{2}(g-2)\mu_B(i\beta\boldsymbol{\alpha}\cdot\boldsymbol{E}-\beta\boldsymbol{\sigma}\cdot\boldsymbol{B})\,, \tag{76}$$

where $\boldsymbol{\pi}_{\text{op}}\equiv\boldsymbol{p}_{\text{op}}-(e/c)\boldsymbol{A}$, $\mu_B\equiv e\hbar/2mc$ and the last (Pauli) term represents the coupling of the anomalous magnetic moment with the field. The Hamiltonian will be put to even form by three successive transformations. First a transformation will be performed [16] with the operator

$$S_{1,\text{op}}\rightleftarrows\frac{E_\pi+mc^2+c\beta\boldsymbol{\alpha}\cdot\boldsymbol{\pi}}{\sqrt{2E_\pi(E_\pi+mc^2)}}\,,\qquad(E_\pi\equiv\sqrt{c^2\boldsymbol{\pi}^2+m^2c^4})\,, \tag{77}$$

where in the right-hand side the Weyl transform (see Appendix III) has been written. If $e=0$, the operator $S_{1,\mathrm{op}}$ reduces to U_{op} of (66). If only terms linear in e and without derivatives of the fields are taken into account we find

$$(78)\qquad (S_1HS_1^\dagger)_{\mathrm{op}} \rightleftarrows S_1HS_1^\dagger + \frac{ie\hbar}{2c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}\frac{\partial H}{\partial p_j}U^\dagger B^k + \frac{ie\hbar}{2c}\varepsilon_{ijk}U\frac{\partial H}{\partial p_i}\frac{\partial U^\dagger}{\partial p_j}B^k + \frac{ie\hbar}{2c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}H\frac{\partial U^\dagger}{\partial p_j}B^k - \frac{ie\hbar}{2}\frac{\partial U}{\partial p_i}\frac{\partial \varphi}{\partial x^i}U^\dagger + \frac{ie\hbar}{2}U\frac{\partial \varphi}{\partial x_i}\frac{\partial U^\dagger}{\partial p^i},$$

since the Weyl transform of $S_{1,\mathrm{op}}$ depends on $\boldsymbol{p}$ and $\boldsymbol{x}$ only through $\boldsymbol{\pi}$. Here the same symbols are used for operators (l.h.s.) and their Weyl transforms (r.h.s.). The operator $S_{1,\mathrm{op}}$ is not unitary since

$$(79)\qquad (S_1S_1^\dagger)_{\mathrm{op}} \rightleftarrows 1 + \frac{ie\hbar}{2c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}\frac{\partial U^\dagger}{\partial p_j}B^k.$$

However the product $U_{1,\mathrm{op}} = S_{2,\mathrm{op}}S_{1,\mathrm{op}}$ is unitary (up to terms linear in e and without field derivatives), if $S_{2,\mathrm{op}}$ is chosen such that

$$(80)\qquad S_{2,\mathrm{op}} \rightleftarrows 1 - \frac{ie\hbar}{4c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}\frac{\partial U^\dagger}{\partial p_j}B^k.$$

The transformed Hamiltonian becomes

$$(81)\qquad (U_1HU_1^\dagger)_{\mathrm{op}} \rightleftarrows S_1HS_1^\dagger + \frac{ie\hbar}{2c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}\frac{\partial H}{\partial p_j}U^\dagger B^k + \frac{ie\hbar}{2c}\varepsilon_{ijk}U\frac{\partial H}{\partial p_i}\frac{\partial U^\dagger}{\partial p_j}B^k + \frac{ie\hbar}{2c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}H\frac{\partial U^\dagger}{\partial p_j}B^k - \frac{ie\hbar}{2}\frac{\partial U}{\partial p_i}\frac{\partial \varphi}{\partial x^i}U^\dagger + \frac{ie\hbar}{2}U\frac{\partial \varphi}{\partial x_i}\frac{\partial U^\dagger}{\partial p^i} - \frac{ie\hbar}{4c}\varepsilon_{ijk}\frac{\partial U}{\partial p_i}\frac{\partial U^\dagger}{\partial p_j}B^k\beta E - \frac{ie\hbar}{4c}\beta E\varepsilon_{ijk}\frac{\partial U}{\partial p_i}\frac{\partial U^\dagger}{\partial p_j}B^k.$$

If we introduce the abbreviation

$$(82)\qquad \boldsymbol{\xi} \equiv (\hbar/i)(\partial U/\partial \boldsymbol{p})U^\dagger = -(\hbar/i)U\partial U^\dagger/\partial\boldsymbol{p} \equiv \boldsymbol{\xi}_e + \boldsymbol{\xi}_o,$$

where e and o indicate even and odd parts, the expression (81) becomes

$$(83)\qquad (U_1HU_1^\dagger)_{\mathrm{op}} \rightleftarrows \beta E_\pi + e\varphi - (ieE/2\hbar c)\varepsilon_{ijk}\{\xi^i_o, \beta\xi^j_o\}B^k - (ec/E)\beta\boldsymbol{\xi}_e\cdot(\boldsymbol{p}\wedge\boldsymbol{B}) + e\boldsymbol{\xi}\cdot\partial\varphi/\partial\boldsymbol{x} + \tfrac{1}{2}(g-2)\mu_B U(i\beta\boldsymbol{\alpha}\cdot\boldsymbol{E} - \beta\boldsymbol{\sigma}\cdot\boldsymbol{B})U^\dagger.$$

Since the time derivative of the transformed wave function is determined by $\{U_1 H U_1^\dagger - (\hbar/i)(\partial U_1/\partial t) U_1^\dagger\}_{\text{op}}$ we also need $\partial U_{1,\text{op}}/\partial t$ of which the Weyl transform is $-(e/c)(\partial U/\partial \boldsymbol{p})\cdot(\partial \boldsymbol{A}/\partial t)$ (up to terms linear in e and without second derivatives of the potentials). We obtain thus

$$(84)\qquad \{U_1 H U_1^\dagger - (\hbar/i)(\partial U_1/\partial t) U_1^\dagger\}_{\text{op}} \rightleftarrows \beta E_\pi + e\varphi - (ieE/2\hbar c)\varepsilon_{ijk}\{\xi^i, \beta\xi_o^j\} B^k - \\ - (ec/E)\beta\boldsymbol{\xi}_e\cdot(\boldsymbol{p}\wedge\boldsymbol{B}) - e\boldsymbol{\xi}\cdot\boldsymbol{E} + \tfrac{1}{2}(g-2)\mu_B U(i\beta\boldsymbol{\alpha}\cdot\boldsymbol{E} - \beta\boldsymbol{\sigma}\cdot\boldsymbol{B})U^\dagger .$$

Here the odd terms which depend on the fields may be transformed away by means of a final unitary (up to terms linear in e and without field derivatives) transformation

$$(85)\qquad U_{2,\text{op}} \rightleftarrows 1 - (ie/4\hbar c)\varepsilon_{ijk}\{\xi_e^i, \xi_o^j\} B^k - \\ - (e/2E)\beta\boldsymbol{\xi}_o\cdot\boldsymbol{E} + \{(g-2)\mu_B/4E\}\beta\{U(i\beta\boldsymbol{\alpha}\cdot\boldsymbol{E} - \beta\boldsymbol{\sigma}\cdot\boldsymbol{B})U^\dagger\}_o .$$

