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Numerical Solution of the Spinor Bethe–Salpeter Equation and the Goldstein Problem

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The spinor Bethe–Salpeter equation describing bound states of a fermion–antifermion pair with massless-boson exchange reduces to a single (uncoupled) partial differential equation for special combinations of the fermion–boson couplings. For spinless bound states with positive or negative parity this equation is a generalization to nonvanishing bound-state masses of the equations studied by Kummer and Goldstein, respectively. In the tight-binding limit the Kummer equation has a discrete spectrum, in contrast to the Goldstein equation, while for loose binding only the generalized Goldstein equation has a nonrelativistic limit. For intermediate binding energies the equations are solved numerically. The generalized Kummer equation is shown to possess a discrete spectrum of coupling constants for all bound-state masses. For the generalized Goldstein equation a discrete spectrum of coupling constants is found only if the binding energy is smaller than a critical value.

1. INTRODUCTION

Relativistic bound states of two particles with spin $\frac{1}{2}$ interacting by the exchange of massless bosons may be described by a spinor Bethe–Salpeter equation [1, 2]. In the ladder approximation this Bethe–Salpeter equation is equivalent to a set of coupled partial differential equations. The values of the coupling constants that determine the strength of the interaction necessary to build a bound state of a given mass are given by the eigenvalues of the differential equations.

In the case of strong binding, with binding energy equal to the sum of the masses of the constituent particles, symmetry arguments in momentum space may be used to reduce the partial differential equations to ordinary differential equations; moreover the equations become partly uncoupled. A discrete series of isotropic solutions describing spinless bound states with positive parity has been obtained recently [3, 4]. The corresponding coupling constants depend continuously on the parameters characterizing the type of fermion–boson interaction. For spinless bound states with negative parity, however, the situation is completely different: in the strong binding limit no isotropic solutions that form part of a discrete series exist for that case. This property of the spinor Bethe–Salpeter equation, which was discovered by Goldstein [5], has been discussed extensively in the literature [6–11].

The solutions in the strong-binding limit for which the bound-state mass ($2\epsilon_B$)

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vanishes have a physical meaning only if they can be considered as the limits of the solutions for small but finite ϵ_B. The behavior of the coupling constants in that limiting process may be investigated with the help of perturbation theory methods [12–14]. For larger values of ϵ_B, however, such a perturbation approach does not give accurate results for the coupling constants.

For general values of the bound-state mass the coupling constant spectrum of the Bethe–Salpeter equation can be obtained only by solving the complete set of coupled partial differential equations to which it is equivalent; owing to the complex structure of the equations no general solutions have been found as yet. However, for special choices of the fermion–boson interactions the coupled differential equations reduce to a single equation, as has been shown previously [14]. In fact, the equation which one may obtain in this way for 0^+ bound states is a generalization of that considered by Kummer [15], while for 0^- states one is led to a generalized Goldstein equation. Both these equations are found to have a structure analogous to that of the equation for bound states of bosons in the Wick–Cutkosky model [16, 17]. It is the purpose of the present paper to determine the eigenvalue spectra of the generalized Kummer and Goldstein equations and to compare them with that of the Wick–Cutkosky model. The results will show how the discrete spectra of the generalized fermion equations behave when the strength of the binding forces is varied.

To study the eigenvalue spectra use will be made of both analytical and numerical methods. Earlier numerical investigations of the bound-state solutions of the Bethe–Salpeter equation have been concerned mainly with the boson case [18–22]. Numerical studies of the fermion equation are rather few in number. Some of these [23, 24] introduce cut-off or regulator parameters in the equation and find eigenvalue spectra depending on these parameters; furthermore the Bethe–Salpeter equation has been considered in the framework of nonlinear spinor theory [25].

In Section 2 the Bethe–Salpeter equation for a fermion–antifermion pair with massless-boson exchange will be used as a starting point to establish the generalized Kummer and Goldstein equations. Some of their properties will be derived and compared with those of the Wick–Cutkosky model in Section 3. The solutions of the Kummer and Wick–Cutkosky equations in the strong-binding limit will be summarized in Section 4, while in Section 5 it will be shown that the generalized Kummer equation does not reduce to a Schrödinger equation in the nonrelativistic limit, in contrast to the Goldstein and Wick–Cutkosky equations. Finally, in Section 6 the results of a numerical investigation of the eigenvalue spectrum of both the fermion and boson equations will be presented.

2. The Spinor Bethe–Salpeter Equation and the Equations of Kummer and Goldstein

The spinor Bethe–Salpeter equation for the wavefunction $\chi(q)$ that describes bound states of a fermion and an antifermion exchanging massless bosons reads in the ladder approximation:

$$(\not{q} + \tfrac{1}{2}\not{P} - 1)\,\chi(q)(\not{q} - \tfrac{1}{2}\not{P} - 1)$$
$$= -i(2\pi)^{-2} \sum_{i=1}^{5} \lambda_i \int d^4q'\,(q - q')^{-2}\,\Gamma^i \chi(q')\,\Gamma^i. \tag{1}$$

Here P^μ is the momentum four vector of the bound state, which may be written as $(2\epsilon_B, 0)$ in its rest frame; the mass of the constituent fermions is chosen as the unit of mass. The interactions characterized by the matrices Γ^i, with $\Gamma^S = 1$, $\Gamma^V = \gamma^\mu$, $\Gamma^T = \sigma^{\mu\nu}$, $\Gamma^A = \gamma^\mu\gamma_5$, and $\Gamma^P = i\gamma_5$, have a strength determined by the coupling constants λ_i.

The wavefunction may be expanded in a complete set of Dirac matrices. The scalar structure functions that are the coefficients of the various matrices in such an expansion satisfy a set of coupled integral equations. Upon performing a Wick rotation, one may transform these equations into partial differential equations, which are difficult to solve for general values of the binding energy. Much simpler uncoupled equations may be obtained, however, if the coupling constants λ_i are chosen so as to satisfy special constraints [14]. In particular, if the coupling constants fulfill the relations

$$\lambda_V + \lambda_A = \tfrac{1}{2}(\lambda_S + \lambda_P), \qquad \lambda_T = \tfrac{1}{4}(\lambda_S - \lambda_P), \tag{2}$$

the wavefunction for a spinless bound state with parity $+1$ may be expressed in terms of a single scalar structure function $\chi_K(q)$ that is determined by the equation

$$\Box_q[(N/D)\chi] = \Lambda\chi\,. \tag{3}$$

The numerator and denominator functions are given by

$$N = (1 - \epsilon_B{}^2 + q^2)^2 + 4\epsilon_B{}^2 q_4{}^2, \tag{4}$$

$$D = D_K = 1 - \epsilon_B{}^2 - q^2, \tag{5}$$

in the rest frame of the bound state, while the coupling constant Λ is defined as

$$\Lambda = \Lambda_K = 4(\lambda_S - \lambda_P + \lambda_V - \lambda_A). \tag{6}$$

Equation (3) with (4)–(6) is a generalization for arbitrary bound-state mass $(2\epsilon_B)$ of the equation studied by Kummer [15].

For spinless states with negative parity bound by boson-exchange interactions with coupling constants λ_i satisfying

$$\lambda_V + \lambda_A = -\tfrac{1}{2}(\lambda_S + \lambda_P), \qquad \lambda_T = \tfrac{1}{4}(\lambda_S - \lambda_P), \tag{7}$$

an equation of the same form (3) may be derived. The numerator function is again given by (4), while the denominator function D_G and the coupling constant Λ_G read

$$D = D_G = 1 + \epsilon_B{}^2 + q^2, \tag{8}$$

$$\Lambda = \Lambda_G = 4(\lambda_S - \lambda_P - \lambda_V + \lambda_A). \tag{9}$$

In this way a generalization for $\epsilon_B \neq 0$ of the equation of Goldstein [5] is found.

