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LONG-TIME TAILS OF TIME CORRELATION FUNCTIONS FOR AN IONIC MIXTURE IN A MAGNETIC FIELD AND THE VALIDITY OF MAGNETO-HYDRODYNAMICS

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Mode-coupling theory is used to determine the long-time behaviour of the Green-Kubo integrands for the heat conductivity and the diffusion coefficients of an ionic mixture in a magnetic field. It is shown that the presence of several species of particles with a different ratio of charges and masses is a prerequisite for the validity of dissipative magnetohydrodynamics.

1. INTRODUCTION

In recent years the collective modes for a classical one-component plasma in a magnetic field have been studied by using a projection operator formalism[1,2]. With the help of mode-coupling theory the long-time behaviour of the velocity autocorrelation function[3] and the heat-conductivity time correlation function (or Green-Kubo integrand for the heat conductivity)[4] have been determined. The long-time tail of the heat-conductivity time correlation function has been found to decay as $t^{-1/2}$, owing to a coupling of two 'gyro-plasmon' modes. The same slowly-decaying long-time tail has been obtained by using a method based on kinetic theory[5,6]. This slow decay implies that the static heat conductivity coefficient for a one-component plasma in a magnetic field is divergent.

To investigate whether this divergency is a peculiarity of the one-component plasma we have recently studied[7] the long-time tails of the heat-conductivity time correlation function for an ionic mixture in a uniform magnetic field. The mixture consists of particles of several species, all with charges of the same sign, that move in an inert uniform background of opposite charge.

2. MODE SPECTRUM OF AN IONIC MIXTURE

The collective modes of a multicomponent ionic mixture are linear combinations of the partial particle densities $n_\sigma(\mathbf{k})$, with σ labelling the s components, the total momentum density $\mathbf{g}(\mathbf{k})$ and the total energy density $\epsilon(\mathbf{k})$ in Fourier space. The time development of these quantities is governed by the Liouville operator L acting in phase space. Employing a projection operator $P = 1 - Q$ that projects an arbitrary function in phase space on the set of linear combinations of n_σ, $\mathbf{g}$ and ϵ, one finds the collective modes as the eigenvectors of the $(s+4) \times (s+4)$–dimensional matrix

$$\Omega_{ij}(\mathbf{k}, z) = -\frac{1}{V} < \overline{a}_i^*(\mathbf{k}) L a_j(\mathbf{k}) > + \frac{1}{V} < \overline{a}_i^*(\mathbf{k}) LQ \frac{1}{z + QLQ} QLa_j(\mathbf{k}) > , \tag{1}$$

for small values of the wave vector $\mathbf{k}$. The angular brackets denote an equilibrium ensemble average, with V the volume. Furthermore, $a_j(\mathbf{k})$ are chosen from $\{n_\sigma, \mathbf{g}, \epsilon\}$, while $\overline{a}_i(\mathbf{k})$ are the adjoints satisfying the relation $V^{-1} < \overline{a}_i^*(\mathbf{k}) a_j(\mathbf{k}) >= \delta_{ij}$. The mode frequencies that govern the time evolution of the modes follow as the eigenvalues z of the frequency matrix. For an unmagnetized mixture the dispersion relation that determines the mode frequencies reads:

$$z^{s+2}\left[z^2 - zc(z) - \omega_{\rm p}^2\right] = 0\,, \tag{2}$$

in lowest order of the wavenumber. Here $\omega_{\rm p} = q_v/m_v^{1/2}$ is the collective plasma frequency, with q_v the total charge density and m_v the total mass density. Furthermore, $c(z)$ is given by:

$$c(z) = \beta \lim_{\mathbf{k}\to\mathbf{0}} \frac{1}{V} < \frac{q_v^*(\mathbf{k})}{k} LQ \frac{1}{z + QLQ} QL \frac{q_v(\mathbf{k})}{k} >\,, \tag{3}$$

