ISOSPIN SUM RULES FOR INCLUSIVE CROSS-SECTIONS

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A systematic analysis of isospin sum rules is presented for the distribution functions of strong, electromagnetic and weak inclusive processes. The general expression for these sum rules is given and some new examples are presented.

Recently a number of papers [1-3] have appeared in which isospin sum rules and inequalities for inclusive processes are discussed. We shall present in this paper a systemaitc derivation, with the aid of the graphical technique of Yutsis, Levinson and Vanagas [4] (YLV), of all isospin sum rules (i.e. isospin equalities for distribution functions). Our procedure can be used, moreover, to derive all inequalities of the Lipkin-Peshkin type [1], but since these are incomplete and not necessarily of the optimal kind we shall restrict ourselves to the equalities.

The general *strong* interaction process of interaction process of interest is given by

$$A_1 + A_2 \rightarrow A_3 + A_4 + \dots + A_n + X \,. \tag{1}$$

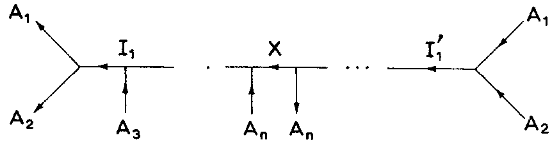
The number of incoming particles has for obvious reasons been chosen as two, although the isospin analysis is independent of this feature of the scattering process. The symbol X stands for the sum over all unobserved particles. The distribution function** dσ for process (1) may be written as:

$$\mathrm{d}\sigma(a_1, \dots, a_n) =$$

$$\sum_{X,x,\eta} \left| \sum_{\substack{I_1,\dots,I_{n-2}\\ i_1,\dots,i_{n-2}}} (-)^{I_1-i_1+\dots+I_{n-2}-i_{n-2}} \begin{pmatrix} A_1 & A_2 & I_1 \\ a_1 & a_2 & -i_1 \end{pmatrix} \times \begin{pmatrix} I_1 & A_3 & I_2 \\ i_1 & -a_3 & -i_2 \end{pmatrix} \dots \begin{pmatrix} I_{n-2} & A_n & X \\ i_{n-2} & -a_n & -x \end{pmatrix} T(I_1, \dots, I_{n-2}, X, \eta) \right|^2 , \tag{2}$$

where the isospin quantum number and its third component for particle i are A_i and a_i, respectively; the X-particle is characterized by isospin quantum numbers (X, x) and an additional quantum number η indicating the channel; I_j are intermediate isospins with third components i_j. The amplitudes $T(I_1, \dots, I_{n-2}, X, \eta)$ are to be considered as independent parameters. Working out the square and using the YLV graphical technique, one may write (2) in the form

$$\mathrm{d}\sigma(a_1, \dots, a_n) = \sum_{\substack{I_1,\dots,I_{n-2}\\ I'_1,\dots,I'_{n-2},X}} (-)^{A_1-a_1+\dots+A_n-a_n} F(I_1, \dots, I_{n-2}, I'_1, \dots, I'_{n-2}, X)$$



(3)

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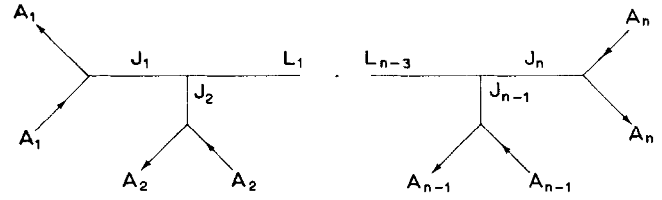
** By dσ we mean the Lorentz-invariant inclusive distribution function for any particular kinematical point; thus all our sum rules may equally be interpreted for the integrated cross-sections, etc.

with parameters F satisfying, at most, polygon inequalities. Recoupling the isospins in a more convenient way we get

$$\mathrm{d}\sigma(a_1, ..., a_n) = \sum_{\substack{J_1,...,J_n \\ L_1,...,L_{n-3}}} (-)^{A_1-a_1+...+A_n-a_n} G(J_1,...,J_n, L_1,...,L_{n-3}) \tag{4}$$