The explicit transformed Hamiltonian is obtained if we substitute the expressions

$$(86)\qquad \boldsymbol{\xi}_e = \frac{\hbar c^2 \boldsymbol{p}\wedge\boldsymbol{\sigma}}{2E(E+mc^2)}, \qquad \boldsymbol{\xi}_o = -\frac{\hbar ic\beta\boldsymbol{\alpha}}{2E} + \frac{\hbar ic^3\beta\boldsymbol{\alpha}\cdot\boldsymbol{p}\boldsymbol{p}}{2E^2(E+mc^2)},$$

which follow from (82) with (66). The result is—up to terms linear in e and without field derivatives—

$$(87)\qquad \hat{H}_{\text{op}} \equiv \left\{U_2 U_1 H U_1^\dagger U_2^\dagger - \frac{\hbar}{i}\frac{\partial(U_2U_1)}{\partial t} U_1^\dagger U_2^\dagger\right\}_{\text{op}} \rightleftarrows \\ \rightleftarrows \beta E_\pi + e\varphi - \mu_B\frac{mc^2}{E}\beta\boldsymbol{\sigma}\cdot\boldsymbol{B} - \mu_B\frac{mc^3}{E(E+mc^2)}(\boldsymbol{p}\wedge\boldsymbol{\sigma})\cdot\boldsymbol{E} - \\ - \frac{1}{2}(g-2)\mu_B\left\{\beta\boldsymbol{\sigma}\cdot\boldsymbol{B} - \frac{\beta c^2\boldsymbol{p}\cdot\boldsymbol{\sigma}\boldsymbol{p}\cdot\boldsymbol{B}}{E(E+mc^2)} + \frac{c(\boldsymbol{p}\wedge\boldsymbol{\sigma})\cdot\boldsymbol{E}}{E}\right\},$$

which is the relativistic generalization of the expression derived by Foldy-Wouthuysen [14]. Apart from the anomalous terms it has been found by Blount [16]. We shall call it the Hamiltonian in the Blount picture.

7. – Covariant position and spin operators for the Dirac particle in the presence of a field.

In order to obtain the equation of motion (Sect. **8**) up to first order in the derivatives of the fields an expression for the position operator including terms with the potentials will be needed. Since the Hamiltonian (87) in the Blount picture is even again—as in Sect. 5—only the even part $\boldsymbol{X}_{\text{op}}$ of the position operator is relevant. This part is fixed by a set of conditions. In the first place the expression for $\boldsymbol{X}_{\text{op}}$ in the Blount picture should reduce to the form (70) for the field-free case. If furthermore the transformation properties of $\boldsymbol{X}_{\text{op}}$ under translations (*viz.* $[\boldsymbol{P}_{i,\text{op}}, \boldsymbol{X}_{j,\text{op}}] = (\hbar/i)\delta_{ij}$), rotations (cf. (A.17) of Appendix I), inversion (cf. (A.21) for a polar vector) and pure Lorentz transformations (cf. (A.10)) are given it follows that in the Blount-picture $\boldsymbol{X}_{\text{op}}$ has the form

$$(88)\qquad \hat{\boldsymbol{X}}_{\text{op}} \rightleftarrows \boldsymbol{x} + \frac{\hbar\boldsymbol{\sigma}\wedge\boldsymbol{\pi}}{2m(E_\pi + mc^2)} + (a_1 + \beta a_2)\left(\boldsymbol{A} - \frac{c}{E}\beta\varphi\boldsymbol{p}\right) + \\ + (a_3 + \beta a_4)\left\{\frac{c}{E}\boldsymbol{p}\wedge\boldsymbol{\sigma}\boldsymbol{p}\cdot\boldsymbol{A} + \frac{m^2c^3}{E}\boldsymbol{A}\wedge - \boldsymbol{\sigma}\frac{mc^3\boldsymbol{p}\wedge\boldsymbol{A}\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E(E+mc^2)} - \beta\boldsymbol{p}\wedge\boldsymbol{\sigma}\varphi\right\}.$$

The velocity operator (up to terms with the potentials) corresponding to this position operator follows with (87):

$$(89)\qquad \hat{\boldsymbol{V}}_{\text{op}} \equiv \frac{i}{\hbar}[\hat{H}_{\text{op}}, \hat{\boldsymbol{X}}_{\text{op}}] \rightleftarrows \beta c^2\boldsymbol{\pi}/E_\pi \equiv \hat{\boldsymbol{V}}.$$

In an analogous way the even part $\hat{\boldsymbol{\Sigma}}_{\text{op}}$ of the spin operator in the Blount picture up to terms with the potentials (which should reduce to (72) in the field-free case) is found by fixing its transformation properties under translations (*viz.* $[\boldsymbol{P}_{i,\text{op}}, \boldsymbol{\Sigma}_{j,\text{op}}] = 0$), rotations (cf. (A.17)), inversion (cf. (A.21) for an axial vector), and pure Lorentz transformations (cf. (A.15)-(A.16)). We obtain

$$(90)\qquad \hat{\boldsymbol{\Sigma}}_{\text{op}} \rightleftarrows \frac{E_\pi}{mc^2}\boldsymbol{\sigma} - \frac{\boldsymbol{\pi}\boldsymbol{\pi}\cdot\boldsymbol{\sigma}}{m(E_\pi + mc^2)} + (b_1 + \beta b_2)\boldsymbol{p}\wedge\boldsymbol{A} + \\ + (b_3 + \beta b_4)\left(\boldsymbol{A}\boldsymbol{p}\cdot\boldsymbol{\sigma} - mc\beta\boldsymbol{\sigma}\varphi - \frac{\beta c\boldsymbol{p}\boldsymbol{p}\cdot\boldsymbol{\sigma}\varphi}{E + mc^2}\right) + \\ + (b_5 + \beta b_6)\left(\frac{c\boldsymbol{p}\boldsymbol{p}\cdot\boldsymbol{A}\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E + mc^2} - c^{-1}E\boldsymbol{\sigma}\boldsymbol{p}\cdot\boldsymbol{A} - \frac{\beta E\boldsymbol{p}\boldsymbol{p}\cdot\boldsymbol{\sigma}\varphi}{E + mc^2} + c^{-2}\beta E^2\boldsymbol{\sigma}\varphi\right).$$

This axial vector forms the space-space part Σ^{ij}_{op} of an antisymmetric quasi-four-tensor $\Sigma^{\alpha\beta}_{\text{op}}$ of which the space-time components $\Sigma^{i0}_{\text{op}} \equiv \boldsymbol{T}^i_{\text{op}}$ are

$$(91)\qquad \hat{\boldsymbol{T}}_{\text{op}} \rightleftarrows \frac{\beta\boldsymbol{\pi}\wedge\boldsymbol{\sigma}}{mc} + (b_1 + \beta b_2)(\boldsymbol{p}\varphi - \beta c^{-1}E\boldsymbol{A}) + \\ + (b_3 + \beta b_4)\left(mc\beta\boldsymbol{\sigma}\wedge\boldsymbol{A} + \frac{c\beta\boldsymbol{p}\wedge\boldsymbol{A}\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E + mc^2}\right) + (b_5 + \beta b_6)(\beta\boldsymbol{\sigma}\wedge\boldsymbol{p}\boldsymbol{p}\cdot\boldsymbol{A} + c^{-1}E\varphi\boldsymbol{p}\wedge\boldsymbol{\sigma}).$$

A further constraint on the spin operator follows from the orthogonality condition

$$c^{-1}\,\hat{\boldsymbol{V}}_{\text{op}}\wedge\hat{\boldsymbol{\Sigma}}_{\text{op}}=\hat{\boldsymbol{T}}_{\text{op}}\,, \tag{92}$$

which is the quantum-mechanical counterpart (up to terms with the potentials) of the classical condition (4) with (46). It is satisfied if b_1, b_2, b_3 and b_4 vanish.