It will be instructive in the following to compare the generalized Kummer and Goldstein equations with the boson Bethe–Salpeter equation for the Wick–Cutkosky model [16, 17]. That equation is likewise of form (3), with the numerator function given by (4); the denominator D_W is equal to unity in this case.

The bound-state function $\chi(q)$ which is a solution of the partial differential equation (3) should satisfy a normalization condition [7, 26–28]. The finiteness of the integral occurring in this condition sets bounds on the behavior of $\chi(q)$ for small and large values of q, viz.,

$$\lim_{q\to 0} q^2\chi(q) = 0, \qquad \lim_{q\to\infty} q^p\chi(q) = 0, \tag{10}$$

with $p = 3$ for the boson and $p = 2$ for the fermion cases. Although the normalization integral will be finite if these requirements are met, it is not necessarily positive. Indeed one may show that the Bethe–Salpeter equation has solutions with a negative norm, corresponding to "ghost" states. For tightly bound states (i.e., for $\epsilon_B \to 0$) one may prove (see Section 4) that the non-ghost-state solutions of (3) are characterized by a value $+1$ of the quantum number η_C ; the latter is defined by writing $\chi(q, -q_4) = \eta_C\chi(\boldsymbol{q}, q_4)$. Furthermore, only the $\eta_C = +1$ solutions have a nonrelativistic limit, as will be discussed in Section 5. For these reasons we will confine ourselves in the following to a study of the solutions with $\eta_C = 1$.

3. General Properties of the Kummer, Goldstein, and Wick–Cutkosky Equations

The differential operator $\Box_q$ occuring in (3) is a negative-definite operator in the space of functions χ satisfying (10). Inspection of the numerator and denominator functions N and D shows then that the eigenvalue spectrum of the Goldstein equation is negative definite, like that of the Wick–Cutkosky equation, while both signs may occur in the spectrum of the Kummer equation.

A change of the coupling constant Λ will result in a change of the bound-state mass $2\epsilon_B$. The derivative $\partial\epsilon_B{}^2/\partial\Lambda$ may be found by differentiating the Bethe–Salpeter equation with respect to ϵ_B [12–14, 29]. For nonghost normalized states with $\eta_C = 1$ one may derive in this way:

$$\Lambda \frac{\partial\epsilon_B{}^2}{\partial\Lambda} = (2\pi)^3 \rho \int d^4q \frac{N}{D} |\chi|^2, \tag{11}$$

with ρ equal to 1, -1, and $-\frac{1}{4}$ for the Kummer, Goldstein, and Wick–Cutkosky cases, respectively. The integral in (11) will be finite only if the wavefunction satisfies the requirements

$$\lim_{q\to 0} q^2\chi(q) = 0, \qquad \lim_{q\to\infty} q^{p+1}\chi(q) = 0, \tag{12}$$

where the same parameter p as that in (10) has been employed. Clearly the conditions found here impose stronger bounds on the asymptotic behavior of χ than those

following from (10). Since a finite value of the derivative $\partial\epsilon_B^2/\partial\Lambda$ is expected on physical grounds, only wavefunctions satisfying (12) will be considered in the following.

The positive-definite character of the functions N and D for the Goldstein and Wick–Cutkosky equations implies in view of (11) that the sign of $\Lambda\partial\epsilon_B^2/\partial\Lambda$ is negative (and hence that of $\partial\epsilon_B^2/\partial\Lambda$ positive) for the non-ghost-state solutions with $\eta_C = 1$. The sign of $\partial\epsilon_B^2/\partial\Lambda$ for the solutions of the Kummer equation may be obtained by writing (11) with (3) in an alternative form [3, 30]:

$$\Lambda^2 \frac{\partial\epsilon_B^2}{\partial\Lambda} = (2\pi)^3 \rho \int d^4q \left(\frac{N}{D}\chi^*\right) \Box_q \left(\frac{N}{D}\chi\right). \tag{13}$$

Clearly the sign of $\partial\epsilon_B^2/\partial\Lambda$ is the opposite of the sign of ρ; in particular it is negative for the non-ghost-state solutions with $\eta_C = 1$ of the Kummer equation.

An auxiliary integral relation for the solutions of the three equations contained in (3) can be derived by making use of the homogeneous form of the potential for massless-boson exchange in the ladder approximation. In fact, if the Bethe–Salpeter equation (3) in its integral form, viz.,

$$\frac{N}{D}\chi = -\frac{\Lambda}{4\pi^2}\int \frac{d^4q'}{(q-q')^2}\chi(q'), \tag{14}$$

is acted upon by the operator $q^\mu\partial/\partial q^\mu - 2$ and a partial integration is performed at the right-hand side, one finds

$$\left[\left(q^\mu\frac{\partial}{\partial q^\mu} - 2\right)\frac{N}{D}\right]\chi + \frac{N}{D}q^\mu\frac{\partial}{\partial q^\mu}\chi = -\frac{\Lambda}{4\pi^2}\int \frac{d^4q'}{(q-q')^2}q'^\mu\frac{\partial}{\partial q'^\mu}\chi(q'). \tag{15}$$

Multiplying this equation by $\chi^*(q)$, integrating over q^μ, and using (14) once more, one arrives at the result [14]

$$\int \left[\left(q^\mu\frac{\partial}{\partial q^\mu} - 2\right)\frac{N}{D}\right] |\chi|^2 d^4q = 0. \tag{16}$$

This relation is of special interest when it is applied to the solutions of the Goldstein equation; in that case one has

$$\left(q^\mu\frac{\partial}{\partial q^\mu} - 2\right)\frac{N}{D_G} = -2\frac{(1-\epsilon_B^2+q^2)[(1-3\epsilon_B^2)q^2 + 1 - \epsilon_B^4] + 4\epsilon_B^2 q^2 q_4^2}{(1+q^2+\epsilon_B^2)^2}. \tag{17}$$

Since the right-hand side is negative definite for $\epsilon_B^2 \leqslant \frac{1}{3}$, the auxiliary relation (16) for the Goldstein case can be satisfied only for loosely bound states with $\epsilon_B^2 > \frac{1}{3}$: strong-binding solutions of the Goldstein equation do not exist (see also Section 4). For the Kummer and Wick–Cutkosky cases, relation (16) does not give rise to similar conclusions.