It follows immediately from the structure of the generalized Wigner symbol presented by the graph in (4) that the dependence on a_i appears only in the external vertices involving two A's and one J. Hence we may perform the sum over all L's for a fixed set of J's. Sum rules will now follow if the generalized Wigner symbol vanishes identically for some sets $\{J_1,...,J_n\}$ with J_i integers satisfying $0 \leqslant J_i \leqslant 2A_i$. Since in (4) each internal line carries zero third component of isotopic spin, the "internal" vertices with $j_1 + j_2 + j_3$ odd identically vanish [4]. In addition, the isospins meeting at a vertex must satisfy the triangular conditions. Consequently the generalized Wigner symbol vanishes if the set $\{J_1,...,J_n\}$ is "odd", i.e. if

$$\sum_{i=1}^{n} J_i = \text{odd} \tag{5}$$

and/or if the polygon conditions, viz.

$$\sum_{i=1}^{n} J_i \geqslant 2J_j \qquad (\text{for } j = 1, ..., n)\,. \tag{6}$$

are violated, whereas it may be proved that for "even" sets $\{J_1,...,J_n\}$ satisfying the polygon conditions there exists at least one set $\{L_1,...,L_{n-3}\}$ such that the generalized Wigner function is non-vanishing [5]. Thus we may write

$$\mathrm{d}\sigma(a_1,...,a_n) = \sum_{J_1,...,J_n}{}' \left[\prod_{i=1}^{n} (-)^{A_i-a_i} \begin{pmatrix} J_i & A_i & A_i \\ 0 & a_i & -a_i \end{pmatrix}\right] H(J_1,...,J_n)\,, \tag{7}$$

where the sum $\sum'$ runs over all allowed sets.

Now from the property of the 3j-symbol

$$\begin{pmatrix} J_i & A_i & A_i \\ 0 & a_i & -a_i \end{pmatrix} = (-)^{J_i+2A_i} \begin{pmatrix} J_i & A_i & A_i \\ 0 & -a_i & a_i \end{pmatrix} \tag{8}$$

it follows directly that, since only "even" sets occur in (7), $\mathrm{d}\sigma$ satisfies the charge-independence (CI) conditions

$$\mathrm{d}\sigma(a_1,...,a_n) = \mathrm{d}\sigma(-a_1,...,-a_n)\,. \tag{9}$$

To extract from (7) the non-charge-independence (NCI) relations we use the orthogonality properties of the 3j-symbols [4] and obtain the sum rules

$$\sum_{a_1,...,a_n} \left\{\prod_{i=1}^{n} (-)^{A_i-a_i} \begin{pmatrix} \bar{J}_i & A_i & A_i \\ 0 & a_i & -a_i \end{pmatrix}\right\} \mathrm{d}\sigma(a_1,...,a_n) = 0\,, \tag{10}$$

where $\{\bar{J}_1, ..., \bar{J}_n\}$ is an even set (with $\bar{J}_i$ integer and $0 \leqslant \bar{J}_i \leqslant 2A_i$) violating conditions (6). These form together the *complete* set of NCI sum rules that follow from isospin considerations.

The simplest examples of (10) involve one or more of the following $3j$-symbols:

$$\begin{pmatrix} 1 & A & A \\ 0 & a & -a \end{pmatrix} \propto (-)^{a-A} a; \quad \begin{pmatrix} 2 & A & A \\ 0 & a & -a \end{pmatrix} \propto (-)^{a-A} \{3a^2 - A(A+1)\}; \quad \begin{pmatrix} 3 & A & A \\ 0 & a & -a \end{pmatrix} \propto (-)^{a-A} \{5a^3 + a[1 - 3A(A+1)]\}, \tag{11}$$

where factors independent of a have been omitted. When we limit ourselves to reactions for particles with isospin $\leqslant 3/2$, only two types of NCI relations exist, those with $\bar{J}_i = 2, \bar{J}_j = 0$ (all $j \neq i$) or those with $\bar{J}_i = 3, \bar{J}_j = 1, \bar{J}_k = 0$ (all $k \neq i, j$); explicitly,

$$\sum_{a_1, ..., a_n} (3a_i^2 - 2)\, d\sigma(a_1, ..., a_n) = 0, \qquad \text{when } A_i = 1, \tag{12a}$$

$$\sum_{a_1, ..., a_n} (4a_i^2 - 5)\, d\sigma(a_1, ..., a_n) = 0, \qquad \text{when } A_i = \tfrac{3}{2}, \tag{12b}$$

$$\sum_{a_1, ..., a_n} (20a_i^3 - 41a_i) a_j\, d\sigma(a_1, ..., a_n) = 0, \qquad \text{when } A_i = \tfrac{3}{2}, A_j \geqslant \tfrac{1}{2}. \tag{13}$$

These equations clearly show that some sum rules may be built from those involving fewer particles by summing over the third components of isospin of the extra particles; this can be understood as an extraction of the extra particles from the X state.