The position operator and the spin operator are not independent of each other. As a generalization of the field-free case we shall require that the sum of the orbital angular momentum $\boldsymbol{X}_{\text{op}}\wedge\boldsymbol{\pi}_{\text{op}}$ and the spin $\frac{1}{2}\hbar\boldsymbol{\Sigma}_{\text{op}}$ is equal to the operator $\boldsymbol{x}\wedge\boldsymbol{\pi}_{\text{op}}+\frac{1}{2}\hbar\boldsymbol{\sigma}$ in the Dirac picture. As shown in Appendix I, formulae (A.17)-(A.20), this quantity is the generator of rotations for a special class of operators. The requirement reads written in the Blount picture

$$\hat{\boldsymbol{X}}_{\text{op}}\wedge\boldsymbol{\pi}_{\text{op}}+\tfrac{1}{2}\hbar\hat{\boldsymbol{\Sigma}}_{\text{op}}=\boldsymbol{x}\wedge\boldsymbol{\pi}_{\text{op}}+\tfrac{1}{2}\hbar\boldsymbol{\sigma}\,. \tag{93}$$

If (88) and (90) are inserted we get the result that the remaining constants a_1, a_2, a_3, a_4, b_5 and b_6 vanish as well, so that finally we obtain—in the Blount picture

$$\hat{\boldsymbol{X}}_{\text{op}}\rightleftarrows\boldsymbol{x}+\hbar\boldsymbol{\sigma}\wedge\boldsymbol{\pi}/2m(E_\pi+mc^2)\,, \tag{94}$$

$$\hat{\boldsymbol{\Sigma}}_{\text{op}}\rightleftarrows(E_\pi/mc^2)\boldsymbol{\sigma}-\boldsymbol{\pi}\boldsymbol{\pi}\cdot\boldsymbol{\sigma}/m(E_\pi+mc^2) \tag{95}$$

and in the Dirac picture

$$\boldsymbol{X}_{\text{op}}\rightleftarrows\boldsymbol{x}+(i\hbar/2mc)\beta(\boldsymbol{\alpha}-c^2\boldsymbol{\alpha}\cdot\boldsymbol{\pi}\boldsymbol{\pi}/E_\pi^2)\,, \tag{96}$$

$$\boldsymbol{\Sigma}_{\text{op}}\rightleftarrows\boldsymbol{\sigma}-i\beta\boldsymbol{\alpha}\wedge\boldsymbol{\pi}/mc \tag{97}$$

as position and spin operators. The spin operator $\boldsymbol{\Sigma}_{\text{op}}$ forms the space-space part Σ^{ij}_{op} of an antisymmetric quasi-four-tensor $\Sigma^{\alpha\beta}_{\text{op}}$ of which the space-time components $\Sigma^{i0}_{\text{op}}\equiv T^i_{\text{op}}$ read in the Blount picture (cf. (91))

$$\hat{\boldsymbol{T}}_{\text{op}}\rightleftarrows\beta\boldsymbol{\pi}\wedge\boldsymbol{\sigma}/mc\,. \tag{98}$$

If in the expressions (95) and (98) the quantity $\boldsymbol{\pi}$ is expressed in terms of the velocity $\hat{\boldsymbol{V}}$ with the help of (89) one obtains formulae like the classical expressions mentioned after (62); here $(\hbar/2)\boldsymbol{\sigma}$ turns out to play the role of the rest frame spin.

8. – Equations of motion and of spin.

The equation of motion for the Dirac particle is obtained by taking twice the total time derivative (with the use of the Hamiltonian (87)) of the position operator (94) in the Blount picture.

In the first place we have to evaluate the velocity operator. The Weyl transform of the commutator $[\hat{H}_{op}, \hat{X}_{op}]$ can be expressed in terms of the Weyl transforms of $\hat{H}_{op}$ and $\hat{X}_{op}$ with the use of (A.37). Up to terms linear in e and without field derivatives one obtains for the velocity operator (cf. (89))

$$(99)\qquad \hat{V}_{op} \equiv \frac{\mathrm{d}\hat{X}_{op}}{\mathrm{d}t} \equiv \frac{i}{\hbar}[\hat{H}_{op}, \hat{X}_{op}] + \frac{\partial \hat{X}_{op}}{\partial t} \rightleftarrows$$

$$\rightleftarrows \frac{\beta \boldsymbol{\pi} c^2}{E_\pi} - \frac{e\hbar c\beta \boldsymbol{\sigma p}\cdot \boldsymbol{B}}{2mE(E+mc^2)} + \frac{e\hbar c\beta \boldsymbol{p\sigma}\cdot\boldsymbol{B}(E^2+Emc^2+m^2c^4)}{2mE^3(E+mc^2)} +$$

$$+ \frac{e\hbar c^2\boldsymbol{p}(\boldsymbol{p}\wedge\boldsymbol{\sigma})\cdot\boldsymbol{E}}{2mE^3} + \frac{1}{2}(g-2)\mu_B\left\{\frac{\beta\boldsymbol{p}\cdot\boldsymbol{\sigma B}}{mE} - \frac{\beta c^2\boldsymbol{pp}\cdot\boldsymbol{\sigma p}\cdot\boldsymbol{B}}{mE^3} + \right.$$

$$\left. + \frac{c\boldsymbol{E}\wedge\boldsymbol{\sigma}}{E} + \frac{c^3\boldsymbol{p}(\boldsymbol{p}\wedge\boldsymbol{\sigma})\cdot\boldsymbol{E}}{E^3} - \frac{c\boldsymbol{p}\cdot\boldsymbol{\sigma p}\wedge\boldsymbol{E}}{mE(E+mc^2)}\right\}.$$

Likewise the acceleration operator may be calculated up to terms linear in e and with first derivatives of the field

$$(100)\qquad \frac{\mathrm{d}^2\hat{X}_{op}}{\mathrm{d}t} \equiv \frac{\mathrm{d}\hat{V}_{op}}{\mathrm{d}t} \equiv \frac{i}{\hbar}[\hat{H}_{op}, \hat{V}_{op}] + \frac{\partial\hat{V}_{op}}{\partial t} \rightleftarrows \frac{c^2}{E}\left(\mathsf{U} - \frac{c^2\boldsymbol{pp}}{E^2}\right)\cdot$$

$$\cdot\left[e\beta\boldsymbol{E} + \frac{ec}{E}\boldsymbol{p}\wedge\boldsymbol{B} + \mu_B\left\{\frac{mc^2}{E}(\boldsymbol{\nabla B})\cdot\boldsymbol{\sigma} + \frac{\beta mc^3}{E(E+mc^2)}(\boldsymbol{\nabla E})\cdot(\boldsymbol{p}\wedge\boldsymbol{\sigma})\right\} + \right.$$

$$+ \mu_B\left(\frac{\partial}{\partial t} + \frac{\beta c^2\boldsymbol{p}\cdot\boldsymbol{\nabla}}{E}\right)\left\{\frac{\boldsymbol{p}(\boldsymbol{p}\wedge\boldsymbol{\sigma})\cdot\boldsymbol{E}}{m^2c^3} - \frac{\beta\boldsymbol{\sigma p}\cdot\boldsymbol{B}}{E+mc^2} + \beta\frac{\boldsymbol{p\sigma}\cdot\boldsymbol{B}(E^2+Emc^2+m^2c^4)}{m^2c^4(E+mc^2)} - \right.$$

$$\left. - \frac{\beta\boldsymbol{pp}\cdot\boldsymbol{\sigma p}\cdot\boldsymbol{B}}{m^2c^2(E+mc^2)}\right\} + \frac{1}{2}(g-2)\mu_B\left\{(\boldsymbol{\nabla B})\cdot\boldsymbol{\sigma} - \frac{c^2(\boldsymbol{\nabla B})\cdot\boldsymbol{pp}\cdot\boldsymbol{\sigma}}{E(E+mc^2)} - \beta\frac{c(\boldsymbol{\nabla E})\cdot(\boldsymbol{\sigma}\wedge\boldsymbol{p})}{E}\right\} +$$

$$\left. + \frac{1}{2}(g-2)\mu_B\left(\frac{\partial}{\partial t} + \frac{\beta c^2\boldsymbol{p}\cdot\boldsymbol{\nabla}}{E}\right)\left\{c^{-1}\boldsymbol{E}\wedge\boldsymbol{\sigma} + \frac{\beta\boldsymbol{\sigma}\cdot\boldsymbol{pB}}{mc^2} - \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma p}\wedge\boldsymbol{E}}{mc(E+mc^2)}\right\}\right].$$