The integral form (14) of the Bethe–Salpeter equation may be employed to show that the spectrum of inverse coupling constants Λ^{-1} is bounded for the Goldstein and Wick–Cutkosky equations. In fact, defining $\psi(q) = [N(q)/D(q)]^{1/2}\,\chi(q)$, so that $\psi(q)$ is square integrable in view of (11) and (12), one may write (14) as

$$\int M(q, q')\,\psi(q')\,d^4q' = \Lambda^{-1}\psi(q) \tag{18}$$

with the integral operator

$$M(q, q') = -\tfrac{1}{4}\pi^{-2}[D(q)\,D(q')/N(q)\,N(q')]^{1/2}\,(q - q')^{-2}. \tag{19}$$

The boundedness of this operator may be proved with the help of the theorem [31]:

$$\| M \| \leqslant \sup_{(q^\mu)} \int | M(q, q')|\,[g(q')/g(q)]\,d^4q', \tag{20}$$

valid for arbitrary positive-definite $g(q)$. The integral is majorized if $(q - q')^{-2}$ is replaced by $(\boldsymbol{q} - \boldsymbol{q}')^{-2}$ in $M(q, q')$. In the case of the Goldstein equation a convenient choice of the weight function $g(q)$ is

$$g_{\mathrm{G}}(q) = \left[\frac{D_{\mathrm{G}}(q)}{N(q)}\right]^{1/2} \frac{\boldsymbol{q}^2 + 1 - \epsilon_{\mathrm{B}}{}^2}{(\boldsymbol{q}^2 + a^2)(\boldsymbol{q}^2 + 1)^{1/2}}, \tag{21}$$

with arbitrary constant a. (The use of a nondefinite weight function in Ref. [31] is not justified.) After evaluation of the integral (20) one gets then

$$\| M_{\mathrm{G}} \| \leqslant \frac{1}{2}\,\pi \sup_{(q>)} \left[\operatorname{arctg}\left(\frac{q}{a}\right) \frac{(q^2 + a^2)(q^2 + 1)^{1/2}}{q(q^2 + 1 - \epsilon_{\mathrm{B}}{}^2)}\right], \tag{22}$$

with $q = |\,\boldsymbol{q}\,|$. This result is valid for all $a > 0$ so that one may take the limit $a \to 0$. Then (22) gives an upper bound for $\| M_{\mathrm{G}} \|$; since $|\,\Lambda_{\mathrm{G}}^{-1}\,| \leqslant \| M_{\mathrm{G}} \|$, this leads immediately to an upper bound for $|\,\Lambda_{\mathrm{G}}^{-1}\,|$, viz,

$$|\,\Lambda_{\mathrm{G}}^{-1}\,| \leqslant \theta\left(\epsilon_{\mathrm{B}}{}^2 - \frac{1}{2}\right) \frac{\pi^2}{8\epsilon_{\mathrm{B}}(1 - \epsilon_{\mathrm{B}}{}^2)^{1/2}} + \theta\left(\frac{1}{2} - \epsilon_{\mathrm{B}}{}^2\right) \frac{\pi^2}{4}, \tag{23}$$

with $\theta(x)$ the unit step function. In an analogous way an upper bound for the spectrum of the Wick–Cutkosky equation may be obtained. In fact, upon choosing the weight function

$$g_{\mathrm{W}}(q) = [N(q)]^{-1/2} \frac{(\boldsymbol{q}^2 + 1 - \epsilon_{\mathrm{B}}{}^2)(\boldsymbol{q}^2 + 1)^{1/2}}{\boldsymbol{q}^2 + a^2}, \tag{24}$$

one derives the inequality

$$|\,\Lambda_{\mathrm{W}}^{-1}\,| \leqslant \theta\left(\epsilon_{\mathrm{B}}{}^2 - \frac{1}{2}\right) \frac{\pi^2}{16\epsilon_{\mathrm{B}}(1 - \epsilon_{\mathrm{B}}{}^2)^{1/2}} + \theta\left(\frac{1}{2} - \epsilon_{\mathrm{B}}{}^2\right) \frac{\pi^2}{8}. \tag{25}$$

Since the integral operator $M_W^2(q, q') = \int M_W(q, q'') M_W(q'', q') d^4q''$ for the Wick–Cutkosky equation has the Hilbert–Schmidt property, the boundedness of $M_W(q, q')$ implies its compactness so that it has a discrete spectrum Λ_W^{-1} for all ϵ_B. For the Goldstein equation one may prove in the same way as in Ref. [31] that the part of the spectrum of Λ_G^{-1} satisfying the inequality $|\Lambda_G^{-1}| > \frac{1}{4}\pi^2$ is discrete for all ϵ_B.

An alternative form of the differential equation (3) is obtained by writing $\phi = (N/D)\chi$, with ϕ satisfying the equation

$$(D/N)\phi = \Lambda^{-1} \Box_q \phi. \tag{26}$$

The requirements (12) may be translated to conditions on ϕ:

$$\lim_{q\to 0} q^2\phi(q) = 0, \qquad \lim_{q\to\infty} q^r\phi(q) = 0 \tag{27}$$

with $r = 0$ for the boson and $r = 1$ for the fermion cases. Form (26) is suitable for a comparison of the eigenvalue spectrum of the Kummer, Goldstein, and Wick–Cutkosky equations. The denominator functions D_K, D_G, and D_W fulfill the identity

$$D_K + D_G = 2D_W. \tag{28}$$

If the spectrum of eigenvalues Λ^{-1} has an infimum Λ_m^{-1} for all three equations (at a fixed value of ϵ_B^2), these will satisfy the inequality

$$\Lambda_{m,K}^{-1} + \Lambda_{m,G}^{-1} \leqslant 2\Lambda_{m,W}^{-1}. \tag{29}$$

Since Λ_G^{-1} and Λ_W^{-1} are negative definite, one may write this inequality as

$$\Lambda_{m,K}^{-1} \leqslant |\Lambda_{m,G}^{-1}| - 2|\Lambda_{m,W}^{-1}|. \tag{30}$$

This relation will be used as a check for the numerical results in Section 6.

Additional information about the possible values of the coupling constants that yield bound-state solutions is obtained by considering the differential equation that follows from (26) in the limit of large q^2:

$$\sigma q^{-s}\phi_\infty = \Lambda^{-1} \Box_q \phi_\infty \tag{31}$$

with (s, σ) equal to $(2, -1)$, $(2, 1)$, and $(4, 1)$ for the Kummer, Goldstein, and Wick–Cutkosky cases, respectively. This asymptotic equation for ϕ_∞ is isotropic in the four-dimensional q^μ-space and may be separated by introducing four-dimensional polar coordinates. The wavefunctions ϕ_∞ for spinless bound states will have the form $\sum_l f_l(q)\, C_l^1(q_4/q)$. If for large q^2 the function ϕ (or ϕ_∞) is dominated by a contribution of the form $q^m C_l^1(q_4/q)$ (with $m < 0$ for the boson and $m < -1$ for the fermion cases, according to (27)), the indicial equation of (31) gives for the boson and fermion cases

$$(m - l)(m + l + 2) = 0, \tag{32}$$

$$(m - l)(m + l + 2) = \sigma\Lambda, \tag{33}$$

respectively. The relation (32) with $m < 0$ shows that $m = -l - 2$ for the Wick–Cutkosky equation; in particular one has $m = -2$ if the bound-state wavefunction is isotropic in the limit of large q^2. For the Kummer equation one gets from (33), with $\sigma = -1$, and the restriction $m < -1$ the inequality $\Lambda_K < (l+1)^2$. Negative values of the coupling constant are found only if $m < -l - 2$, and hence $m < -2$ for the isotropic case. For the Goldstein equation, relation (33), with $\sigma = 1$ and $m < -1$, implies that $\Lambda_G > -(l+1)^2$. Since the Goldstein equation has a negative-definite spectrum, the eigenvalues Λ_G are restricted now by an upper and a lower bound: $-(l+1)^2 < \Lambda_G < 0$, corresponding to values of m satisfying the inequalities $-l - 2 < m < -1$. For bound-state wavefunctions that are asymptotically isotropic one gets in particular $-1 < \Lambda_G < 0$ and $-2 < m < -1$.