A recoupling of the external legs in (4) does not yield new independent sum rules for $d\sigma$*. However, recoupled versions may be used to derive sum rules involving composite isospin states from which all inequalities of the Lipkin-Peshkin [1] type follow.

By adding and subtracting the sum rules (10) and the CI relations (9), a subset of equalities may sometimes be constructed in which one or more of the particles has fixed charge. For instance, when $A_i < A_j$, sum rules for a particular charge of i may be omitted; when $A_i + A_j < A_k$, sum rules in which both the charges of i and j are kept fixed may be found.

For the reaction $NN \to \pi\pi X$ two NCI sum rules exist, associated with the $\bar{J}$ sets $\{0, 0, 2, 0\}$ and $\{0, 0, 0, 2\}$. These involve distribution functions containing $\pi^0\pi^0$, which are experimentally difficult to measure, a difficulty that becomes even more troublesome when considering a system with more outgoing pions. However, in this case, by subtracting the two NCI relations and using the CI equations as well, we obtain the equation

$$d\sigma(pp\pi^0\pi^+) + d\sigma(pn\pi^0\pi^+) + d\sigma(pn\pi^0\pi^-) + d\sigma(pp^0\pi^-) = d\sigma(pp\pi^+\pi^0) + d\sigma(pn\pi^+\pi^0) + d\sigma(pn\pi^-\pi^0) + d\sigma(pp\pi^-\pi^0) \tag{14}$$

which does not contain distribution functions for π^0 pairs and which, to the best of our knowledge, has not previously been given. This sum rule, which involves proton-proton and proton-neutron incoming systems, may be useful in the analysis of inclusive proton-deuteron data.

For inclusive *weak* processes we have to construct sum rules without involving neutral spurions. It turns out that both for $\Delta S = 0$ and $\Delta S = \pm 1$ this is equivalent to considering sum rules with spurions of fixed charge. Consequently, as previously discussed, for $\Delta S = \pm 1$ at least one of the particles involved must have isospin $\geqslant 1$, while for $\Delta S = 0$ it must have isospin $\geqslant \frac{3}{2}$. The explicit sum rules then follow by use of (12a), (12b) and (13) together with the CI relations (9). The simplest examples have already been quoted elsewhere [2, 3].

The *electromagnetic* processes involve neutral spurions with $I = 0$ and $I = 1$, and interference terms between

* Unless it is known that certain resonance production dominates, in which case (4), with the appropriate external legs labelled by the isospin quantum numbers of resonance, may be used as well

these two may occur in $d\sigma$. Three graphical representations of generalized Wigner symbols must be considered, viz., for spurion external legs with isospins (0, 0), (0, 1) and (1, 1). In the presence of the interference term (9) no longer holds; only violations of the polygon conditions common to all three graphs will yield sum rules*. Since the spurous wave have a definite charge there must be, in this case, at least one particle with isospin $\geqslant \frac{3}{2}$ in order to get a sum rule. Examples are

$$d\sigma(\gamma D \to \Delta^{++}X) + 3d\sigma(\gamma D \to \Delta^{0}X) = 3d\sigma(\gamma D \to \Delta^{+}X) + d\sigma(\gamma D \to \Delta^{-}X) \tag{15}$$

and the related sum rule obtainable from this equation by extracting an antinucleon from the X state, line-reversing it and suppressing the deuteron:

$$\sum_{N=p,n} d\sigma(\gamma N \to \Delta^{++}X) - 3d\sigma(\gamma N \to \Delta^{+}X) + 3d\sigma(\gamma N \to \Delta^{0}X) - d\sigma(\gamma N \to \Delta^{-}X) = 0. \tag{16}$$

In this paper we have discussed all isospin sum rules that may be derived for strong, electromagnetic and weak interactions. The general form of the non-charge-independence relations has been written in (10) and the specific cases for particles with isospin $\leqslant \frac{3}{2}$ have been given in (12) and (13). A couple of new sum rules have been presented in (14) and (16). In the general analysis the graphical technique gave a considerable simplification as compared to a purely algebraic method, particularly in the proof of the completeness of the set of sum rules obtained.

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* For processes where, due to dynamical considerations, one of the graphs dominates, more sum rules may be obtained For example, in inclusive ρ photoproduction a one-pion-exchange model, in which the ρ is assumed to be produced at the photon vertex, involves only the $I = 0$ spurion and leads to sum rules such as $d\rho(\gamma p \to \rho^{+}) + d\sigma(\gamma p \to \rho^{-}) = 2d\sigma(\gamma p \to \rho^{0})$.

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