The first terms at the right-hand side contain the fields $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ as functions of the space co-ordinate in the Blount picture. Since the position of the particle is given by $\hat{\boldsymbol{X}}$ (94), we now wish to introduce the fields as functions of $\hat{\boldsymbol{X}}$. Then we obtain for the first two terms of the right-hand side of (100) up to terms linear in e and with first derivatives of the fields

$$(101)\qquad e\beta\boldsymbol{E}(\boldsymbol{X}, t) + \frac{ec}{E}\boldsymbol{p}\wedge\boldsymbol{B}(\boldsymbol{X}, t) - \mu_B\left\{\frac{\beta c(\boldsymbol{\sigma}\wedge\boldsymbol{p})\cdot\boldsymbol{\nabla E}}{E+mc^2} + \frac{c^2(\boldsymbol{\sigma}\wedge\boldsymbol{p})\cdot\boldsymbol{\nabla}(\boldsymbol{p}\wedge\boldsymbol{B})}{E(E+mc^2)}\right\}.$$

(The noncommutative character of the components of $\hat{\boldsymbol{X}}$ does not cause trouble here because of the limitation to first derivatives of the fields.) Furthermore we introduce the spin operator $\hat{\boldsymbol{\Sigma}}$ (95) instead of $\boldsymbol{\sigma}$; since in (100) $\boldsymbol{\sigma}$ is only needed up to order e^0 we write

$$\boldsymbol{\sigma} = \frac{mc^2}{E}\hat{\boldsymbol{\Sigma}} + \frac{c^2 \boldsymbol{p}\boldsymbol{p}\cdot\hat{\boldsymbol{\Sigma}}}{E(E+mc^2)} . \tag{102}$$

Substituting (101) and (102) into (100), using the Maxwell equation $\nabla\wedge\boldsymbol{E} = -\partial_0\boldsymbol{B}$ and introducing the abbreviations $\boldsymbol{\beta}\equiv c\boldsymbol{p}/E$, $\gamma\equiv(1-\beta^2)^{-\frac{1}{2}}$ and $\partial_0\equiv c^{-1}\partial/\partial t$ we obtain

$$\begin{aligned} m\,\mathrm{d}\hat{\boldsymbol{V}}_{\mathrm{op}}/\mathrm{d}t \rightleftarrows{}& \gamma^{-2}(\mathbf{U}-\boldsymbol{\beta}\boldsymbol{\beta})\cdot[\gamma e\boldsymbol{E}(\boldsymbol{X},t)+\gamma e\boldsymbol{\beta}\wedge\boldsymbol{B}(\boldsymbol{X},t)+ \\ &+\tfrac{1}{2}g\mu_B\{(\nabla\boldsymbol{B})\cdot\hat{\boldsymbol{\Sigma}}+(\nabla\boldsymbol{E})\cdot(\boldsymbol{\beta}\wedge\hat{\boldsymbol{\Sigma}})+\gamma^2(\partial_0+\boldsymbol{\beta}\cdot\nabla)\boldsymbol{\beta}\hat{\boldsymbol{\Sigma}}\cdot(\boldsymbol{B}-\boldsymbol{\beta}\wedge\boldsymbol{E})\}- \\ &-\tfrac{1}{2}(g-2)\mu_B\gamma^2(\partial_0+\boldsymbol{\beta}\cdot\nabla)\{\hat{\boldsymbol{\Sigma}}\wedge(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B})-\hat{\boldsymbol{\Sigma}}\wedge\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{E}\}] . \end{aligned} \tag{103}$$

Here we have limited ourselves to the « upper left » part of the matrix expression (*i.e.* β replaced by 1) which is the relevant part if expectation values for positive-energy solutions are evaluated.

The spin equation follows by taking the total derivative of the spin operator (95) in the Blount picture. Using the Hamiltonian (87) we obtain for its Weyl transform

$$\begin{aligned} \frac{\hbar}{2}\frac{\mathrm{d}\hat{\boldsymbol{\Sigma}}_{\mathrm{op}}}{\mathrm{d}t} &\equiv \frac{i}{\hbar}\left[\hat{H}_{\mathrm{op}}, \frac{\hbar}{2}\hat{\boldsymbol{\Sigma}}_{\mathrm{op}}\right]+\frac{\hbar}{2}\frac{\partial\hat{\boldsymbol{\Sigma}}_{\mathrm{op}}}{\partial t} \rightleftarrows \\ &\rightleftarrows \mu_B\left\{\beta\boldsymbol{\sigma}\wedge\boldsymbol{B}-\frac{\beta c^2\boldsymbol{p}\wedge\boldsymbol{B}\boldsymbol{p}\cdot\boldsymbol{\sigma}}{E(E+mc^2)}+\left(\frac{c\boldsymbol{p}}{E}\wedge\boldsymbol{\sigma}\right)\wedge\boldsymbol{E}\right\}+ \\ &+\frac{1}{2}(g-2)\mu_B\left\{\beta\boldsymbol{\sigma}\wedge\boldsymbol{B}+\frac{\beta\boldsymbol{p}\wedge\boldsymbol{B}\boldsymbol{p}\cdot\boldsymbol{\sigma}}{m(E+mc^2)}-\frac{c\boldsymbol{p}\boldsymbol{\sigma}\cdot\boldsymbol{E}}{E}+\frac{E\boldsymbol{p}\cdot\boldsymbol{\sigma}}{mc}-\frac{c\boldsymbol{p}\boldsymbol{p}\cdot\boldsymbol{\sigma}\boldsymbol{p}\cdot\boldsymbol{E}}{mE(E+mc^2)}\right\}, \end{aligned} \tag{104}$$

where only terms linear in e and without derivatives of the fields have been included. Upon introduction of $\hat{\boldsymbol{\Sigma}}$ with the help of (102) the equation becomes

$$\begin{aligned} \frac{\hbar}{2}\frac{\mathrm{d}\hat{\boldsymbol{\Sigma}}_{\mathrm{op}}}{\mathrm{d}t} \rightleftarrows{}& \mu_B\frac{g}{2}\frac{mc^2}{E}\left\{\beta\hat{\boldsymbol{\Sigma}}\wedge\boldsymbol{B}+\left(\frac{c\boldsymbol{p}}{E}\wedge\hat{\boldsymbol{\Sigma}}\right)\wedge\boldsymbol{E}\right\}+ \\ &+\frac{1}{2}(g-2)\mu_B\left\{\beta\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{\Sigma}}}{mE}\boldsymbol{p}\wedge\boldsymbol{B}-\frac{mc^3}{E^2}\boldsymbol{p}\wedge(\hat{\boldsymbol{\Sigma}}\wedge\boldsymbol{E})-\frac{c}{mE^2}\boldsymbol{p}\wedge(\boldsymbol{p}\wedge\boldsymbol{E})\boldsymbol{p}\cdot\hat{\boldsymbol{\Sigma}}\right\}. \end{aligned} \tag{105}$$