4. Tightly-Bound-State Solutions of the Kummer and Wick–Cutkosky Equations

For tightly bound states the mass $2\epsilon_B$ is negligibly small, compared with 1. Their wavefunctions may be obtained from (3) or (26) by putting $\epsilon_B = 0$. Of course this limiting case is nonphysical. Nevertheless it is useful to study the bound-state wavefunctions in this limit since they may be considered as zeroth-order approximations of the wavefunctions in a perturbative scheme that treats ϵ_B as a small parameter. Such a perturbation treatment may be set up only for wavefunctions with a behavior for small and large q that fulfills condition (12) or (27).

When ϵ_B vanishes, the partial differential equation (26), with (4) and (5) or (8) inserted, becomes isotropic in four-dimensional q^μ-space. Choosing polar coordinates one can reduce it to a hypergeometric differential equation for both the fermion and boson cases. This equation has to be solved with boundary conditions (27).

The Kummer equation is found to have the following solutions for spinless bound states [15, 3]:

$$\phi_K^{l,r}(q) = C_{l,r}(1+q^2)^{-\mu-l}\, q^l P_r^{(l+1,2\mu+l-1)}\left(\frac{1-q^2}{1+q^2}\right) C_l^1\left(\frac{q_4}{q}\right). \tag{34}$$

Here the parameters r and l are nonnegative integers, while μ is given by

$$\mu = r + \tfrac{3}{2} + (2r^2 + 2rl + l^2 + 4r + 3l + \tfrac{9}{4})^{1/2}. \tag{35}$$

The coupling constants Λ corresponding to solutions (34) are

$$\Lambda_K = -4(\mu + l)(\mu - 1), \tag{36}$$

in agreement with (33) for $m = -2\mu - l$. Evaluation of the normalization integral shows that the solutions with odd l correspond to ghost states, while the even l solutions may be properly normalized by adjusting the constant $C_{l,r}$ [14]. By cal-

culating the perturbation integral (11) for the normalized wavefunctions an expression is obtained for the derivative $\Lambda^{-1}\partial\Lambda/\partial\epsilon_B{}^2$, viz.,

$$\Lambda_K^{-1}\frac{\partial\Lambda_K}{\partial\epsilon_B{}^2}=\left(1-\frac{1}{2}\,\delta_{l,0}\right)\frac{[2\mu^2+2\mu(l-1)+4l^2+6l+3](2\mu+l-1)}{2(2\mu-2r-3)(2\mu+2l+2r+3)(2\mu+2l+2r-1)}\,; \tag{37}$$

of course it applies only to the non-ghost-state solutions with even l. When (35) is substituted, it turns out that $\Lambda_K^{-1}\partial\Lambda_K/\partial\epsilon_B{}^2$ is positive; since Λ_K is negative, one gets $\partial\Lambda_K/\partial\epsilon_B{}^2<0$, in accordance with the general result proved in Section 3.

The hypergeometric equation to which the Goldstein equation may be reduced for $\epsilon_B=0$ is found to have no solutions that are compatible with conditions (27). This is no surprise, since it was proved in Section 3 that the Goldstein equation does not possess bound-state solutions for $\epsilon_B{}^2\leqslant\frac{1}{3}$. In the literature it has been stated that the Goldstein equation for $\epsilon_B=0$ leads to a continuous spectrum of eigenvalues Λ. This paradoxical situation, sometimes referred to as the Goldstein problem, results indeed if one tries to solve the equation under the weaker boundary conditions (10). However, since for $\epsilon_B=0$, the solutions make sense only insofar as they may be used in a perturbation scheme for small ϵ_B, one is obliged to impose the stronger conditions (12) or (27), thereby dismissing the continuous spectrum of eigenvalues.

The Wick–Cutkosky equation possesses solutions for $\epsilon_B=0$ that have the same structure as those of the Kummer equation, viz.,

$$\phi_W^{l,r}(q)=C_{r,l}(1+q^2)^{-l-1}\,q^l C_r^{l+3/2}\left(\frac{1-q^2}{1+q^2}\right)C_l^1\left(\frac{q_4}{q}\right), \tag{38}$$

with $r=0,1,2,\ldots$ and $l=0,1,2,\ldots$; the corresponding coupling constants are

$$\Lambda_W=-4(l+r+1)(l+r+2). \tag{39}$$

The eigenvalue spectrum is degenerate for $l+r\geqslant 1$. As is well known [29], the polar coordinates used here do not lead to solutions that can be considered as the limit of the solutions for $\epsilon_B\neq 0$. The correct zeroth-order solutions are obtained by solving the Wick–Cutkosky equation in bipolar coordinates; in these coordinates the equation separates for all ϵ_B. A convenient set of coordinates (z,t) which is closely related to the bipolar coordinates is

$$t=(q^2-1+\epsilon_B{}^2)[(1-\epsilon_B{}^2-q^2)^2+4q^2(1-\epsilon_B{}^2)]^{-1/2}, \tag{40}$$

$$z=2(1-\epsilon_B{}^2)^{1/2}(1-\epsilon_B{}^2+q^2)^{-1}\,q_4\,, \tag{41}$$

with $-1\leqslant t\leqslant 1$, $-1\leqslant z\leqslant 1$. When a wavefunction ϕ_W of the form

$$\phi_W(t,z)=[(1-z^2)^{-1/2}-t]f_1(t)f_2(z) \tag{42}$$

is substituted into (26), a pair of separate equations for f_1 and f_2 is found. The first equation is a Chebichef differential equation with solutions $f_1(t) = U_n(t)$, for $n = 0, 1, 2,...$ The equation for $f_2(z)$ becomes

$$(1 - z^2) f_2''(z) - 2zf_2'(z) - [(n + 1)^2 (1 - z^2)^{-1} + \tfrac{1}{4}\Lambda_W(1 - \epsilon_B{}^2 + \epsilon_B{}^2 z^2)^{-1}] f_2(z) = 0. \tag{43}$$

It has not yet been solved for general values of ϵ_B. For $\epsilon_B = 0$ its solutions are

$$f_2(z) = (1 - z^2)^{(1/2)(n+1)} C_r^{n+3/2}(z), \tag{44}$$

with $r = 0, 1, 2,...$ The eigenvalues Λ_W are found to be

$$\Lambda_W = -4(n + r + 1)(n + r + 2). \tag{45}$$

The solutions (42) with (44) for $\epsilon_B = 0$ are indeed the limits of those valid for $\epsilon_B \neq 0$. The evaluation of the normalization integral shows that the non-ghost-state solutions are characterized by even r [29]; in view of (41), (42), and (44) these correspond to wavefunctions even in q_4 (with $\eta_C = 1$). From the perturbation integral (11) one may derive then for the non-ghost-state solutions

$$\Lambda_W^{-1} \frac{\partial \Lambda_W}{\partial \epsilon_B{}^2} = -2 \frac{(n + r)^2 + 3(n + r) + n^2 + 2n + 2}{[2(n + r) + 1][2(n + r) + 5]}; \tag{46}$$

hence the derivative $\partial \Lambda_W / \partial \epsilon_B{}^2$ is positive, as has been proved generally in Section 3.