with β the inverse temperature and $q_v(\mathbf{k}) = \sum_\sigma e_\sigma n_\sigma(\mathbf{k})$ the fluctuation of the charge density. As (2) shows the system supports $s+2$ modes with vanishing frequency in the long-wavelength limit. Two of these are viscous modes; the remaining s are mixed heat-diffusion modes. Furthermore, (2) possesses two complex solutions z_ρ, with $\rho = \pm 1$, which are generalized plasmon modes. Both modes are damped as a result of friction between the various components that oscillate out of phase. If all components have equal ratios of charge and mass this damping mechanism is absent. Indeed, it follows from (3) that $c(z) = 0$ in this case, since the electric current density is then proportional to the total momentum density. As a consequence the frequencies of the plasmon modes are then simply $\rho\omega_{\rm p}$, as in the one-component plasma. We shall call a mixture consisting of particles with equal charge-mass ratios a 'well-poised' mixture.

If the ionic mixture is magnetized the dispersion relation that determines the mode frequencies in order k^0 gets a more complicated form:

$$z^s\left\{(z^2 - zc - \omega_{\rm p}^2\hat{k}_\parallel^2)[z^2 - (\omega_B + {\rm i}a)^2] - z^2[(\omega_{\rm p} + b)^2 - (b')^2]\hat{k}_\perp^2 - 2{\rm i}z(\omega_B + {\rm i}a)(\omega_{\rm p} + b)b'\hat{k}_\perp^2\right\} = 0\,, \tag{4}$$

with $\omega_B = q_v B/(m_v c)$ the collective Larmor frequency. The symbols $\hat{k}_\parallel$ and $\hat{k}_\perp$ denote the components of the unit vector $\hat{\mathbf{k}} = \mathbf{k}/k$ parallel and perpendicular to the magnetic field, respectively. The coefficients a, b, b' and c depend on z. The last of these has been defined already in (3), while the other ones have a similar form, with one or both of the factors $q_v(\mathbf{k})/k$ replaced by $\mathbf{g}(\mathbf{k})$. In the magnetized case only s modes, viz. the mixed heat-diffusion modes, have a vanishing frequency in the long-wavelength limit. The viscous modes merge with the generalized plasmon modes so as to yield four generalized 'gyro-plasmon' modes. Their frequencies $z_{\lambda\rho}$, with $\lambda = \pm 1$ and $\rho = \pm 1$, are complex in general, so that these modes are damped. As in the unmagnetized case the dispersion relation becomes simpler if the mixture is 'well-poised'. In that case the coefficients a, b, b' and c all vanish, so that the gyro-plasmon mode frequencies are the solutions of the relation:

$$z^4 - (\omega_{\rm p}^2 + \omega_B^2)z^2 + \omega_{\rm p}^2\omega_B^2\hat{k}_\parallel^2 = 0\,. \tag{5}$$

This relation has the same form as that valid for the one-component plasma in a magnetic field. Again the modes are no longer damped in the long-wavelength limit.

The modes are linear combinations of the particle densities, the total momentum density and the energy density. For the unmagnetized mixture the plasmon modes are:

$$a_{\rho}(\mathbf{k}) = \beta^{1/2}\frac{q_v(\mathbf{k})}{k} + \left(\frac{\beta}{m_v}\right)^{1/2}\frac{\omega_{\mathrm{p}}}{z_{\rho}}\hat{\mathbf{k}}\cdot\mathbf{g}(\mathbf{k})\,. \tag{6}$$

It should be noted that $q_v(\mathbf{k})$ is divided by a factor k, so that its contribution seems to grow with decreasing values of the wavenumber. However, large-scale charge fluctuations are strongly suppressed owing to the long-range Coulomb forces.

The gyro-plasmon modes in a magnetized mixture also contain a term proportional to $q_v(\mathbf{k})/k$:

$$a_{\lambda\rho}(\mathbf{k}) = \beta^{1/2}\frac{q_v(\mathbf{k})}{k} + \left(\frac{\beta}{m_v}\right)^{1/2}\mathbf{v}_{\lambda\rho}(\hat{\mathbf{k}})\cdot\mathbf{g}(\mathbf{k})\,. \tag{7}$$

The vector $\mathbf{v}_{\lambda\rho}$ depends on the coefficients a, b and b', for $z = z_{\lambda\rho}$.