Using the same abbreviations as in (103) and replacing again the matrix β by 1 (*i.e.* limiting ourselves to the part occurring in the expectation value for the positive-energy solutions) one gets finally

$$\begin{aligned} \tfrac{1}{2}\hbar\,\mathrm{d}\hat{\boldsymbol{\Sigma}}_{\mathrm{op}}/\mathrm{d}t \rightleftarrows{}& \tfrac{1}{2}g\mu_B\gamma^{-1}\{\hat{\boldsymbol{\Sigma}}\wedge\boldsymbol{B}+(\boldsymbol{\beta}\wedge\hat{\boldsymbol{\Sigma}})\wedge\boldsymbol{E}\}+ \\ &+\tfrac{1}{2}(g-2)\mu_B\gamma^{-1}\{\gamma^2\boldsymbol{\beta}\cdot\hat{\boldsymbol{\Sigma}}(\boldsymbol{E}+\boldsymbol{\beta}\wedge\boldsymbol{B})-\hat{\boldsymbol{\Sigma}}\boldsymbol{\beta}\cdot\boldsymbol{E}-\gamma^2\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{E}\boldsymbol{\beta}\cdot\hat{\boldsymbol{\Sigma}}\} . \end{aligned} \tag{106}$$

Furthermore let us consider the time evolution of the « rest frame spin » $\frac{1}{2}\hbar\boldsymbol{\sigma}$ in the Blount picture. One finds

$$(107)\qquad \frac{\hbar}{2}\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} \rightleftarrows \mu_B\left\{\beta\frac{mc^2}{E}\,\boldsymbol{\sigma}\wedge\boldsymbol{B} - \frac{mc^3}{E(E+mc^2)}\,\boldsymbol{\sigma}\wedge(\boldsymbol{p}\wedge\boldsymbol{E})\right\} + \\ + \frac{1}{2}(g-2)\mu_B\left\{\beta\boldsymbol{\sigma}\wedge\boldsymbol{B} - \frac{\beta c^2\boldsymbol{p}\cdot\boldsymbol{B}\boldsymbol{\sigma}\wedge\boldsymbol{p}}{E(E+mc^2)} - \frac{c}{E}\boldsymbol{\sigma}\wedge(\boldsymbol{p}\wedge\boldsymbol{E})\right\}.$$

Replacing again the matrix β by 1 we obtain

$$(108)\qquad \tfrac{1}{2}\hbar\,\mathrm{d}\boldsymbol{\sigma}/\mathrm{d}t \rightleftarrows \boldsymbol{\sigma}\wedge(\boldsymbol{\omega}_{\mathrm{L}} + \boldsymbol{\omega}_{\mathrm{Th}}) + \tfrac{1}{2}(g-2)\boldsymbol{\sigma}\wedge\boldsymbol{\omega}_{\mathrm{L}},$$

where we introduced the angular frequency vectors of the Larmor and Thomas precessions

$$(109)\qquad \boldsymbol{\omega}_{\mathrm{L}} \equiv \mu_B\{\boldsymbol{B} - \gamma(\gamma+1)^{-1}\boldsymbol{\beta}\boldsymbol{\beta}\cdot\boldsymbol{B} - \boldsymbol{\beta}\wedge\boldsymbol{E}\},$$

$$(110)\qquad \boldsymbol{\omega}_{\mathrm{Th}} \equiv \mu_B\gamma(\gamma+1)^{-1}\boldsymbol{\beta}\wedge(\boldsymbol{E} + \boldsymbol{\beta}\wedge\boldsymbol{B}).$$

Equations (103) and (106) are the quantum-mechanical counterparts of the classical equations (63) and (64) since one should notice that

$$(111)\qquad \mathrm{d}\boldsymbol{v}/\mathrm{d}t = \gamma^{-1}(\mathbf{U} - \boldsymbol{v}\boldsymbol{v}/c^2)\cdot\mathrm{d}(\gamma\boldsymbol{v})/\mathrm{d}t.$$

The magnetodynamic effect in (103) contains indeed the anomalous part of the magnetic moment only.

The resulting eqs. (103) and (106) are operator equations which have the same form as the corresponding classical equations. The equations which result if expectation values for a wave packet are taken have the same form as the classical equations only if the expectation value of products of operators can be written as the product of expectation values. This is approximately true for narrow wave packets in the limit of $\hbar \to 0$. In that approximation the quasi-covariance of the operators leads to the covariance of the expectation values (cf. Appendix I).

Appendix I

Conditions of covariance.

In Sect. **7** a position operator $\boldsymbol{X}_{\mathrm{op}}$ was determined by relation (A.10) (cf. Sect. **5** formula (69) for the field-free case). This relation may be obtained in the following way. The expectation value of $\boldsymbol{X}_{\mathrm{op}}$ (depending on the co-ordinate $\boldsymbol{x}$, the momentum $\boldsymbol{p}_{\mathrm{op}} = (\hbar/i)\,\partial/\partial\boldsymbol{x}$ and the potentials $\boldsymbol{A}$ and φ) at time t in the

($x^0 = ct$, $\boldsymbol{x}$)-reference frame is

$$\int \psi^\dagger(x^0, \boldsymbol{x}) \boldsymbol{X}_{op}(\boldsymbol{x}, \boldsymbol{p}_{op}, \boldsymbol{A}, \varphi)\, \psi(x^0, \boldsymbol{x})\, d\boldsymbol{x} \tag{A.1}$$

in the Dirac picture (*). Let us consider the expectation value in a different frame which is obtained by an infinitesimal pure Lorentz transformation

$$x^{0\prime} = x^0 + \boldsymbol{\epsilon}\cdot\boldsymbol{x}\,, \qquad \boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\epsilon} x^0\,. \tag{A.2}$$

We want to take the expectation value at a time $\hat{x}^{0\prime}$ which is numerically equal to x^0

$$\int \psi^{\dagger\prime}(\hat{x}^{0\prime}, \hat{\boldsymbol{x}}') \boldsymbol{X}'_{op}\{\hat{\boldsymbol{x}}', (\hbar/i)\,\partial/\partial\hat{\boldsymbol{x}}', \boldsymbol{A}', \varphi'\}\, \psi'(\hat{x}^{0\prime}, \hat{\boldsymbol{x}}')\, d\hat{\boldsymbol{x}}'\,. \tag{A.3}$$

The spinor field is transformed according to

$$\psi'(\hat{x}^{0\prime}, \hat{\boldsymbol{x}}') = (1 + \tfrac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{\alpha})\, \psi(\hat{x}^0, \hat{\boldsymbol{x}})\,, \tag{A.4}$$

where the circumflexed co-ordinates without primes follow from (A.2) and $\hat{x}^{0\prime} = x^0$ as

$$\hat{x}^0 = x^0 - \boldsymbol{\epsilon}\cdot\hat{\boldsymbol{x}}'\,, \qquad \hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}' - \boldsymbol{\epsilon}\cdot x^0\,. \tag{A.5}$$

From (A.4) with (A.5) and the equation $H_{op}\psi = -(\hbar/i)\,\partial\psi/\partial t$ one has

$$\begin{aligned}\psi'(\hat{x}^{0\prime}, \hat{\boldsymbol{x}}') = \psi(x^0, \hat{\boldsymbol{x}}') + \tfrac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{\alpha}\psi(x^0, \hat{\boldsymbol{x}}') - x^0\boldsymbol{\epsilon}\cdot\partial\psi(x^0, \hat{\boldsymbol{x}}')/\partial\hat{\boldsymbol{x}}' +&\\ + (i/\hbar c)\,\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{x}}' H_{op}\{\hat{\boldsymbol{x}}', (\hbar/i)\,\partial/\partial\hat{\boldsymbol{x}}', \boldsymbol{A}, \varphi\}\, \psi(x^0, \hat{\boldsymbol{x}}')\,.&\end{aligned} \tag{A.6}$$