5. The Nonrelativistic Approximation

The nonrelativistic approximation of the boson and spinor Bethe–Salpeter equation studied here may be obtained by starting from the integral equation (14) and taking the limit of no retardation, i.e., replacing $(q - q')^{-2}$ by $(\boldsymbol{q} - \boldsymbol{q}')^{-2}$ [32, 33]. By integrating over q_4, one arrives then at an equation for the nonrelativistic wave function $\psi(\boldsymbol{q}) = \int dq_4 \chi(q)$ (which is nonvanishing for functions χ even in q_4), viz.,

$$\left(\int dq_4 \frac{D}{N}\right)^{-1} \psi(\boldsymbol{q}) = -\frac{\Lambda}{4\pi^2} \int d\boldsymbol{q}' \frac{\psi(\boldsymbol{q}')}{(\boldsymbol{q} - \boldsymbol{q}')^2}. \tag{47}$$

The wavefunction $\chi(q)$ may be recovered from its nonrelativistic counterpart $\psi(\boldsymbol{q})$, since one has

$$\chi(q) = \frac{D}{N} \left(\int dq_4 \frac{D}{N}\right)^{-1} \psi(\boldsymbol{q}). \tag{48}$$

The Wick–Cutkosky equation in the nonrelativistic limit follows by inserting

expression (4) for N and $D = 1$ into (47) and evaluating the integral at the left-hand side by complex contour integration; the result is

$$(1 + \boldsymbol{q}^2)^{1/2}\,(1 + \boldsymbol{q}^2 - \epsilon_B{}^2)\,\psi_W(\boldsymbol{q}) = -\frac{\Lambda_W}{8\pi}\int d\boldsymbol{q}'\,\frac{\psi_W(\boldsymbol{q}')}{(\boldsymbol{q} - \boldsymbol{q}')^2}. \tag{49}$$

In the nonrelativistic limit the momentum transfer $\boldsymbol{q}^2$ between the constituent particles is small compared to their masses (which have been put equal to unity) so that the factor $(1 + \boldsymbol{q}^2)^{1/2}$ may be suppressed. One is left then with the momentum-space representation of the Schrödinger equation for particles bound by a potential equal to $\frac{1}{4}\pi\Lambda|\,\boldsymbol{x}\,|$ in coordinate space. For bound states with vanishing angular momentum the coupling constant is given by $\Lambda_W = -8\pi^{-1}(n + 1)(1 - \epsilon_B{}^2)^{1/2}$ with n a nonnegative integer.

In an analogous way one may derive the nonrelativistic limit of the Goldstein equation:

$$(1 + \boldsymbol{q}^2)^{-1/2}\,(1 + \boldsymbol{q}^2 - \epsilon_B{}^2)\,\psi_G(\boldsymbol{q}) = -\frac{\Lambda_G}{4\pi}\int d\boldsymbol{q}'\,\frac{\psi_G(\boldsymbol{q}')}{(\boldsymbol{q} - \boldsymbol{q}')^2}, \tag{50}$$

which for $\boldsymbol{q}^2 \ll 1$ again reduces to the nonrelativistic Schrödinger equation with a spectrum $\Lambda_G = -4\pi^{-1}(n + 1)(1 - \epsilon_B{}^2)^{1/2}$ for isotropic bound states.

The Kummer equation leads to a completely different equation if the same line of reasoning is followed, viz.,

$$(1 + \boldsymbol{q}^2)^{1/2}\,\boldsymbol{q}^{-2}(1 + \boldsymbol{q}^2 - \epsilon_B{}^2)\,\psi_K(\boldsymbol{q}) = \frac{\Lambda_K}{4\pi}\int d\boldsymbol{q}'\,\frac{\psi_K(\boldsymbol{q}')}{(\boldsymbol{q} - \boldsymbol{q}')^2}. \tag{51}$$

In view of the presence of the extra factor $\boldsymbol{q}^{-2}$ at the left-hand side one must conclude that the Kummer equation does not reduce to an ordinary Schrödinger equation in the nonrelativistic limit. This anomalous behavior of the Kummer equation will be corroborated by the numerical results of Section 6.

6. Numerical Solution of the Kummer and Goldstein Equations

A suitable starting point for a numerical solution of the Bethe–Salpeter equations considered in the preceding sections has been given in Eq. (26), which may be written in a slightly different form by introducing the new veriables $\bar{q}^\mu = (1 - \epsilon_B{}^2)^{-1/2}\,q^\mu$, for $\epsilon_B \neq 1$:

$$(\bar{D}/\bar{N})\,\phi = \bar{\Lambda}^{-1}\,\Box_{\bar{q}}\,\phi. \tag{52}$$

Here the abbreviations

$$\bar{N} = (1 + \bar{q}^2)^2 + 4\bar{\epsilon}_B{}^2\bar{q}_4{}^2, \tag{53}$$

$$\bar{D}_K = 1 - \bar{q}^2, \qquad \bar{D}_G = 1 + 2\bar{\epsilon}_B{}^2 + \bar{q}^2, \qquad \bar{D}_W = 1, \tag{54}$$

with $\tilde{\epsilon}_B^2 = \epsilon_B^2/(1 - \epsilon_B^2)$, have been introduced; furthermore $\bar{\Lambda}$ equals Λ for the spinor cases, while $\bar{\Lambda} = (1 - \epsilon_B^2)^{-1}\Lambda$ for the boson equation. (In the following the bars over q, N, and D will again be omitted.)

A convenient method of obtaining numerical estimates for the coupling constants of the Bethe–Salpeter equation consists in the use of a variational principle [18, 20–22, 25]. In fact, the inverse coupling constant $\bar{\Lambda}^{-1}$ will be approximated by determining in accordance with the Rayleigh–Ritz variational principle the stationary values of the quotient

$$\frac{\int d^4q(D/N) \mid \phi \mid^2}{\int d^4q\ \phi^* \Box_q \phi}. \tag{55}$$

The numerator is finite for functions ϕ satisfying (27); nonvanishing results for $\bar{\Lambda}^{-1}$ will follow if the integral in the denominator is finite as well.

An alternative form for the Rayleigh–Ritz quotient is found by transforming the denominator to an integral in coordinate space. In fact, introducing $\tilde{\phi}(x)$ as

$$\tilde{\phi}(x) = (2\pi)^{-2} \int d^4q\ e^{-iq\cdot x}\phi(q), \tag{56}$$

one gets from (55) the variational quotient

$$-\frac{\int d^4q\ (D/N) \mid \phi \mid^2}{\int d^4x\ V^{-1}(x) \mid \tilde{\phi} \mid^2}, \tag{57}$$

with $V^{-1}(x) = x^2$ the inverse of the potential function in coordinate space. In this form the variational principle has a wider applicability than that given in (55); it is often called Schwinger's variational principle in the literature [18, 20, 21].

The stationary value of the variational quotient will be determined in the usual way by inserting for $\phi(q)$ (and its Fourier transform $\tilde{\phi}(x)$) a linear combination of trial functions. An optimal value of $\bar{\Lambda}^{-1}$ is obtained then by solving a generalized matrix eigenvalue problem of the type $A \cdot x = \lambda B \cdot x$.

A useful set of trial functions may be found by inspection of the ($\epsilon_B = 0$)-solutions (34) and (38) for the Kummer and Wick–Cutkosky equations; the following functions have been employed in the numerical calculations:

$$\phi_{n,l}^{(\mu)}(q) = q^l(1 + q^2)^{-\mu-l-n}\, C_l^1(q_4/q). \tag{58}$$

Here n is a nonnegative integer; for l, even integers have been selected since that choice leads to wavefunctions ϕ that are even in q_4. Indeed the non-ghost-state solutions of the Bethe–Salpeter equation in the tightly bound limit are even in q_4; furthermore only such wavefunctions have a nonrelativistic limit.