To determine the mode frequencies of the mixed heat-diffusion modes in order k^2 one may use perturbation theory with respect to the wavenumber. In this way one proves that these frequencies are proportional to the eigenvalues of the $(s\times s)$–dimensional matrix:

$$\mathsf{M}_{ij}(\hat{\mathbf{k}},z) = \sum_{n=1}^{s}\mathsf{N}_{in}\lim_{\mathbf{k}\to\mathbf{0}}\frac{1}{k^2V} < \left[a_n^{(0)}(\mathbf{k})\right]^* LQ\frac{1}{z+L}QLa_j^{(0)}(\mathbf{k}) > \,. \tag{8}$$

for $z\to \mathrm{i}0$, provided this limit exists. The elements of the matrix N are trivial thermodynamic derivatives which need not be specified here. The basis set $a_i^{(0)}(\mathbf{k})$ is given by:

$$a_1^{(0)}(\mathbf{k}) = \epsilon(\mathbf{k}) - \frac{h_v}{q_v}q_v(\mathbf{k})\;,\; a_{\sigma}^{(0)}(\mathbf{k}) = n_{\sigma}(\mathbf{k}) - \frac{n_{\sigma}}{q_v}q_v(\mathbf{k})\,, \tag{9}$$

with $\sigma = 2,\cdots,s$ and h_v the enthalpy per unit of volume. The derivation of (8) for a magnetized mixture is rather complicated[7]. In fact, perturbation theory first yields an expression containing the resolvent of QLQ and a few supplementary terms depending on the coefficients $a(z)$, $b(z)$ and $b'(z)$. Upon introducing the resolvent of L one finds that these supplementary terms drop out.

3. LONG-TIME TAILS OF TIME CORRELATION FUNCTIONS

To assess whether the limit of (8) for $z\to \mathrm{i}0$ exists one studies the asymptotic behaviour of the related time correlation functions:

$$F_{\alpha\beta}(\hat{\mathbf{k}},t) = \lim_{\mathbf{k}\to\mathbf{0}}\frac{1}{V} < \left[Q\hat{\mathbf{k}}\cdot\mathbf{j}_{\alpha}(\mathbf{k})\right]^* \mathrm{e}^{\mathrm{i}Lt}Q\hat{\mathbf{k}}\cdot\mathbf{j}_{\beta}(\mathbf{k}) > \,. \tag{10}$$

Here α en β take the values $1,\cdots,s$, with $\mathbf{j}_1\equiv\mathbf{j}_{\epsilon}$ the energy-current density and $\mathbf{j}_{\sigma}\equiv\mathbf{g}_{\sigma}/m_{\sigma}$ the particle-current density of component $\sigma = 2,\cdots s$. The long-time behaviour of these correlation functions can be obtained by using mode-coupling theory. In fact, we may write for large t:

$$F_{\alpha\beta}(\hat{\mathbf{k}},t) \simeq \lim_{\mathbf{k}\to\mathbf{0}}\frac{1}{2V}\sum_{ij}\sum_{\mathbf{q}}A_{ij}^{\alpha}(\mathbf{k},\mathbf{q})\left[\overline{A}_{ij}^{\beta}(\mathbf{k},\mathbf{q})\right]^*\exp\left\{-\mathrm{i}\left[z_i(\mathbf{q})+z_j(\mathbf{k}-\mathbf{q})\right]t\right\}\,, \tag{11}$$

where the summations are extended over all collective modes and over all values of the wave vector $\mathbf{q}$ of these modes. The mode-coupling amplitudes $A_{ij}^{\alpha}(\mathbf{k},\mathbf{q})$ are given by

$$A_{ij}^{\alpha}(\mathbf{k},\mathbf{q}) = \frac{1}{V} < \left[Q\hat{\mathbf{k}}\cdot\mathbf{j}_{\alpha}(\mathbf{k})\right]^* a_i(\mathbf{q})a_j(\mathbf{k}-\mathbf{q}) > \,. \tag{12}$$