Taking into account the dependence of $\boldsymbol{X}'_{op}$ on $\boldsymbol{A}'$ and φ' (and their Lorentz transformation, cf. (A.2)), introducing (A.6) into (A.3) and using the translation property $[\partial/\partial\boldsymbol{x}_i, \boldsymbol{X}^j_{op}] = \delta^{ij}$ we obtain

$$\begin{aligned}\int \psi^{\dagger\prime} \boldsymbol{X}'_{op}\psi'\, d\boldsymbol{x}' - \int \psi^\dagger \boldsymbol{X}_{op}\psi\, dx = \boldsymbol{\epsilon} x^0 - (i/\hbar)\int \psi^\dagger[\boldsymbol{\epsilon}\cdot\boldsymbol{N}_{op}, \boldsymbol{X}_{op}]\,\psi\, dx +&\\ + \int \psi^\dagger\{(\partial\boldsymbol{X}_{op}/\partial\boldsymbol{A})\cdot\boldsymbol{\epsilon}\varphi + (\partial\boldsymbol{X}_{op}/\partial\varphi)\,\boldsymbol{\epsilon}\cdot\boldsymbol{A}\}\,\psi\, dx\,,&\end{aligned} \tag{A.7}$$

where we have introduced the abbreviation

$$\boldsymbol{N}_{op} \equiv c^{-1}\boldsymbol{x}H_{op} - \tfrac{1}{2}i\hbar\boldsymbol{\alpha} = \tfrac{1}{2}c^{-1}\{\boldsymbol{x}, H_{op}\}\,. \tag{A.8}$$

(*) The Dirac picture is used throughout this Appendix.

On the other hand the left-hand side is to be the expectation value of (*)

(A.9)
$$\boldsymbol{\epsilon}x^0 - (i/2\hbar c)\{\boldsymbol{\epsilon}\cdot\boldsymbol{X}_{\text{op}}, [H_{\text{op}}, \boldsymbol{X}_{\text{op}}]\},$$

so that the covariance condition for the position operator $\boldsymbol{X}_{\text{op}}$ reads finally

(A.10)
$$[N^i_{\text{op}}, X^j_{\text{op}}] - \frac{\hbar}{i}\left(\frac{\partial X^j_{\text{op}}}{\partial A_i}\varphi + \frac{\partial X^j_{\text{op}}}{\partial\varphi}A^i\right) = \frac{1}{2}c^{-1}\{X^i_{\text{op}}, [H_{\text{op}}, X^j_{\text{op}}]\}.$$

Conditions of a similar type may be established to characterize the covariant properties of arbitrary physical operators. Of course in general one cannot expect that operators for physical quantities can be defined in such a way that their expectation values transform simply as four-vectors, four-tensors etc.; this is only possible for conserved operators (Klein's theorem). For nonconserved quantities we shall introduce the notion of quasi-four-vectors and quasi-four-tensors in the following way. The condition for a vector operator $\boldsymbol{V}_{\text{op}}$ (depending on $\boldsymbol{p}_{\text{op}}$, $\boldsymbol{A}$ and φ) to be the space part of a quasi-four-vector reads

(A.11)
$$[N^i_{\text{op}}, V^j_{\text{op}}] - \frac{\hbar}{i}\left(\frac{\partial V^j_{\text{op}}}{\partial A_i}\varphi + \frac{\partial V^j_{\text{op}}}{\partial\varphi}A^i\right) = \frac{1}{2}c^{-1}\{X^i_{\text{op}}, [H_{\text{op}}, V^j_{\text{op}}]\} - \frac{\hbar}{i}\delta^{ij}V^0_{\text{op}},$$

where V^0_{op} should satisfy

(A.12)
$$[N^i_{\text{op}}, V^0_{\text{op}}] - \frac{\hbar}{i}\left(\frac{\partial V^0_{\text{op}}}{\partial A_i}\varphi + \frac{\partial V^0_{\text{op}}}{\partial\varphi}A^i\right) = \frac{1}{2}c^{-1}\{X^i_{\text{op}}, [H_{\text{op}}, V^0_{\text{op}}]\} - \frac{\hbar}{i}V^i_{\text{op}}.$$

In particular, if $\boldsymbol{V}_{\text{op}}$ is independent of φ and depends on $\boldsymbol{A}$ only via $\boldsymbol{\pi}_{\text{op}} \equiv \boldsymbol{p}_{\text{op}} - (e/c)\boldsymbol{A}$ (*i.e.* $\boldsymbol{V}_{\text{op}} = \boldsymbol{V}_{\text{op}}(\boldsymbol{\pi}_{\text{op}})$), the left-hand side of (A.11) becomes

(A.13)
$$[N^i_{\text{op}}, V^j_{\text{op}}] + (e\hbar/ic)(\partial V^j_{\text{op}}/\partial p_{i,\text{op}})\varphi = [N^i_{\text{op}} - (e/c)x^i\varphi, V^j_{\text{op}}],$$

up to terms linear in e; hence for operators $\boldsymbol{V}_{\text{op}}$ of this special type the generator of pure Lorentz transformations is effectively

(A.14)
$$\boldsymbol{N}_{\text{op}} - (e/c)\boldsymbol{x}\varphi = \tfrac{1}{2}c^{-1}\{\boldsymbol{x}, H_{\text{op}} - e\varphi\},$$

where (A.8) has been used.

(*) In the classical theory of a composite particle the set of centres of energy at successive times determines a world-line independent of the Lorentz frame (cf. Sect. **2**). As a result the positions observed in different Lorentz frames are connected in a particular way (cf. (4)). In fact, let us consider the two points $x^0, \boldsymbol{X}(x^0)$ and $\tilde{x}^0, \boldsymbol{X}(\tilde{x}^0)$ on the world line of which the time co-ordinates x^0 in the reference frame and $\tilde{x}^{0\prime}$ in an infinitesimally different frame have the same numerical value. Thus from (A.2), $x^0 \equiv \tilde{x}^{0\prime} = \tilde{x}^0 + \boldsymbol{\epsilon}\cdot\boldsymbol{X}(\tilde{x}^0)$ and $\boldsymbol{X}'(x^0) \equiv \boldsymbol{X}'(\tilde{x}^{0\prime}) = \boldsymbol{X}(\tilde{x}^0) + \boldsymbol{\epsilon}\tilde{x}^0$. From the first of these equations one has up to first order in $\boldsymbol{\epsilon}$ that $\tilde{x}^0 = x^0 - \boldsymbol{\epsilon}\cdot\boldsymbol{X}(x^0)$. With the help of this relation the second equation becomes upon Taylor expansion up to first order $\boldsymbol{X}'(x^0) - \boldsymbol{X}(x^0) = \boldsymbol{\epsilon}x^0 - \boldsymbol{\epsilon}\cdot\boldsymbol{X}(x^0)\mathrm{d}\boldsymbol{X}(x^0)/\mathrm{d}x^0$. This expression is equal to the expectation value of (A.9) for a narrow wave packet in the limit $\hbar \to 0$, since then the expectation value of a (symmetrized) product of operators is equal to the product of expectation values.