Linear combinations of up to 42 trial functions of form (58) have been employed in the numerical work. For each even l (with $0 \leqslant l \leqslant 10$) the values of n are chosen in the range $0 \leqslant n \leqslant M_l$. The upper bound M_l is taken to be a linearly decreasing function of l, viz., $M_l = M_0 - l$, so that $l \leqslant M_0$. By varying M_0 the matrix size is modified; in practice the values $M_0 = 7, 9$, or 11 have been used. The convergence properties of

the variational quotient for increasing matrix size gives an indication of the error resulting from truncating the infinite set of trial functions (58) to a finite subset.

The exponent μ in (58) is a freely adjustable parameter, which determines the asymptotic behavior of the wavefunction for large q. In fact, if a linear combination of the trial functions (58) is adopted as an approximation for ϕ, one finds that for large q it is dominated by its isotropic part with $l = 0$, so that $\phi_\infty(q) \propto q^{-2\mu} C_0^1(q_4/q)$. From (32) and (33) with $m = -2\mu$ it then follows that for the Wick–Cutkosky case one must choose $\mu = 1$. For the Goldstein equation μ should satisfy the inequalities $\frac{1}{2} < \mu < 1$, while for the Kummer equation one has $\mu > \frac{1}{2}$ (and $\mu > 1$ for solutions with $\Lambda < 0$). In principle relation (33) may be employed for the fermion cases to fix μ in a self-consistent way: a choice of μ leads through the variational principle to a value of Λ from which μ can be determined with the help of (33). However, since a finite number of trial functions has to be used, this scheme does work only if the integrals in (55) are dominated by the contributions of large q. This is the case for the eigenfunctions of the Goldstein equation and for the ($\Lambda > 0$)-eigenfunctions of the Kummer equation which are both characterized by $\frac{1}{2} < \mu < 1$. The negative eigenvalues of the Kummer equation are found for wavefunctions with $\mu > 1$. Since then the large q contributions do not dominate in the variational quotient, the self-consistent scheme is not expected to yield precise values of Λ. In this case μ is fixed by optimizing the convergence of the variational quotient for increasing matrix size.

The Fourier transform (56) of the trial functions (58) may be evaluated by inserting the expansion of the exponential function

$$e^{-iq\cdot x} = 2 \sum_l (-i)^l \frac{J_{l+1}(qx)}{qx} C_l^1\left(\frac{q_4}{q}\right) C_l^1\left(\frac{x_4}{x}\right) + R, \tag{59}$$

which follows from Sonine's formula and the addition theorem for Gegenbauer polynomials [34]; the remainder R stands for a sum of anisotropic terms that depend on both the length and the direction of the spatial part $\boldsymbol{q}$ of the momentum transfer. Employing the orthogonality relation of the Gegenbauer polynomials, one is left with an integral containing $J_{l+1}(qx)$, which is known to be proportional to a modified Bessel function [35]; the final result for $\tilde{\phi}_{n,l}^{(\mu)}(x)$ (with even l) is

$$\tilde{\phi}_{n,l}^{(\mu)}(x) = (-)^{(1/2)l}\, 2^{-\mu-l-n+1}[\Gamma(\mu + l + n)]^{-1}\, x^{\mu+l+n-2} K_{\mu+n-2}(x)\, C_l^1(x_4/x). \tag{60}$$

The denominator of the variational quotient (57) is easily evaluated now, since integrals of products of two modified Bessel functions and a power of x may be expressed in terms of gamma functions [35].

The integral in the numerator of (57) can be calculated by choosing the integration variables q and $\psi = \arccos(q_4/q)$ and employing the identity valid for even l and l' [35]:

$$\int_0^\pi d\psi \frac{\sin^2\psi}{a^2 + \cos^2\psi} C_l^1(\cos\psi)\, C_{l'}^1(\cos\psi)$$
$$= \frac{\pi}{2a} \frac{(-)^{(1/2)(l+l')}}{(1+a^2)^{1/2}} [(a + (1+a^2)^{1/2})^{-|l-l'|} + (a + (1+a^2)^{1/2})^{-l-l'-2}]. \tag{61}$$

The ensuing integral over q may be transformed, by a suitable choice of a new integration variable, into a linear combination of the basis set of integrals:

$$I(k_1, k_2) = \int_0^1 \frac{(1-t)^{2\mu-2}\, t^{k_1-1}}{(1+\bar{\epsilon}_B{}^2 t^2)^{k_2+2\mu}}\, dt, \tag{62}$$

with positive integers k_1 and k_2. Recursion relations for these integrals can then be used to write them as combinations of $I(1, 1)$ and $I(2, 1)$; the latter have to be evaluated numerically, at least if 2μ is noninteger.

The numerical methods described above were applied first to the Wick–Cutkosky equation. In that case the coupling-constant spectrum may be obtained in an alternative way by starting from Eq. (43), which followed from the introduction of bipolar coordinates. Numerical values of the inverse coupling constant Λ^{-1} are found from that equation by determining the stationary values of the corresponding Rayleigh–Ritz quotient for trial functions of the form

$$f_{n,r}(z) = (1-z^2)^{(1/2)(n+)}\, z^r \tag{63}$$

(cf. (44)), with nonnegative even integer r. Since for each value of n the lowest eigenvalue Λ^{-1} corresponds to a solution with a nonrelativistic limit [17], only these have been investigated numerically. The results for various bound-state masses $2\epsilon_B$ have been collected in Table I; attention has been confined to the first few branches of the

TABLE I

Coupling Constants Λ of the Wick–Cutkosky Equation as a Function of the Bound-State Mass Parameter $\bar{\epsilon}_B{}^2$

$\bar{\epsilon}_B{}^2$	$\epsilon_B{}^2$	$-\Lambda_1$	$-\Lambda_2$	$-\Lambda_3$
0.0	0.0	8.000	24.000	48.000
0.1	0.0909	7.414	22.122	44.109
0.2	0.1667	6.920	20.543	40.845
0.4	0.2857	6.129	18.033	35.672
0.6	0.3750	5.524	16.123	31.749
0.8	0.4444	5.043	14.617	28.667
1.0	0.5000	4.651	13.397	26.177
1.2	0.5455	4.325	12.387	24.122
1.5	0.6000	3.927	11.159	21.630
2.0	0.6667	3.426	9.629	18.540
3.0	0.7500	2.772	7.656	14.584
5.0	0.8333	2.072	5.582	10.469
7.0	0.8750	1.693	4.484	8.317
10.0	0.9091	1.360	3.536	6.478
15.0	0.9375	1.058	2.692	4.860
20.0	0.9524	0.885	2.218	3.963
30.0	0.9677	0.689	1.691	2.979
50.0	0.9804	0.504	1.208	2.090

coupling-constant spectrum, with the smallest value of $|\Lambda|$. The values for the coupling constants obtained here are consistent with those following from the curves given in Ref. [17]. In the tight-binding limit both Λ and $\partial\Lambda/\partial\epsilon_B^2$ agree with (45) and (46), while in the weak-binding limit the nonrelativistic theory is confirmed.

The values of Λ given in Table I are found to be reproduced quite accurately if the variational quotient (57) with trial functions (58) (for $\mu = 1$) is employed. An illustration of the fast convergence attained by the latter method is presented in Table II, where the approximate values of Λ for $\epsilon_B^2 = 0.5$ are listed as a function of the number of trial functions of form (58).