The mode-coupling amplitudes $\overline{A}^{\beta}_{ij}(\mathbf{k},\mathbf{q})$ are defined analogously, with adjoint modes $\overline{a}_i$ and $\overline{a}_j$. Since the plasmon modes and the gyro-plasmon modes contain a term $q_v(\mathbf{k})/k$ the mode-coupling amplitudes for these modes can be divergent in the long-wavelength limit. Indeed, one finds:

$$\frac{1}{V} < \left[\hat{\mathbf{k}} \cdot \mathbf{j}_\epsilon(\mathbf{k})\right]^* \frac{q_v(\mathbf{q})}{q} \mathbf{g}(\mathbf{k}-\mathbf{q}) >= \frac{q_v}{q\beta^2}(\hat{\mathbf{k}} - \hat{\mathbf{k}} \cdot \hat{\mathbf{q}}\hat{\mathbf{q}}) \,, \tag{13}$$

for small values of the wavenumbers.

Slowly decaying contributions to the mode-coupling expression for the time correlation function $F_{\alpha\beta}(\hat{\mathbf{k}},t)$ arise if both modes i and j are undamped for small wavenumber. For an unmagnetized ionic mixture the generalized plasmon modes can therefore be excluded from the sum over the modes in (11). The contribution with the slowest decay arises from the coupling of the currents to a viscous mode and to a mixed heat-diffusion mode. Since the mode-coupling amplitudes for this coupling are of zeroth order in the wavenumber the resulting long-time behaviour of $F_{\alpha\beta}(\hat{\mathbf{k}},t)$ is proportional to $t^{-3/2}$, so that the Fourier transform $F_{\alpha\beta}(\hat{\mathbf{k}},z)$ is finite for $z \to \mathrm{i}0$. Hence, the transport coefficients occurring in the frequencies of the heat-diffusion modes are all finite for a general unmagnetized ionic mixture. If the mixture is well-poised, however, the above reasoning is not conclusive, since in that case the plasmon modes are no longer damped for small wavenumbers. The coupling of the energy current to a viscous mode and a plasmon mode is characterized by a mode-coupling amplitude that diverges for small wavenumber, as follows from (13). As a consequence the time correlation function $F_{11}(\hat{\mathbf{k}},t)$ has a tail proportional to $t^{-1/2}\cos(\omega_{\mathrm{p}}t+\theta)$, with θ a phase factor. Owing to the oscillating factor its Fourier transform is still finite as z goes to i0.

If a magnetic field is present the picture changes. For a general mixture the dominant contributions to the tails of the time correlation functions stem from the coupling of the currents to the heat-diffusion modes. The resulting tails are found to be proportional to $t^{-5/2}$. However, for the well-poised mixture in a magnetic field the situation is completely different. The gyro-plasmon modes are no longer damped. The coupling of the energy current to two gyro-plasmon modes in such a way that the zeroth-order mode frequencies in the exponent of (11) compensate each other leads to a slowly decaying tail in $F_{11}(\hat{\mathbf{k}},t)$ proportional to $t^{-1/2}$, without an accompanying oscillating factor. Hence, the transport coefficients appearing in the frequencies of the heat-diffusion modes for a well-poised magnetized ionic mixture are divergent. The conclusion is that the presence of several species of particles with different charge-mass ratios is necessary for the validity of magnetohydrodynamics, at least if dissipative effects are to be included.

REFERENCES

1) L. G. Suttorp and A. J. Schoolderman, Physica 141A (1987) 1.

2) M. C. Marchetti, T. R. Kirkpatrick and J. R. Dorfman, J. Stat. Phys. 46 (1987) 679, 49 (1987) 871.

3) L. G. Suttorp and A. J. Schoolderman, Physica 143A (1987) 494.

4) A. J. Schoolderman and L. G. Suttorp, Physica 144A (1987) 513.

5) A. J. Schoolderman and L. G. Suttorp, J. Stat. Phys. 53 (1988) 1237.

6) A. J. Schoolderman and L. G. Suttorp, Physica A 156(1989)795.

7) A. J. Schoolderman and L. G. Suttorp, in print.