The condition for a vector operator $\boldsymbol{V}_{\text{op}}$ to be the space-space-part T^{ij}_{op} of an antisymmetric tensor $T^{\alpha\beta}_{\text{op}}$ reads

$$(\text{A.15})\qquad [N^i_{\text{op}}, V^j_{\text{op}}]-\frac{\hbar}{i}\left(\frac{\partial V^j_{\text{op}}}{\partial A_i}\varphi+\frac{\partial V^j_{\text{op}}}{\partial\varphi}A^i\right)=\frac{1}{2}c^{-1}\{X^i_{\text{op}}, [H_{\text{op}}, V^j_{\text{op}}]\}-\frac{\hbar}{i}\varepsilon^{ijk}W_{k,\text{op}},$$

where the space-time components $W^i_{\text{op}}=T^{i0}_{\text{op}}$ should satisfy

$$(\text{A.16})\qquad [N^i_{\text{op}}, W^j_{\text{op}}]-\frac{\hbar}{i}\left(\frac{\partial W^j_{\text{op}}}{\partial A_i}\varphi+\frac{\partial W^j_{\text{op}}}{\partial\varphi}A^i\right)=\frac{1}{2}c^{-1}\{X^i_{\text{op}}, [H_{\text{op}}, W^j_{\text{op}}]\}+\frac{\hbar}{i}\varepsilon^{ijk}V_{k,\text{op}}.$$

These relations are easily generalized to the case where the operators depend moreover on the fields and their derivatives.

The condition for a certain transformation character with respect to the rotation group may be formulated in a similar fashion. In particular, the condition for a set of three operators V^i_{op} (depending on $\boldsymbol{p}_{\text{op}}$, $\boldsymbol{A}$ and φ) to form a vector operator reads

$$(\text{A.17})\qquad [M^i_{\text{op}}, V^j_{\text{op}}]+(\hbar/i)\varepsilon^{imn}A_m\partial V^j_{\text{op}}/\partial A^n=-(\hbar/i)\varepsilon^{ijk}V_{k,\text{op}},$$

where

$$(\text{A.18})\qquad \boldsymbol{M}_{\text{op}}=\boldsymbol{x}\wedge\boldsymbol{p}_{\text{op}}+\tfrac{1}{2}\hbar\boldsymbol{\sigma}.$$

If $\boldsymbol{V}_{\text{op}}$ depends only on $\boldsymbol{\pi}_{\text{op}}\equiv\boldsymbol{p}_{\text{op}}-(e/c)\boldsymbol{A}$ the left-hand side of (A.17) becomes

$$(\text{A.19})\qquad [M^i_{\text{op}}, V^j_{\text{op}}]-(e\hbar/ic)\varepsilon^{imn}A_m\partial V^j_{\text{op}}/\partial p^n_{\text{op}}=[M^i_{\text{op}}-(e/c)(\boldsymbol{x}\wedge\boldsymbol{A})^i, V^j_{\text{op}}],$$

up to terms linear in e, so that for these operators the generator of rotations is

$$(\text{A.20})\qquad \boldsymbol{M}_{\text{op}}-(e/c)\boldsymbol{x}\wedge\boldsymbol{A}=\boldsymbol{x}\wedge\boldsymbol{\pi}_{\text{op}}+\tfrac{1}{2}\hbar\boldsymbol{\sigma},$$

where (A.18) has been used.

The condition for a vector operator $\boldsymbol{V}_{\text{op}}$ to have polar or axial character respectively reads finally

$$(\text{A.21})\qquad \beta\boldsymbol{V}_{\text{op}}(\boldsymbol{p}_{\text{op}}, \boldsymbol{A}, \varphi)\beta=\mp\boldsymbol{V}_{\text{op}}(-\boldsymbol{p}_{\text{op}}, -\boldsymbol{A}, \varphi).$$

Appendix II

Covariant position operator for a free Dirac particle.

In the course of the derivations the position and spin operators, both in the absence and presence of fields, were fixed by requiring certain properties with respect to the transformations of the Poincaré group. As an example of such a procedure we show here how the position operator (70) for a free particle may be obtained.

From translation, rotation and inversion invariance it follows (see (68)) that the even part $\boldsymbol{X}_{\text{op}}$ of the position operator could be expressed in terms of four form factors, since the only vectors available are $\boldsymbol{p}_{\text{op}}$ and $\boldsymbol{\sigma}$. The left-hand side of eq. (69), which determines the pure Lorentz transformation character contains the generator $\boldsymbol{N}_{\text{op}}$ (A.8). In the P-FW picture the Hamiltonian $\hat{H}^0_{\text{op}}$ is given by (67) while the Dirac co-ordinate gets the form

$$\hat{\boldsymbol{x}}_{\text{D,op}} = U_{\text{op}} \boldsymbol{x}_{\text{D}} U^\dagger_{\text{op}} = U_{\text{op}} \boldsymbol{x} U^\dagger_{\text{op}} = \boldsymbol{x} + (\hbar/i)(\partial U_{\text{op}}/\partial \boldsymbol{p}_{\text{op}}) U^\dagger_{\text{op}} , \tag{A.22}$$

so that the generator $\hat{\boldsymbol{N}}_{\text{op}}$ in the P-FW picture becomes

$$\hat{\boldsymbol{N}}_{\text{op}} = \tfrac{1}{2} c^{-1} \beta \{\boldsymbol{x}, E_{\text{op}}\} + \tfrac{1}{2} \beta \hbar c (\boldsymbol{p}_{\text{op}} \wedge \boldsymbol{\sigma})/(E_{\text{op}} + mc^2) , \tag{A.23}$$

where (82) and (86) have been used. If (67), (68) and (A.23) are inserted into (69) one obtains the result that a certain linear combination of independent tensors vanishes. Hence all coefficients have to be zero. This leads to the following set of linear differential equations of the first order:

$$f_3'(E_{\text{op}}) + f_3(E_{\text{op}})/(E_{\text{op}} + mc^2) = 0 , \tag{A.24}$$

$$p^2_{\text{op}} f_3'(E_{\text{op}}) + c^{-2} E_{\text{op}} f_3(E_{\text{op}}) - \tfrac{1}{2}\hbar/(E_{\text{op}} + mc^2) = 0 , \tag{A.25}$$

$$f_3'(E_{\text{op}}) + f_3(E_{\text{op}})/E_{\text{op}} - \tfrac{1}{2}\hbar c^2/E_{\text{op}}(E_{\text{op}} + mc^2)^2 = 0 , \tag{A.26}$$

$$f_4'(E_{\text{op}}) + f_4(E_{\text{op}})/(E_{\text{op}} + mc^2) = 0 , \tag{A.27}$$

$$p^2_{\text{op}} f_4'(E_{\text{op}}) + c^{-2} E_{\text{op}} f_4(E_{\text{op}}) = 0 , \tag{A.28}$$

$$f_4'(E_{\text{op}}) + f_4(E_{\text{op}})/E_{\text{op}} = 0 , \tag{A.29}$$

whereas $f_1(E_{\text{op}})$ and $f_2(E_{\text{op}})$ must be zero. The solutions of the equations for $f_3(E_{\text{op}})$ and $f_4(E_{\text{op}})$ are

$$f_3(E_{\text{op}}) = \tfrac{1}{2}\hbar/m(E_{\text{op}} + mc^2) , \qquad f_4(E_{\text{op}}) = 0 . \tag{A.30}$$

Upon substitution into (68) these results lead to (70).

Appendix III

The Weyl transformation.