The numerical results for the coupling constants Λ of the generalized Kummer equation that have been obtained from (57) with (58) are presented in Table III. In Fig. 1, curves giving the ratios Λ/Λ_0 (with Λ_0 the coupling constant for $\epsilon_B = 0$) have

TABLE II

Convergence of the Results for the Coupling Constants Λ of the Wick–Cutkosky Equation with $\epsilon_B^2 = 0.5$, as a Function of the Matrix Size N

N	$-\Lambda_1$	$-\Lambda_2$	$-\Lambda_3$
12	4.6514165	13.397238	26.180326
20	4.6514119	13.397214	26.177228
30	4.6514118	13.397213	26.177165
42	4.6514118	13.397213	26.177163

TABLE III

Coupling Constants Λ of the Kummer Equation as a Function of the Bound-State Mass Parameter $\bar{\epsilon}_B^2$

$\bar{\epsilon}_B^2$	$-\Lambda_1$	$-\Lambda_2$	$-\Lambda_3$
0.0	24.000	93.957	112.00
0.1	24.326	94.977	115.59
0.2	24.637	95.906	119.11
0.4	25.223	97.578	125.90
0.6	25.768	99.075	132.41
0.8	26.280	100.45	138.65
1.0	26.764	101.73	144.65
1.2	27.224	102.94	150.45
1.5	27.874	104.63	158.81
2.0	28.875	107.21	171.97
3.0	30.642	111.72	196.0
5.0	33.579	119.19	236.0
7.0	36.03	125.4	
10.0	39.15	133.2	

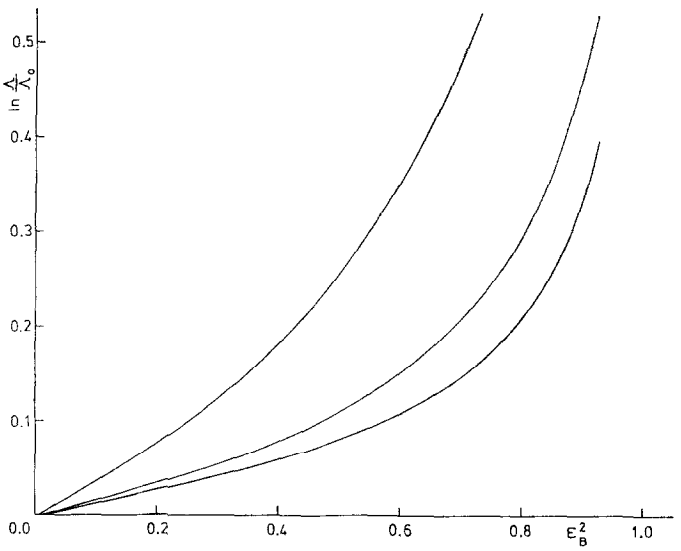

FIG. 1. The coupling constant ratio Λ/Λ_0 as a function of the square of the bound-state mass fraction ϵ_B for the generalized Kummer equation.

been drawn for these branches. It turns out that Λ is a monotonically decreasing negative function of ϵ_B. This property of the Kummer equation, which has been proved already in Section 3, is rather peculiar: it implies that increasing strength of the binding interaction corresponds to a decreasing binding energy.

For small bound-state mass $2\epsilon_B$ the values of Λ and its derivative $\partial\Lambda/\partial\epsilon_B{}^2$ as given by (36) and (37) are recovered by the numerical results. When ϵ_B approaches 1 the coupling constants are found to drop to increasingly large negative values; eventually they appear to tend to $-\infty$. This behavior is in accordance with the fact that the Kummer equation does not possess a nonrelativistic limit, as discussed in Section 5.

The coupling-constant spectrum of the generalized Goldstein equation is completely different from that of the Kummer equation. In Section 3 it has been shown that bound states with wavefunctions satisfying the boundary conditions (12) are not to be expected for $\epsilon_B{}^2 < \frac{1}{3}$; in particular, no such bound states exist for $\epsilon_B = 0$. On the other hand, in Section 4 a nonrelativistic coupling-constant spectrum was found to result from the Goldstein equation for $\epsilon_B{}^2$ close to 1 (i.e., for small binding energies). These predictions are confirmed by the outcome of the numerical analysis.

The lowest branches of the Λ-spectrum are given in Table IV, while the corresponding curves have been drawn in Fig. 2 and, in a more convenient parametrization, in Fig. 3. For bound-state mass parameters $\epsilon_B{}^2$ close to 1 the coupling constants depart from zero in the way described by the nonrelativistic theory (as indicated by the tangent lines in Fig. 3). When the interaction becomes stronger, the binding energy increases monotonically. A critical binding strength is obtained for $\Lambda = -1$, corresponding to $\epsilon_{Bc}^2 = 0.724$ and $\epsilon_{Bc}'^2 = 0.939$ for the branches considered. As Λ approaches -1 the derivative $\partial\Lambda/\partial\epsilon_B{}^2$ tends to zero for both branches. From (11)

it follows that the integral in the numerator of the Rayleigh quotient (57) becomes divergent if the trial function is to represent the eigenfunction accurately. Since Λ remains finite, the denominator of (57) is divergent as well. For that reason the curves

TABLE IV

Coupling Constants Λ of the Goldstein Equation as a Function of the Bound-State Mass Parameter ϵ_B^2

ϵ_B^2	$-\Lambda_1$	$-\Lambda_2$
2.7	0.9997	
2.8	0.9981	
3.0	0.9920	
3.5	0.9661	
4.0	0.9332	
5.0	0.8649	
7.0	0.7481	
10.0	0.6254	
15.0	0.5007	
16.0	0.4827	0.999
17.0	0.4663	0.988
20.0	0.4247	0.934
30.0	0.335	0.765
50.0	0.248	0.573

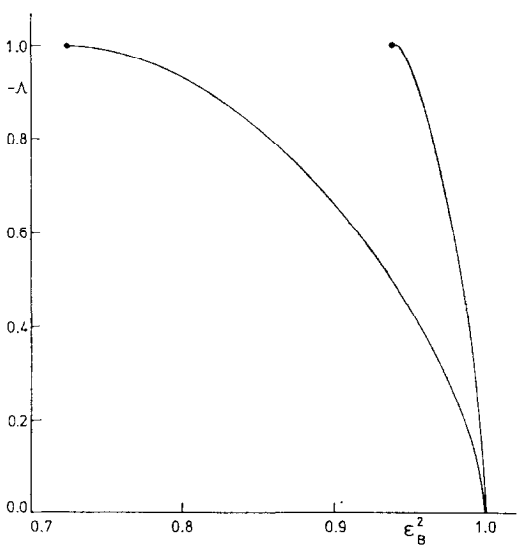

FIG. 2. The coupling constant Λ as a function of the square of the bound-state mass fraction ϵ_B for the generalized Goldstein equation.

giving the numerical results for the coupling constant break off at the critical values of ϵ_B^2; no discrete spectrum has been found for $\epsilon_B^2 \leqslant \epsilon_{Bc}^2$.

Comparison of the numerical results for the lowest branches of the inverse coupling constants of the Goldstein and the Wick–Cutkosky equation (see Tables I and IV) shows that the right-hand side of inequality (30) is negative for all values of the bound-state mass parameter ϵ_B^2 larger than ϵ_{Bc}^2. Since the lowest eigenvalue Λ^{-1} of the Kummer equation is negative, inequality (30) is manifestly satisfied.