For each operator A_{op} acting on a four-component spinor a Weyl transform may be defined in the following fashion. With the use of closure relations for the complete sets $|\boldsymbol{x}, \lambda\rangle$ and $|\boldsymbol{p}, \lambda\rangle$ which are eigenvectors of $\boldsymbol{x}_{\text{op}}$ and $\boldsymbol{p}_{\text{op}}$ and where $\lambda = 1, 2, 3, 4$ one may write the identity ([32])

$$\text{(A.31)}\qquad \begin{cases} A_{op} = \int \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{u}\,\mathrm{d}\boldsymbol{v}\,|\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{v},\varkappa\rangle\langle\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{v},\varkappa|\boldsymbol{p}+\tfrac{1}{2}\boldsymbol{u},\varkappa'\rangle \\ \langle\boldsymbol{p}+\tfrac{1}{2}\boldsymbol{u},\varkappa'|A_{op}|\boldsymbol{p}-\tfrac{1}{2}\boldsymbol{u},\lambda'\rangle\langle\boldsymbol{p}-\tfrac{1}{2}\boldsymbol{u},\lambda'|\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{v},\lambda\rangle\langle\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{v},\lambda| \end{cases}$$

(summation convention). Substituting for the scalar products $\langle\boldsymbol{x},\varkappa|\boldsymbol{p}\varkappa'\rangle = h^{-\frac{3}{2}}\exp[i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar]\,\delta_{\varkappa\varkappa'}$ one obtains

$$\text{(A.32)}\qquad A_{op} = h^{-3}\int \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{x}\,A_{\varkappa\lambda}(\boldsymbol{p},\boldsymbol{x})\,\Delta_{\varkappa\lambda,op}(\boldsymbol{p},\boldsymbol{x})\,,$$

where the Hermitian operator $\Delta_{\varkappa\lambda,op}(\boldsymbol{p},\boldsymbol{x})$ is given by

$$\text{(A.33)}\qquad \Delta_{\varkappa\lambda,op}(\boldsymbol{p},\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{v}\,\exp[i\boldsymbol{p}\cdot\boldsymbol{v}/\hbar]\,|\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{v},\varkappa\rangle\langle\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{v},\lambda|$$

and the matrix

$$\text{(A.34)}\qquad A_{\varkappa\lambda}(\boldsymbol{p},\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{u}\,\exp[i\boldsymbol{x}\cdot\boldsymbol{u}/\hbar]\langle\boldsymbol{p}+\tfrac{1}{2}\boldsymbol{u},\varkappa|A_{op}|\boldsymbol{p}-\tfrac{1}{2}\boldsymbol{u},\lambda\rangle\,,$$

which contains all information on A_{op}, is called the Weyl transform of A_{op} (notation $A_{op} \rightleftarrows A_{\varkappa\lambda}(\boldsymbol{p},\boldsymbol{x})$); an alternative expression is

$$\text{(A.35)}\qquad A_{\varkappa\lambda}(\boldsymbol{p},\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{v}\,\exp[i\boldsymbol{p}\cdot\boldsymbol{v}/\hbar]\langle\boldsymbol{x}-\tfrac{1}{2}\boldsymbol{v},\varkappa|A_{op}|\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{v},\lambda\rangle\,.$$

The Weyl transform of the product of two operators is given by ([33])

$$\text{(A.36)}\qquad A_{op}B_{op} \rightleftarrows \left\{\exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{\partial\boldsymbol{x}_A}\cdot\frac{\partial}{\partial\boldsymbol{p}_B}-\frac{\partial}{\partial\boldsymbol{p}_A}\cdot\frac{\partial}{\partial\boldsymbol{x}_B}\right)\right]\right\}A_{\varkappa\mu}(\boldsymbol{p},\boldsymbol{x})B_{\mu\lambda}(\boldsymbol{p},\boldsymbol{x})\,.$$

Useful corollaries of this formula are

$$\text{(A.37)}\qquad \frac{i}{\hbar}[A_{op},B_{op}] \rightleftarrows -\frac{1}{\hbar}\left[\sin\left\{\frac{\hbar}{2}\left(\frac{\partial}{\partial\boldsymbol{x}_A}\cdot\frac{\partial}{\partial\boldsymbol{p}_B}-\frac{\partial}{\partial\boldsymbol{p}_A}\cdot\frac{\partial}{\partial\boldsymbol{x}_B}\right)\right\}\right]\cdot$$
$$\cdot\{A_{\varkappa\mu}(\boldsymbol{p},\boldsymbol{x})B_{\mu\lambda}(\boldsymbol{p},\boldsymbol{x})+B_{\varkappa\mu}(\boldsymbol{p},\boldsymbol{x})A_{\mu\lambda}(\boldsymbol{p},\boldsymbol{x})\}+$$
$$+\frac{i}{\hbar}\left[\cos\left\{\frac{\hbar}{2}\left(\frac{\partial}{\partial\boldsymbol{x}_A}\cdot\frac{\partial}{\partial\boldsymbol{p}_B}-\frac{\partial}{\partial\boldsymbol{p}_A}\cdot\frac{\partial}{\partial\boldsymbol{x}_B}\right)\right\}\right]\{A_{\varkappa\mu}(\boldsymbol{p},\boldsymbol{x})B_{\mu\lambda}(\boldsymbol{p},\boldsymbol{x})-B_{\varkappa\mu}(\boldsymbol{p},\boldsymbol{x})A_{\mu\lambda}(\boldsymbol{p},\boldsymbol{x})\}\,,$$

([32]) B. LEAF: *Journ. Math. Phys.*, **9**, 65 (1968).
([33]) H. J. GROENEWOLD: *Physica*, **12**, 405 (1946).

$$(A.38)\qquad A_{op}B_{op}C_{op} \rightleftarrows \left\{\exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{\partial \boldsymbol{x}_A}\cdot\frac{\partial}{\partial \boldsymbol{p}_B} - \frac{\partial}{\partial \boldsymbol{p}_A}\cdot\frac{\partial}{\partial \boldsymbol{x}_B} + \frac{\partial}{\partial \boldsymbol{x}_A}\cdot\frac{\partial}{\partial \boldsymbol{p}_C} - \frac{\partial}{\partial \boldsymbol{p}_A}\cdot\frac{\partial}{\partial \boldsymbol{x}_C} + \frac{\partial}{\partial \boldsymbol{x}_B}\cdot\frac{\partial}{\partial \boldsymbol{p}_C} - \frac{\partial}{\partial \boldsymbol{p}_B}\cdot\frac{\partial}{\partial \boldsymbol{x}_C}\right)\right]\right\} A_{\varkappa\mu}(\boldsymbol{p}, \boldsymbol{x}) B_{\mu\nu}(\boldsymbol{p}, \boldsymbol{x}) C_{\nu\lambda}(\boldsymbol{p}, \boldsymbol{x}) \,.$$

These expressions are used in Sect. **6-8**.

RIASSUNTO (*)

Si trasforma la hamiltoniana per una particella di Dirac con momento magnetico anomalo in un campo elettromagnetico ad una forma regolare fino a termini lineari nella costante di accoppiamento e senza derivate del campo. Si derivano le parti regolari degli operatori di spin e di posizione imponendo le condizioni di covarianza. Si deducono le equazioni covarianti del moto e di spin; esse hanno la stessa forma di quelle classiche per una particella composta con momento di dipolo magnetico. Si dimostra che l'effetto magnetico dinamico per una particella in un campo non omogeneo contiene il prodotto vettoriale del campo elettrico e del momento magnetico anomalo.

(*) *Traduzione a cura della Redazione.*

Ковариантные уравнения движения для заряженной частицы с магнитным дипольным моментом.

Резюме (*). — Гамильтониан для дираковской частицы с аномальным магнитным моментом в электромагнитном поле преобразуется к четной форме вплоть до линейных членов по константе связи и без производных от поля. Предполагая условия инвариантности, выводятся четные части операторов положения и спина. Затем выводятся ковариантные уравнения движения и уравнения для спина; оказывается, что они имеют ту же форму, что и классические уравнения для составной частицы с магнитным дипольным моментом. (Показывается, что магнитнодинамический эффект для частицы в неоднородном поле содержит только векторное произведение электрического поля и аномального магнитного момента.)

(*) *Переведено редакцией.*