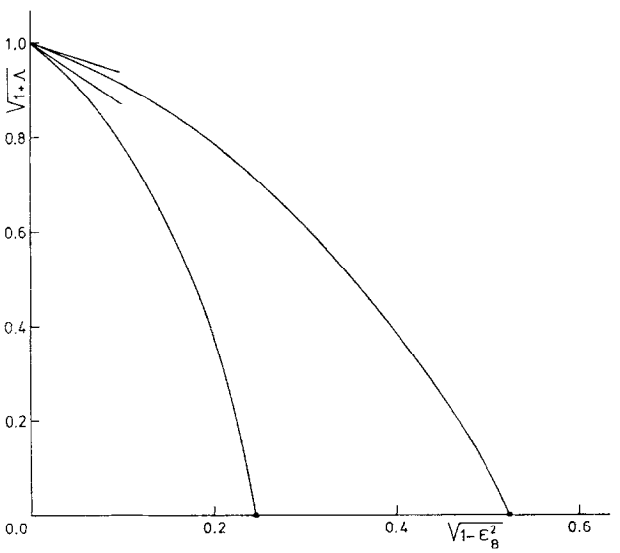

FIG. 3. The coupling constant Λ for the generalized Goldstein equation in an alternative parametrization; the tangent lines at $\epsilon_B^2 = 1$ indicate the nonrelativistic limit.

The character of the coupling-constant spectrum of the Goldstein equation, which is the tight-binding limit of the generalized equation studied here, has been investigated repeatedly in the literature [5–11]. Several methods, such as the introduction of a cut-off [5], the use of symmetry arguments [8], or dimensional continuation [11], have been devised to isolate a discrete spectrum in this special case. The results found here show that the absence of a discrete spectrum is not a unique property of the tight-binding limit: the discrete spectrum breaks off already for much smaller binding energies, in fact as soon as the coupling constant becomes of the order of unity. Discrete spectra of this type are known to occur for interactions with a singular potential [36–39]. An example is the Klein–Gordon equation for a spinless particle in a Coulomb field that has been used to describe mesic atoms [40, 41]. The validity of the one-particle approximation inherent in that equation may be questioned for large values of the coupling constant. A similar remark may be made with respect to the Bethe–Salpeter equation, since the ladder approximation has been crucial in the derivation of the basic equations of the present paper.

7. Conclusion

The study of the generalized Kummer and Goldstein equations, to which the spinor Bethe–Salpeter equation with massless-boson exchange reduces for special combinations of the fermion–boson couplings, has revealed several interesting features. The spectrum of the generalized Kummer equation which describes spinless bound states of positive parity turned out to be discrete for all binding energies. However, this spectrum has an unusual property: an increasing interaction strength leads to decreasing binding energies; as a consequence a nonrelativistic limit does not exist.

The spectrum of the generalized Goldstein equation valid for spinless bound states with negative parity is completely different. It is discrete only if the binding energy and the binding strength are smaller than a critical value. In that range decreasing binding energies correspond to a decreasing interaction strength. In particular, in the nonrelativistic limit the generalized Goldstein equation reduces to a Schrödinger equation with a Coulomb potential. For large binding energies the generalized Goldstein equation does not possess a discrete spectrum of coupling constants.

References

1. E. E. Salpeter and H. A. Bethe, *Phys. Rev.* **84** (1951), 1232.
2. For a review see, e.g., N. Nakanishi, *Progr. Theoret. Phys. Suppl.* **43** (1969), 1.
3. L. G. Suttorp, *Nuovo Cimento A* **29** (1975), 225.
4. K. Higashijima, *Progr. Theoret. Phys.* **55** (1976), 1591.
5. J. S. Goldstein, *Phys. Rev.* **91** (1953), 1516.
6. H. S. Green, *Phys. Rev.* **97** (1955), 540.
7. S. Mandelstam, *Proc. Roy. Soc. A* **233** (1955), 248.
8. R. Delbourgo, A. Salam, and J. Strathdee, *Nuovo Cimento A* **50** (1967), 193.
9. R. F. Keam, *J. Math. Phys.* **11** (1970), 394.
10. K. Higashijima and A. Nishimura, *Nuclear Phys. B* **113** (1976), 173.
11. R. Delbourgo and V. B. Prasad, *J. Phys. A* **10** (1977), 105.
12. R. F. Keam, *J. Math. Phys.* **12** (1971), 515.
13. B. J. Brennan and R. F. Keam, *Progr. Theoret. Phys.* **49** (1973), 1679.
14. L. G. Suttorp, *Nuovo Cimento A* **33** (1976), 257.
15. W. Kummer, *Nuovo Cimento* **31** (1964), 219.
16. G. C. Wick, *Phys. Rev.* **96** (1954), 1124.
17. R. E. Cutkosky, *Phys. Rev.* **96** (1954), 1135.
18. C. Schwartz, *Phys. Rev.* **137** (1965), B717; **141** (1966), 1468; C. Schwartz and C. Zemach, *Phys. Rev.* **141** (1966), 1454.
19. A. Pagnamenta, *Nuovo Cimento A* **53** (1968), 30.
20. K. Ladányi, *Nuovo Cimento A* **56** (1968), 173.
21. W. B. Kaufmann, *Phys. Rev.* **187** (1969), 2051.
22. E. Zur Linden, *Nuovo Cimento A* **63** (1969), 181, 193; E. Zur Linden and H. Mitter, *Nuovo Cimento B* **61** (1969), 389.
23. P. Narayanaswamy and A. Pagnamenta, *Nuovo Cimento A* **53** (1968), 635; *Phys. Rev.* **172** (1968), 1750.
24. A. H. Guth, *Ann. Phys. N.Y.* **82** (1974), 407.
25. W. Bauhoff, *Z. Naturforsch. A* **30** (1975), 656.

26. G. R. ALLCOCK, *Phys. Rev.* **104** (1956), 1799; G. R. ALLCOCK AND D. J. HOOTON, *Nuovo Cimento* **8** (1958), 590.
27. R. E. CUTKOSKY AND M. LEON, *Phys. Rev.* **135** (1964), B1445.
28. D. LURIÉ, A. J. MACFARLANE, AND Y. TAKAHASHI, *Phys. Rev.* **140** (1965), B1091.
29. N. NAKANISHI, *Phys. Rev.* **138** (1965), B1182.
30. M. CIAFALONI, *Nuovo Cimento A* **51** (1967), 1090.
31. G. TIKTOPOULOS, *J. Math. Phys.* **6** (1965), 573.
32. E. E. SALPETER, *Phys. Rev.* **87** (1952), 328.
33. G. FELDMAN, T. FULTON, AND J. TOWNSEND, *Phys. Rev. A* **8** (1973), 1149.
34. A. ERDÉLYI, "Higher Transcendental Functions," Vol. 2, pp. 64, 178, McGraw–Hill, New York, 1953.
35. A. ERDÉLYI, "Tables of Integral Transforms," Vol. 2, pp. 24, 145, 283, McGraw–Hill, New York, 1954.
36. K. M. CASE, *Phys. Rev.* **80** (1950), 797.
37. M. CIAFALONI, *Nuovo Cimento Suppl.* **5** (1967), 680.
38. W. M. FRANK, D. J. LAND, AND R. M. SPECTOR, *Rev. Mod. Phys.* **43** (1971), 36.
39. C. SCHWARTZ, *J. Math. Phys.* **17** (1976), 863.
40. R. E. MARSHAK, "Meson Physics," p. 172, McGraw–Hill, New York, 1952.
41. E. CORINALDESI AND F. STROCCHI, "Relativistic Wave Mechanics," p. 90, North–Holland, Amsterdam, 1963.